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ON LINEAR PLANES

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ABSTRACT. A linear plane over a ground field k is an algebraic surface in affine 3-space over k which is biregular to the affine plane and whose equation is linear in one of the three variables of the 3-space. In this note we give a concrete description of a linear plane over a field of characteristic zero, thereby proving it to be an embedded plane, i.e. we show that by an automorphism of the affine 3-space, it can be transformed to a coordinate plane.

1. **Introduction.** Let $A(n, k)$ denote a polynomial ring in n-variables over a domain k, which we (geometrically) call the *affine n-space* over the ground domain k. By a *hypersurface* in $A(n, k)$ we mean any nonunit principal ideal, say (f), $f \in A(n, k)$. If there is no confusion we will simply say that "f is a hypersurface in $A(n, k)$".

Now let k be a field. A hypersurface f is defined to be

(1) *a hyperplane over k in $A(n, k)$*, if $A(n, k)/(f) \approx A(n-1, k)$;

(2) *a general hyperplane over k*, if $A(n, \bar{k})/(f+\lambda) \approx A(n-1, \bar{k})$ for all $\lambda \in \bar{k}$, where $\bar{k}$ = the algebraic closure of k;

(3) *a generic hyperplane over k*, if $A(n, k(T))/(f-T) \approx A(n-1, k(T))$, where T is an indeterminate over k;

(4) *an embedded hyperplane over k*, if

$$A(n, k) = B[f], \quad \text{where } B \approx A(n-1, k);$$

(5) *a linear hypersurface over k*, if $A(n, k) = B[Z]$, with $B \approx A(n-1, k)$ and $Z \in A$ such that $f = aZ + b$ with $a, b \in B$, $a \neq 0$;

(6) *a linear hyperplane over k*, if f is both a linear hypersurface and a hyperplane.

As usual, when $n \leqslant 3$ we drop "hyper" and for $n = 2$ replace the words surface and plane by curve and line respectively.

Now assume that either

$(*)$ f is a hyperplane, or (as possibly stronger hypothesis)

$(**)$ f is a linear hyperplane.

For each $n \geqslant 1$, the following questions arise naturally.

Q($1.n$). Is f a general hyperplane over k?

Q($2.n$). Is f a generic hyperplane over k?

Q($3.n$) (Epimorphism problem). Is f an embedded hyperplane over k?

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Note that "yes" to Q(3.n) clearly implies "yes" to Q(1.n) and Q(2.n). Also note that for $n = 1$, the answer is (trivially) "yes" to all questions.

For $n = 2$ and char $k = 0$, Abhyankar and Moh gave an affirmative answer to all questions by the Epimorphism Theorem, which is an affirmative answer to Q(3, 2), with hypothesis (*) [**AM**].

For $n = 2$ with hypothesis (*), and char $k = p \neq 0$, Q(3.2) and Q(2.2) are known to have a common counterexample, namely

$$f = y^{p^2} - x - x^{2p} \in k[x, y] \approx A(2, k).$$

It is easy to see that the same example serves as a counterexample to Q(2.n) and Q(3.n) in general. We point out that *as yet no counterexamples to* Q(1.2) *seem to be known.*

With hypothesis (**) and $n = 2$, however, it is trivial to show that the answer to all questions is affirmative even in characteristic $p \neq 0$.

In this note, we prove the first step for $n = 3$, by proving the

THEOREM. *If* char $k = 0$, *then any linear plane in* $A(3, k)$ *is an embedded plane.*

I would like to thank Professor Heinzer and Mr. Gurjar for stimulating conversations on this problem.

In [**R**] Peter Russell has established our Lemma 3 in any characteristic, thereby establishing the Theorem for arbitrary field k.

2. **Proof of the Theorem**. First we introduce some notation to be fixed throughout.

NOTATION. Write $A(3, k) = C[Z]$ such that $H \in C[Z]$ is a linear plane linear in Z and $C \approx A(2, k)$. If K denotes the quotient field of C then $K[Z] = K[H]$. Thus there is a *retract* $\psi: K[Z] \to K$ with kernel H. Write

$$H = gZ - f; \qquad g, f \in C.$$

Then $\psi(Z) = f/g$. Restriction of ψ to $C[Z]$ gives a map $C[Z] \to C[f/g]$ which is the identity map on C. We denote $C[f/g] \approx C[Z]/(H)$ by B. By hypothesis $B \approx A(2, k) \approx C$.

The theorem will be proved, when we show H to be an embedded plane over k.

LEMMA 1. *Assume that k is of arbitrary characteristic p. If f_1, f_2 are two lines in C such that $(f_1, f_2)C = C$ then there exist c, d $\in$ k such that*

$$f_2 = cf_1 + d, \qquad c \neq 0.$$

PROOF. Note that the only units modulo a line are constants (nonzero elements in the image of k modulo the line). Since f_2 is a unit modulo f_1, we get

$$f_2 = c_1 f_1 + d_1, \qquad 0 \neq c_1 \in C, d_1 \in k.$$

Similarly we can write

$$f_1 = c_2 f_2 + d_2, \qquad 0 \neq c_2 \in C, d_2 \in k.$$

Comparing degrees with respect to any choice of x, y with $C = k[x, y]$, we easily see from the above two equations that

$$\text{degree } f_1 = \text{degree } f_2 \quad \text{and} \quad \text{degree } c_1 = \text{degree } c_2 = 0.$$

The desired equation now follows.

COROLLARY 1. *Let* $h \in k[x, y] = C$ *and* k' *a separable extension of* k. *Assume that*

$$h = \prod_1^s u_i^{r_i}, \qquad r_i > 0,$$

such that $u_i \in k'[x, y] \approx A(2, k')$ *are distinct lines over* k'. *Moreover assume that* $(u_i, u_j)k'[x, y] = k'[x, y]$, *for* $i \neq j$. *Then there exist* $c_i, d_i \in k'$ *and* $u \in k[x, y]$ *such that*

$$(*) \qquad u_i = c_i(u + d_i).$$

Further, $\prod_1^s c_i^{r_i} = c$ *is in* k *and hence*

$$h = c\prod_1^s (u + d_i)^{r_i}.$$

PROOF. From Lemma 1 we can certainly find $u \in k'[x, y]$ such that $(*)$ is satisfied. We start with such a choice and modify it to get it to be in $k[x, y]$.

By making an automorphism of $k[x, y]$ we can assume that the coefficient of the top y-degree in u is some nonzero $e \in k'$; and replacing u by u/e we get that *u is monic in y*.

We will now show that all coefficients of u except possibly the constant term belong to k.

Let σ be an isomorphism of k'/k. By $\sigma(u)$ we shall denote the result of acting σ on all coefficients of u.

If $u \in k[x, y]$, then we are finished with the proof. Otherwise we can choose an isomorphism of k'/k such that $\sigma(u) \neq u$. By the expression of h, we get $\sigma(u) = au + b$, $a, b \in k'$.

Since $\sigma(u)$, u are both monic in y, we get $a = 1$. Thus $\sigma(u) - u \in k'$.

Since this is true for any k-isomorphism of k' and since k'/k is separable, we get that all coefficients of u but the constant term d say, belong to k.

Replacing u by $u - d$, we get the desired expression because $c \in k$ is obvious by comparing the top degree coefficient of y on both sides.

LEMMA 2. *Let* k'/k *be a separable algebraic extension. If* $u \in k[x, y] = C \subset k'[x, y]$ *is a line over* k', *then* u *is already a line over* k.

PROOF. Let $R = k[x, y]/(u)$ and $R' = k'[x, y]/(u)$. As usual, we may assume $k \subset R \subset R'$ and $k' \subset R'$ by identifying isomorphic rings. Further, by extending k' if necessary we may assume k' is Galois over k.

Let $a \in R$ be algebraic over k. By assumption $R' = k'[t]$ for some t. Hence $a \in k'$. Choose $h \in k[x, y]$ to be some preimage of a. If $a \neq 0$, then h is unit modulo $(u)k'[x, y]$ and hence a constant modulo $(u)k'[x, y]$, i.e.

$$(1) \qquad h = u(x, y)p(x, y) + a^*, \qquad a^* \in k', p(x, y) \in k'[x, y].$$

Let σ be any k-automorphism of k'. Then applying σ to (1)

$$(2) \qquad h = u(x, y)\sigma(p(x, y)) + \sigma(a^*).$$

Subtracting (1) from (2) we conclude that $u(x, y)$ divides $a^* - \sigma(a^*)$ in $k'[x, y]$. Thus $a^* = \sigma(a^*)$. Since this holds for each σ, a^* is fixed by each member of the Galois group and hence $a^* \in k$.

Thus k is relatively algebraically closed in R. It is easy to see that we can replace k' by some finite extension k^* of k with $k^* \subset k'$ such that $u \in k^*[x, y]$ is a line over k^*, since we need to include in k^* only the coefficients of polynomials expressing images of x, y modulo u in terms of t and t in terms of images of x, y modulo u. Since R is clearly normal (say by the Jacobian criterion) we can apply (2.9) of [**AEH**] to get that $R = k[t^*]$ for some $t^* \in R$. Thus $u \in k[x, y]$ is a line over k.

REMARK. *If K is a field of characteristic zero and $u \in K[x, y] \approx A(2, K)$ is a line over K, then there exists a $v \in K[x, y]$ such that v is in the ring generated by the coefficients of u and $K[u, v] = K[x, y]$.*

This version of the Epimorphism Theorem can be deduced from [**AM**] by observing that v, in their terminology, is an "approximate root" of u and hence has the above stated property.

Thus in the above Lemma 2, we can find $v \in k[x, y]$ such that $k'[u, v] = k'[x, y]$. Writing x, y as polynomials in u, v over k' and taking "trace" it is easy to check that then $k[x, y] = k[u, v]$.

LEMMA 3. *With the basic notations introduced at the beginning of this section we get the following. If* char $k = 0$, *then there is an embedded line u in C over k such that*

$$g = c\prod_{1}^{s}(u + d_i)^{r_i}, \qquad r_i > 0,$$

where $c \in k$ and $\{d_i\}$ are distinct elements algebraic over k.

PROOF. First suppose that the result is already true when k is replaced by its algebraic closure, say k'. In view of Corollary 1, u may be assumed to be in $k[x, y]$ and in view of Lemma 2, u is an embedded line over k. Thus we *may assume that k is algebraically closed.*

Let $\mathfrak{M} = \{M_1, \ldots, M_s\}$ be the set of all maximal ideals in C containing f, g. Clearly $B/M_iB \approx (C/M_i)[Z] \approx k[Z]$. Hence by the Epimorphism Theorem [**AM**] each M_iB is an embedded line over k (in B). Let $u_iB = M_iB$. Clearly g has a factorization in B,

$$g = \prod_{1}^{s} u_i^{r_i}, \qquad r_i > 0.$$

Also, it is clear from $u_iB = M_iB$ that $(u_i, u_j)B = B$, if $i \neq j$.

The expression for g follows by applying Corollary 1 (with $k = k'$). It remains to check that u is already a line over k in C.

Choose $\lambda \in k$ such that $u + \lambda \neq u + d_i$ for $i = 1, \ldots, s$. We see that the image of f/g modulo $(u + \lambda)B$ is contained in the image of C modulo $C \cap (u + \lambda)B$, and hence $B/(u + \lambda)B = C/(u + \lambda)C$.

Now $u + \lambda$ is clearly an embedded line over k in $B \approx A(2, k)$, and hence we get

$$B/(u + \lambda)B \approx A(1, k) \approx C/(u + \lambda)C.$$

Then $u + \lambda$ is an embedded line in C over k and hence so is u.

LEMMA 4. *Let g be as in the conclusion of Lemma 3. Let $v \in C$ such that $k[u, v] = C$. Let $f = a_0 + a_1v + \cdots + a_nv^n$, $a_i \in k[u]$. Then*:

(1) *a_1 is a unit in $k[u]$ modulo g.*

(2) $a_2, \ldots, a_n$ *are nilpotent in* $k[u]$ *modulo* g.

PROOF. Let k' be the algebraic closure of k. Clearly, we only need to show that

$$(1^*) \qquad a_1 \notin (u + d_i)k'[u], \text{ for } i = 1, \ldots, s; \text{ and}$$

$$(2^*) \qquad a_j \in (u + d_i)k'[u], \text{ for } i = 1, \ldots, s \text{ and } j = 2, \ldots, n.$$

Since the hypothesis about f, g, u and H is unchanged if k is replaced by k' we may *assume that* $k = k'$, *i.e.* k *is algebraically closed.*

Let, as in Lemma 3, $M_i = (u + d_i)B \cap C$ and let $M_i^* \in C[Z]$ be the preimages of M_i under ψ. Then it is clear that

$$M_i^* = (u + d_i, gZ - f)C[Z] = (u + d_i, f)C[Z].$$

Hence we get that

$$M_i = (u + d_i, f)C = M_i^* \cap C.$$

Thus f generates a maximal ideal in C modulo$(u + d_i)$ and hence the image of f in the canonical homomorphism $C \to C/(u + d_i)$ is a ring generator of $C/(u + d_i)$ over k. Since image of v has the same property (1*) and (2*) are easily seen.

REMARK. In the above proof, if we simply assume $u \in C$ to be a line over k (not necessarily embedded, as may happen in characteristic $p \neq 0$) then we still get that the image of f modulo $(u + d_i)$ in C is a ring generator.

Moreover, if $Q(1, 2)$ could be answered in the affirmative in characteristic $p \neq 0$, then one could modify Lemma 3 to prove that $u + d_i$ is a line in $k[x, y]$ over k (if k is algebraically closed).

Now let S be any ring and $S[V]$ a polynomial ring over S. A sequence $(c_0, c_1, \ldots, c_n)$ in S is said to be a *generator sequence* if

(1) c_1 is a unit in S, and

(2) $c_2, \ldots, c_n$ are nilpotent in S.

LEMMA 5. *Let* $S[V]$ *be as above and let* $h = c_0 + c_1V + \cdots + c_nV^n \in S[V]$ *where* $(c_0, c_1, \ldots, c_n)$ *is a generator sequence. Then there exists a generator sequence* $(0, c_1^*, \ldots, c_n^*)$ *in* S *and some* $c \in S$ *such that*

$$h = c_1^*(V + c) + \cdots + c_n^*(V + c)^n.$$

In other words,

$$h = (V + c)\ (\textit{unit in } S[V])$$

and hence

$$(V + c) = h\ (\textit{unit in } S[V]).$$

REMARK. The reason to call $(c_0, c_1, \ldots, c_n)$ a generator sequence is that the corresponding $h = c_0 + c_1V + \cdots + c_nV^n$ is a ring generator of $S[V]$ over S.

If h is a ring generator then it is easy to check that $(c_0, c_1, \ldots, c_n)$ is a generator sequence. To see the converse, let T be an indeterminate over $S[V]$. Apply the Lemma to $h - T$ and $S[T]$ in place of h and S, to get some $c \in S[T]$ such that $h - T \in (V + c)S[T][V]$.

Let σ: $S[V][T] \to S[T]$ be defined by $\sigma(V) = -c$, $\sigma(T) = T$ and $\sigma(s) = s$

for $s \in S$. The restriction of σ to $S[V]$ is an S-isomorphism taking h to T. It follows that $S[V] = S[h]$.

PROOF. For any generator sequence $(e_0, \ldots, e_n)$ in S we define a new generator sequence $(\tau(e_0), \ldots, \tau(e_n))$ as follows.

Let $e = e_0 e_1^{-1}$. Write $\tau(V) = V + e$. Then $h(V)$ has a unique expression

$$h(V) = e_0' + e_1'(V + e) + \cdots + e_n'(V + e)^n, \qquad e_i' \in S.$$

We define $(\tau(e_0), \ldots, \tau(e_n)) = (e_0', \ldots, e_n')$, which can be checked to be a generator sequence.

By iteration of τ on $(c_0, \ldots, c_n)$ we see that for large enough m we have,

$$\tau^m(c_0) = 0, \quad \tau^m(V) = V + c \quad \text{for some } c \in S.$$

The proof is done by taking $(c_0^*, \ldots, c_n^*) = (\tau^m(c_0), \ldots, \tau^m(c_n))$ and $V + c = \tau^m(V)$.

COROLLARY 2. *Let T be an indeterminate over $C[Z]$. Let Φ: $C[T][Z] \to C[T][(f + T)/g]$ be the unique epimorphism which is identity on $C[T]$ and carries Z to $(f + T)/g$. Clearly Φ has kernel $H - T$.*

Let $C = k[u, v]$ as in Lemma 4. Then there exists $c \in k[u, T]$ such that $v + c = P(f + T) + Qg$ where $P = P(T)$, $Q = Q(T) \in C[T]$.

PROOF. Apply Lemma 5 with $S = k[u, T]/g$, $V = v$ and $h = f + T$; using Lemmas 3 and 4 to check the hypothesis.

COROLLARY 3. *In the notation of Corollary 2 let $z^* = \Phi(Z)$, $v^* = v + c$. Then we have*

$$\Phi(C[T][Z]) = k[T, u, Pz^* + Q].$$

PROOF. Let $R = k[T, u, Pz^* + Q]$. Since $g \in k[u] \subset R$ we get that $v^* = g(Pz^* + Q) \in R$. Also write $f + T = b_0^* + b_1^*(v^*) + \cdots + b_n^*(v^*)^n$ where $b_i^* \in k[T, u]$ and $b_0^* \in (g)k[T, u]$, as obtained by the application of Lemma 5 performed in Corollary 2. Then

$$z^* = \sum_0^n b_i^* g^{i-1}(Pz^* + Q)^i \in R$$

since $g, b_0^* g^{-1} \in k[T, u] \subset R$.

It is now clear that

$$\Phi(C[T][Z]) = \Phi(k[u, v, T, Z]) = k[u, v, T, z^*] = k[u, v^*, T, z^*] \subset R.$$

Since the opposite containment is clear, the proof is finished.

PROOF OF THE THEOREM. We *claim that*

$$C[Z] = k[H, u, P(H)Z + Q(H)]$$

and this will clearly establish the theorem. For proof, let $F \in C[Z]$. Then there exists a polynomial $F^*(X, Y, Z) \in k[X, Y, Z] \approx A(3, k)$ such that $\Phi(F) = F^*(T, u, Pz^* + Q)$.

But then using $\Phi(T) = \Phi(H)$ we get that

$$\Phi(F - F^*(H, u, P(H)Z + Q(H))) = 0,$$

i.e.

$$F - F^*(H, u, P(H)Z + Q(H)) = (H - T)F^{**},$$

where $F^{**} \in C[T][Z]$.

Putting $T = H$ on both sides we get $F = F^*(H, u, P(H)Z + Q(H))$. The claim now follows.

REMARK. The above proof can be easily generalized to the following criterion for embedded hyperplanes. Let T be an indeterminate over $A(n, k)$.

$H \in A(n, k)$ is an embedded hyperplane over $k \Leftrightarrow A(n, k)[T]/(H - T) \approx A((n-1), k[T]) \Leftrightarrow A(n, k[T])/(H - T) \approx A(n-1, k[T])$.

This also provides an alternative definition (4) in the introduction.

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