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On linear operators having supercyclic vectors

by

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Abstract. We show that for a real separable Banach space X there are operators in $B(X)$ having supercyclic vectors if and only if $\dim X \le 2$ or $\dim X = \infty$.

1. Introduction. Let $(X, \|\cdot\|)$ be a real (or complex) Banach space and $B(X)$ the set of linear continuous mappings from X into itself. Let $T \in B(X)$. A vector $x \in X$ is called (a) *cyclic*, (b) *supercyclic*, (c) *hypercyclic* if the orbit

$$\mathrm{Orb}(T,x) := \{T^n x : n \in \mathbb{N}_0\}$$

satisfies

(a) $\overline{\mathrm{span}(\mathrm{Orb}(T,x))} = X$,

(b) $\overline{\{\lambda y : y \in \mathrm{Orb}(T,x),\ \lambda \in \mathbb{R}(\mathbb{C})\}} = X$,

(c) $\overline{\mathrm{Orb}(T,x)} = X$

(see [5]).

As far as we know it is still an open problem whether there is an operator with hypercyclic vectors in every separable infinite-dimensional Banach space, and it is well known that there are none in finite dimensions (see [8]). In this paper we will characterize those separable Banach spaces which have operators with supercyclic vectors. Of course, a Banach space having such operators is separable. The main result is:

THEOREM 1. *Let $(X, \|\cdot\|)$ be a real separable Banach space. Then there exist operators in $B(X)$ having supercyclic vectors if and only if*

$$\dim X \in \{0,1,2\} \quad \textit{or} \quad \dim X = \infty.$$

To prove Theorem 1 we will use methods of the theory of universal functions developed by K.-G. Groß-Erdmann [4].

For further properties of the operator classes defined above compare, e.g., [1], [2], [5], [6] and [8].

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2. Universal elements. Let X_1 and Y_1 be topological spaces and $\Lambda = (L_n)$ a sequence of continuous mappings $L_n : X_1 \to Y_1$ ($n \in \mathbb{N}$). Then $x \in X_1$ is called *Λ-universal* if the set $\{L_n x : n \in \mathbb{N}\}$ is dense in Y_1. In [4, p. 11] Große-Erdmann showed that if X_1 is a Baire space and Y_1 is a metrizable and separable space, then the set of Λ-universal elements is residual in X_1 if and only if the set $\{(\xi, L_n\xi) : \xi \in X_1,\ n \in \mathbb{N}\}$ is dense in the topological product of X_1 and Y_1. Therein a subset of a Baire space is called *residual* if its complement is of first category.

3. Proof of Theorem 1. In the proof of Theorem 1 we will use the following lemma (see [8]).

LEMMA. *Let $(X, \|\cdot\|)$ be a finite-dimensional Banach space, $T \in B(X)$ and $x \in X$. Then, for the sequence $(T^n x)_{n=0}^{\infty}$, one of the following three possibilities holds*:

(a) $\lim_{n\to\infty} T^n x = 0$;

(b) $\lim_{n\to\infty} \|T^n x\| = \infty$;

(c) $\overline{\mathrm{Orb}(T,x)}$ *is compact and* $0 \notin \overline{\mathrm{Orb}(T,x)}$.

Proof of Theorem 1.

1) $\dim X < \infty$. If $\dim X \in \{0,1\}$, then the identity has supercyclic vectors. If $\dim X = 2$, then for $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ the operator

$$T = \begin{pmatrix} \cos(2\pi\alpha) & \sin(2\pi\alpha) \\ -\sin(2\pi\alpha) & \cos(2\pi\alpha) \end{pmatrix}$$

has supercyclic vectors, since for irrational α the sequence $(n\alpha)_{n=0}^{\infty}$ is uniformly distributed modulo 1 (see [7, p. 71]).

Now let $k = \dim X \geq 3$. Assume that $T \in B(X)$ has supercyclic vectors. Then T must be regular and since cT has supercyclic vectors for every $c \in \mathbb{R}\setminus\{0\}$ if T has, we can assume without loss of generality that T has an eigenvalue λ with $|\lambda| = 1$. So we can find a one(two)-dimensional T-invariant subspace U if λ is (not) real and a $(k-1)$(resp. $(k-2)$)-dimensional T-invariant subspace W such that $X = U \oplus W$, and $\overline{\{T^n u : n \in \mathbb{N}_0\}}$ is compact and does not contain 0 for every $u \in U\setminus\{0\}$. If λ is not real this follows from the fact that together with λ also $\overline{\lambda}$ is an eigenvalue, since X is a real vector space. Since T has supercyclic vectors, there must be an $x \in X$ of the form $x = u + w$, $u \in U\setminus\{0\}$, $w \in W\setminus\{0\}$, such that $\{\lambda y : y \in \mathrm{Orb}(T,x),\ \lambda \in \mathbb{R}\}$ is dense in X.

By our lemma, there are three possibilities for the sequence $(T^n w)_{n=0}^{\infty}$:

(a) $\lim_{n\to\infty} T^n w = 0$. There must be a subsequence $(T^{n_m} x)_{m=0}^{\infty}$ and a sequence of real numbers $(\mu_m)_{m=0}^{\infty}$ such that $\mu_m T^{n_m} u + \mu_m T^{n_m} w \to u + w$ as $m \to \infty$. But from this it follows that $(\mu_m)_{m=0}^{\infty}$ is bounded and, since $T^{n_m} w \to 0$ as $m \to 0$, we have a contradiction.

(b) $\lim_{n\to\infty} \|T^n w\| = \infty$. In this case it follows from $\mu_m T^{n_m} u + \mu_m T^{n_m} w \to u$ as $m \to \infty$ that $\mu_m T^{n_m} w \to 0$ as $m \to \infty$ and so $\mu_m T^{n_m} x \to 0$, which is also a contradiction.

(c) $0 \notin \overline{\mathrm{Orb}(T,w)}$ compact. Then from $\mu_m T^{n_m} u + \mu_m T^{n_m} w \to w$ we conclude $\mu_m \to 0$ as $m \to \infty$ and so $\mu_m T^{n_m} x \to 0$ as $m \to \infty$, which is again a contradiction.

So T cannot have supercyclic vectors.

2) $\dim X = \infty$. In this case, by a theorem due to Ovsepian and Pełczyński there exists a sequence $(x_k)_{k=1}^{\infty}$ in X and a sequence $(\varphi_k)_{k=1}^{\infty}$ in X^* with the following properties:

(1) $\varphi_k(x_l) = \delta_{k,l}$, $k, l \in \mathbb{N}$.

(2) $\overline{\mathrm{span}\{x_k : k \in \mathbb{N}\}} = X$.

(3) $\varphi_k(x) = 0$, $k \in \mathbb{N} \Rightarrow x = 0$.

(4) $\|x_k\| = 1$, $k \in \mathbb{N}$; $\sup_{k\in\mathbb{N}} \|\varphi_k\| = C < \infty$.

We define $T : X \to X$ by $Tx = \sum_{k=1}^{\infty} (1/2)^k \varphi_{k+1}(x) x_k$ and so $T \in B(X)$ as a consequence of (4). We will show that T has supercyclic vectors to do that it is enough to show that there is an $x \in X$ and a sequence $(\mu_n)_{n=0}^{\infty}$ of real numbers such that $\{\mu_n T^n x : n \in \mathbb{N}_0\}$ is dense in X.

We choose $\mu_n = 2^{n^2}$, $n \in \mathbb{N}_0$, and will show now that the sequence $\Lambda = (2^{n^2} T^n)_{n=0}^{\infty}$ has Λ-universal elements. We find that

$$T^n x = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{kn + n(n-1)/2} \varphi_{k+n}(x) x_k, \qquad n \in \mathbb{N},\ x \in X.$$

Let $\varepsilon > 0$, $u = \sum_{j=1}^{\nu} \alpha_j x_j$, $v = \sum_{j=1}^{\nu} \beta_j x_j$ with $\alpha_1, \ldots, \alpha_\nu, \beta_1, \ldots, \beta_\nu \in \mathbb{R}$. We will prove that there is a $w \in X$ and an $n_0 \in \mathbb{N}_0$ such that $\|w - u\| \leq \varepsilon$ and $\|2^{n_0^2} T^{n_0} w - v\| \leq \varepsilon$. From (2) we then deduce that $\{(\xi, 2^{n^2} T^n \xi) : \xi \in X,\ n \in \mathbb{N}_0\}$ is dense in $X \times X$ and, using the results of Große-Erdmann, we are done. Let

$$w = u + \sum_{j=1}^{\nu} 2^{j n_0 + n_0(n_0-1)/2 - n_0^2} \beta_j x_{j+n_0}$$

with $n_0 \geq \nu$ such that $\|w - u\| \leq \varepsilon$. Since $n_0 \geq \nu$, we get from (1)

$$T^{n_0} u = 0$$

and therefore

$$2^{n_0^2}T^{n_0}w = T^{n_0}\Big(\sum_{j=1}^{\nu} 2^{jn_0+n_0(n_0-1)/2}\beta_j x_{j+n_0}\Big)$$
$$= \sum_{j=1}^{\nu}\Big(\frac{1}{2}\Big)^{jn_0+n_0(n_0-1)/2} 2^{jn_0+n_0(n_0-1)/2}\beta_j x_j = v\,.$$

So $\|2^{n_0^2}T^{n_0}w - v\| = 0 \le \varepsilon$. ∎

4. Final remarks. 1) By analogy with the proof of Theorem 1, one can prove that for a complex separable Banach space there are operators with supercyclic vectors if and only if $\dim X \in \{0,1\}$ or $\dim X = \infty$.

2) In the infinite-dimensional case, the operator T in the proof of Theorem 1 is compact. An operator with hypercyclic vectors cannot be compact (see [6]).

3) In [2, p. 42] a supercyclic vector $x \in X$ for $T \in B(X)$ and X real is defined by $\overline{\{\lambda y : y \in \mathrm{Orb}(T,x),\ \lambda > 0\}} = X$. Also with this definition, Theorem 1 holds with exactly the same proof with the only difference that in the case $\dim X = 1$ we take minus identity.

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Functionals on transient stochastic processes with independent increments

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Abstract. The paper is devoted to the study of integral functionals $\int_0^\infty f(X(t,\omega))\,dt$ for a wide class of functions f and transient stochastic processes $X(t,\omega)$ with stationary and independent increments. In particular, for nonnegative processes a random analogue of the Tauberian theorem is obtained.

1. Notation and preliminaries. Let μ be a Borel measure on the real line $(-\infty,\infty)$. We denote by $L(\mu)$ the space of all complex-valued Borel functions f with the finite norm

$$\|f\|_\mu = \int_{-\infty}^{\infty} |f(x)|\,\mu(dx)\,.$$

The measure μ is said to be *shift-bounded* if

$$\sup\{\mu([a+x, b+x]) : x \in (-\infty,\infty)\} < \infty$$

for every bounded interval $[a,b]$. All measures under consideration in the sequel will tacitly be assumed to be shift-bounded and not identically equal to 0. The support of a function f is denoted by $\operatorname{supp} f$. The indicator of a set A is denoted by 1_A.

Put $\gamma(dx) = e^{-|x|}\,dx$. The space $L_\infty(\gamma)$ consists of all complex-valued Borel functions f with the finite norm

$$\|f\|_\infty = \operatorname{vrai\,sup}\{|f(x)| : x \in (-\infty,\infty)\}\,.$$

In the sequel we shall briefly say "almost everywhere" instead of "γ-almost everywhere".

If the integral $\int_{-\infty}^{\infty} |f(x+y)|\,\mu(dy)$ is finite for almost all x, then the convolution $f * \mu$ is defined by the formula

$$(f*\mu)(x) = \int_{-\infty}^{\infty} f(x+y)\,\mu(dy)\,.$$

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