ON LINEAR TRANSFORMATIONS PRESERVING
THE PÓLYA FREQUENCY PROPERTY

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ABSTRACT. We prove that certain linear operators preserve the Pólya frequency property and real-rootedness, and apply our results to settle some conjectures and open problems in combinatorics proposed by Bóna, Brenti and Reiner-Welker.

1. INTRODUCTION

Many sequences encountered in various areas of mathematics, statistics and computer science are known or conjectured to be unimodal or log-concave; see [7, 30, 32]. A sufficient condition for a sequence to enjoy these properties is that it is a Pólya frequency (PF for short) sequence, or equivalently for finite sequences, that its generating function has only real and non-positive zeros. It is often the case that the generating function of a finite PF-sequence has more transparent properties when expanded in a basis other than the standard basis $\{x^i\}_{i\geq 0}$ of $\mathbb{R}[x]$. Therefore it is natural to investigate how PF-sequences translate when expressed in various bases. This amounts to studying properties of the linear operator that maps one basis to another. A systematic study of this was first pursued by Brenti in [6]. This is also the theme of this paper.

In Section 3 we will study linear operators of the type

$$\phi_F = \sum_{k=0}^{n} Q_k(x) \frac{d^k}{dx^k},$$

where $F(x,z) = \sum_{k=0}^{n} Q_k(x) z^k \in \mathbb{R}[x,z]$. Here we will give sufficient conditions on F for ϕ_F to preserve the PF-property. The results attained generalize and unify theorems of Hermite, Poulain, Pólya and Schur. In this section we will also give a sufficient condition for a family of natural $\mathbb{R}$-bilinear forms to preserve the PF-property in both arguments. This generalizes results of Wagner [10, 35, 36].

An important linear operator in combinatorics is the operator defined by $\mathcal{E}(\binom{x}{i}) = x^i$, for all $i \in \mathbb{N}$. In Section 4 we will prove that whenever a polynomial f of degree d has non-negative coefficients when expanded in the basis $\{x^i(x+1)^{d-i}\}_{i=0}^{d}$, the polynomial $\mathcal{E}(f)$ will have only real, non-positive and simple zeros.

In the remainder of the paper we use the theory developed to settle some conjectures and open problems raised in combinatorics. In Section 5 we prove that the numbers $\{W_t(n,k)\}_{k=0}^{n-1}$ of t-stack sortable permutations in $\mathcal{S}_n$ with k descents form PF-sequences when $t = 2, n-2$, thereby settling two new cases of an open problem proposed by Bóna [2, 3].

In Section 6 we prove that the q-Eulerian polynomials $A_n(x;q)$, defined by Foata and Schützenberger [16] and further studied by Brenti in [9], have only real zeros for all integers q. This settles a conjecture raised by Brenti. Here we also continue the study of the W-Eulerian polynomials, defined for any finite Coxeter group W, and the q-analog $B_n(x;q)$, initiated by Brenti in [8].

In Section 7 we prove that the h-vectors of a family of simplicial complexes associated to finite Weyl groups defined by Fomin and Zelevinski [17] are PF, thus settling an open problem raised by Reiner and Welker [28].

2. Notation and preliminaries

In this section we collect definitions, notation and results that will be used frequently in the rest of the paper. Let $\{a_i\}_{i=0}^{\infty}$ be a sequence of real numbers. It is *unimodal* if there is a number p such that $a_0 \leq a_1 \leq \cdots \leq a_p \geq a_{p+1} \geq \cdots$, and *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $i > 0$.

An infinite matrix $A = (a_{ij})_{i,j\geq 0}$ of real numbers is *totally positive*, TP, if all minors of A are non-negative. An infinite sequence $\{a_i\}_{i=0}^{\infty}$ of real numbers is a *Pólya frequency sequence* (PF-sequence) if the matrix $(a_{i-j})_{i,j\geq 0}$ is TP. Thus a PF-sequence is by definition log-concave and therefore also unimodal. A finite sequence $a_0, a_1, a_2, \ldots, a_n$ is said to be PF if the infinite sequence $a_0, a_1, a_2, \ldots, a_n, 0, 0, \ldots$ is PF. A sequence $\{a_i\}_{i=0}^{\infty}$ is said to be PF_r if all minors of size at most r of $(a_{i-j})_{i,j\geq 0}$ are non-negative. If the polynomials $\{b_i(x)\}_{i=0}^d$ are linearly independent over $\mathbb{R}$ and $r \in \mathbb{N}$ we define the set $PF_r[\{b_i(x)\}_{i=0}^d]$ to be

$$PF_r[\{b_i(x)\}_{i=0}^d] = \{\sum_{i=0}^{d} \lambda_i b_i(x) : \{\lambda_i\}_{i=0}^{\infty} \text{ is } PF_r\},$$

and $PF[\{b_i(x)\}_{i=0}^d] = \bigcap_{r=1}^{\infty} PF_r[\{b_i(x)\}_{i=0}^d]$.

The following theorem characterizes PF-sequences. It was conjectured by Schoenberg and proved by Edrei [15]; see also [22].

Theorem 2.1. *Let $\{a_i\}_{i=0}^{\infty}$ be a sequence of real numbers with $a_0 = 1$. Then it is a PF-sequence if and only if the generating function can be expanded in a neighborhood of the origin as*

$$\sum_{i\geq 0} a_i z^i = e^{\gamma z} \frac{\prod_{i\geq 0}(1 + \alpha_i z)}{\prod_{i\geq 0}(1 - \beta_i z)},$$

where $\gamma \geq 0$, $\alpha_i, \beta_i > 0$ and $\sum_{i\geq 0}(\alpha_i + \beta_i) < \infty$.

A consequence of this theorem is that a finite sequence is PF if and only if its generating function is a polynomial with only real non-positive zeros.

Let $f, g \in \mathbb{R}[x]$ be real-rooted with zeros: $\alpha_1 \leq \cdots \leq \alpha_i$ and $\beta_1 \leq \cdots \leq \beta_j$, respectively. We say that f *interlaces* g, denoted $f \preceq g$, if $j = i + 1$ and

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \cdots \leq \beta_{j-1} \leq \alpha_{j-1} \leq \beta_j.$$

We say that f *alternates left of* g, denoted $f \ll g$, if $i = j$ and

$$\alpha_1 \le \beta_1 \le \cdots \le \beta_{i-1} \le \alpha_i \le \beta_i.$$

If in addition f and g have no common zero, then we say that f *strictly interlaces* g and f *strictly alternates left of* g, respectively. We also say that two polynomials f and g *alternate* if one of the polynomials alternates left of or interlaces the other. We will need two simple lemmata concerning these concepts. A polynomial is said to be *standard* if its leading coefficient is positive.

Lemma 2.2. *Let g and $\{f_i\}_{i=1}^n$ be real-rooted standard polynomials.*
  (i) *If for each $1 \le i \le n$ we have either $g \ll f_i$ or $g \preceq f_i$. Then the sum $F = f_1 + f_2 + \cdots + f_n$ is real-rooted with $g \preceq F$ or $g \ll F$, depending on the degree of F.*
  (ii) *If for each $1 \le i \le n$ we have either $f_i \ll g$ or $f_i \preceq g$. Then the sum $F = f_1 + f_2 + \cdots + f_n$ is real-rooted with $F \preceq g$ or $F \ll g$, depending on the degree of F.*

Proof. The lemma follows easily by counting the sign-changes of F at the zeros of g; see, e.g., [37, Prop. 3.5]. □

The next lemma is obvious.

Lemma 2.3. *If $f_0, f_1, \ldots, f_n$ are real-rooted polynomials with $f_0 \ll f_n$ and $f_{i-1} \ll f_i$ for all $1 \le i \le n$, then $f_i \ll f_j$ for all $0 \le i \le j \le n$.*

The following theorem is a characterization of alternating polynomials due to Obreschkoff [24] and Dedieu [13].

Theorem 2.4. *Let $f, g \in \mathbb{R}[x]$. Then f and g alternate (strictly alternate) if and only if all polynomials in the space*

$$\{\alpha f + \beta g : \alpha, \beta \in \mathbb{R}\}$$

have only real (real and simple) zeros.

An immediate but non-trivial consequence of this theorem is

Corollary 2.5. *Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. Then ϕ preserves the real-rootedness property (real- and simple-rootedness property) only if ϕ preserves the alternating property (strictly alternating property).*

We denote by $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, \ldots\}$. The symmetric group of bijections $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is denoted by $\mathcal{S}_n$. A *descent* in a permutation $\pi \in \mathcal{S}_n$ is an index $1 \le i \le n-1$ such that $\pi(i) > \pi(i+1)$. Let $\mathrm{des}(\pi)$ denote the number of descents in π. The *Eulerian polynomials*, $A_n(x)$, are defined by $A_n(x) = \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{des}(\pi)+1}$ and satisfy (see e.g. [11])

$$\sum_{k \ge 0} k^n x^k = \frac{A_n(x)}{(1-x)^{n+1}}.$$

The binomial polynomials are defined by $\binom{x}{0} = 1$ and $\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$ for $k \ge 1$.

In several proofs we will implicitly use the fact that the zeros of a polynomial are continuous functions of the coefficients of the polynomial. In particular the limit of a sequence of real-rooted polynomials is again real-rooted. For a treatment of these matters we refer the reader to [23].

3. A class of linear operators preserving the PF-property

For any polynomial $F(x,z) = \sum_{k=0}^{n} Q_k(x)z^k \in \mathbb{R}[x,z]$ we define a linear operator $\phi_F : \mathbb{R}[x] \to \mathbb{R}[x]$ by

$$\phi_F(f) := \sum_{k=0}^{n} Q_k(x)\frac{d^k}{dx^k}f(x).$$

In this section we will investigate for which $F \in \mathbb{R}[x,z]$ the linear operator ϕ_F preserves the real-rootedness and the PF-properties.

We will need some terminology and a theorem from [5]. For $\xi \in \mathbb{R}$ let $T_\xi : \mathbb{R}[x] \to \mathbb{R}[x]$ be the translation operator defined by $T_\xi(f(x)) = f(x + \xi)$. For any linear operator $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ we define a linear transform $\mathcal{L}_\phi : \mathbb{R}[x] \to \mathbb{R}[x,z]$ by

(3.1)
$$
\begin{aligned}
\mathcal{L}_\phi(f) &:= \phi(T_z(f)) \\
&= \sum_n \phi(f^{(n)})(x)\frac{z^n}{n!} \\
&= \sum_n \frac{\phi(x^n)}{n!}f^{(n)}(z).
\end{aligned}
$$

Definition 3.1. Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. Define a function $d_\phi : \mathbb{R}[x] \to \mathbb{N} \cup \{-\infty\}$ by: If $\phi(f^{(n)}) = 0$ for all $n \in \mathbb{N}$, we let $d_\phi(f) = -\infty$. Otherwise let $d_\phi(f)$ be the smallest integer d such that $\phi(f^{(n)}) = 0$ for all $n > d$. Hence $d_\phi(f) \leq \deg f$ for all $f \in \mathbb{R}[x]$.

The set $\mathcal{A}(\phi)$ is defined as follows: If $d_\phi(f) \in \{-\infty, 0\}$ and $\phi(f)$ is standard real- and simple-rooted, then $f \in \mathcal{A}(\phi)$. Moreover, $f \in \mathcal{A}(\phi)$ if $d = d_\phi(f) \geq 1$ and all of the following conditions are satisfied:

  (i) $\phi(f^{(i)})$ all have leading coefficients of the same sign and $\deg(\phi(f^{(i-1)})) = \deg(\phi(f^{(i)})) + 1$ for $1 \leq i \leq d$,
  (ii) $\phi(f)$ and $\phi(f')$ have no common real zero,
  (iii) $\phi(f^{(d)})$ strictly interlaces $\phi(f^{(d-1)})$,
  (iv) for all $\xi \in \mathbb{R}$ the polynomial $\mathcal{L}_\phi(f)(\xi, z)$ is real-rooted.

The following theorem is proved in [5].

Theorem 3.2. Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. If $f \in \mathcal{A}(\phi)$, then $\phi(f)$ is real- and simple-rooted.

We will also need the following classical theorem of Hermite and Poulain. For a proof see [24].

Theorem 3.3. Let $f = a_0 + a_1 x + \cdots + a_n x^n$ and g be real-rooted polynomials. Then the polynomial

$$f(\frac{d}{dx})g := a_0 g(x) + a_1 g'(x) + \cdots + a_n g^{(n)}(x)$$

is real-rooted. Moreover, if $f(\frac{d}{dx})g \neq 0$, then any multiple zero of $f(\frac{d}{dx})g$ is a multiple zero of g.

The following theorem gives a sufficient condition for a polynomial to be mapped onto a real-rooted polynomial.

Theorem 3.4. Let $F = \sum_{k=0}^{n} Q_k(x)z^k$ be such that $Q_0 \neq 0$ and
  (I) For all $\xi \in \mathbb{R}$, $F(\xi, z)$ is real-rooted,

(II) Q_0 *strictly interlaces or strictly alternates left of* Q_1, *and* $\deg Q_0 = 0$ *or* Q_0 *and* Q_1 *have leading coefficients of the same sign.*

Suppose that

(III) f *is real- and simple-rooted and that for* $0 \leq k \leq \deg f$ *the polynomials* $\phi_F(f^{(k)})$ *have their leading term of the same sign with*

$$\deg \phi_F(f^{(k)}) = \deg Q_0 + \deg f - k.$$

Then $\phi_F(f)$ *is real- and simple-rooted.*

Proof. We will show that the set of real- and simple-rooted polynomials satisfying (III) is a subset of $\mathcal{A}(\phi_F)$ by verifying conditions (i)-(iv) of Definition 3.1. Condition (i) follows immediately from (III). For condition (iv) note that

$$\mathcal{L}_\phi(f)(\xi, z) = \sum_{k=0}^{n} Q_k(\xi) f^{(k)}(\xi + z),$$

so by Theorem 3.3 condition (iv) is satisfied. Suppose that η is a common zero of $\phi_F(f)$ and $\phi_F(f')$. By (3.1) we have that 0 is a multiple zero of $\mathcal{L}_\phi(f)(\eta, z)$. Moreover, since $\mathcal{L}_\phi(f)(\eta, z)$ is not identically equal to zero, by (II), Theorem 3.3 tells us that 0 is a multiple zero of $f(\eta + z)$. This means that η is a multiple zero of f contrary to the assumption that f is simple-rooted, and verifies condition (ii).

For condition (iii) we have to show that for all $\alpha \in \mathbb{R}$ such that $x + \alpha$ satisfies (III) the polynomial $\phi_F(1) = Q_0$ strictly interlaces $f(x) := \phi_F(x + \alpha) = (x + \alpha)Q_0 + Q_1$. This follows from (II) when analyzing the sign of $f(x) := \phi_F(x + \alpha)$ at the zeros of Q_0: Let $\alpha_k < \alpha_{k-1} < \cdots < \alpha_1$ be the zeros of Q_0 ordered by size. Suppose that Q_0 and Q_1 are standard and that Q_0 strictly interlaces or strictly alternates left of Q_1. Then $\operatorname{sgn} f(\alpha_i) = \operatorname{sgn} Q_1(\alpha_i) = (-1)^i$ for $1 \leq i \leq k$. By Rolle's theorem we know that f has a zero in each interval (α_i, α_{i+1}). This accounts for $k-1$ real zeros of f. Since Q_0 has positive sign, so does f by condition (III). Now, because $f(\alpha_1) < 0$ and f is standard, f must have a zero to the right of α_1. We now know that f has k real zeros. The signs at α_i force the remaining zero to be in the interval $(-\infty, \alpha_k)$. Thus Q_0 strictly interlaces f as was to be shown.

Now, if $Q_0 = A \in \mathbb{R}$, then $\deg Q_1 \leq 1$. Suppose that $Q_1 = B \in \mathbb{R}$. Then clearly A strictly interlaces $(x + \alpha)A + B$. If $Q_0 = A$ and $Q_1 = Cx + D$ where $A, C, D \in \mathbb{R}$, then $f = (A + C)x + A\alpha + D$, so by (III) we have that Q_0 strictly interlaces f. This concludes the proof. $\qquad\square$

In some cases it may be convenient to have a sharper hypothesis. Therefore we state the following form of the theorem.

Corollary 3.5. *Let* $d \in \mathbb{N}$ *be given and let* $F = \sum_{k=0}^{n} Q_k(x) z^k$ *be such that* $Q_0 \neq 0$ *and*

    (i) *For all* $\xi \in \mathbb{R}$, $F(\xi, z)$ *is real-rooted.*
    (ii) Q_0 *strictly interlaces or strictly alternates left of* Q_1, *and* $\deg Q_0 = 0$ *or* Q_0 *and* Q_1 *have leading coefficients of the same sign.*
    (iii) *The polynomials* $\phi_F(x^k)$, $0 \leq k \leq d$ *have the same sign and*

$$\deg \phi_F(x^k) = \deg Q_0 + k.$$

Then $\phi_F(f)$ *is real-rooted (real- and simple-rooted) if* f *is real-rooted (real and simple-rooted) and* $\deg(f) \leq d$.

Proof. The case of real- and simple-rooted f follows immediately from Theorem 3.4 since (iii) implies (III). If f is a real-rooted polynomial of degree at most d, then f is the limit of a sequence $\{f_k\}_{k=0}^{\infty}$ of real- and simple-rooted polynomials of degree at most d. It follows that $\phi_F(f)$ is the limit of $\phi_F(f_k)$, and the thesis follows by continuity. $\qquad\square$

In the language of PF-sequences we have

Theorem 3.6. *Let $d \in \mathbb{N}$ be given and let $F = \sum_{k=0}^{n} Q_k(x)z^k \in \mathbb{R}[x,z]$ be such that $Q_0 \neq 0$ and*

(i) *For all $\xi \in \mathbb{R}$, $F(\xi, z)$ is real-rooted.*
(ii) *$\phi_F(1)$ strictly interlaces $\phi_F(x)$.*
(iii) *For all $0 \leq k \leq d$*
$$\deg \phi_F(x^k) = \deg Q_0 + k,$$
and $\phi_F(x^k) \in PF_1$.
Then $PF[\{\phi_F(x^i)\}_{i=0}^{d}] \subseteq PF[x^i]$.

Several old results can be derived from these last few theorems. In [25, p. 163] Pólya gave a theorem which he states was probably the most general theorem on real-rootedness known at the time. "Dieser Satz gehört wohl zu den allgemeinsten bekannten Sätzen über Wurzelrealität".

Theorem 3.7. *Let $f(x)$ be a real-rooted polynomial of degree n, and let*
$$b_0 + b_1 x + \cdots + b_{n+m} x^{n+m} \quad (m \geq 0)$$
be a real-rooted polynomial such that $b_i > 0$ for $0 \leq i \leq n$. Then the equation
$$G(x,y) := b_0 f(y) + b_1 x f'(y) + b_2 x^2 f''(y) + \cdots + b_n x^n f^{(n)}(y) = 0$$
has n real intersection points (counted with multiplicity) with the line
$$sx - ty + u = 0,$$
provided that $s, t \geq 0$, $s + t > 0$ and $u \in \mathbb{R}$.

Proof. We may assume that $s, t > 0$ since the other cases follow by continuity when s and/or t tends to zero. Thus we may write the equation as
$$a_0 g(x) + a_1 x g'(x) + a_2 x^2 g''(x) + \cdots + a_n x^n g^{(n)}(x) = 0,$$
where $g(x) = f(st^{-1}x + ut^{-1})$ and $a_i = s^i t^{-i} b_i$. Now, we see that all hypotheses of Corollary 3.5 are satisfied for
$$F(x,z) = a_0 + a_1 xz + a_2 x^2 z^2 + \cdots + a_{n+m} x^{n+m} z^{n+m},$$
when $d = n$. $\qquad\square$

We will later need one famous consequence of this theorem, $t = 1, s = u = 0$, due to Schur [29].

Theorem 3.8. *Let $f = \sum_{k=0}^{n} a_k x^k$ and $g = \sum_{k=0}^{m} b_k x^k$ be two real-rooted polynomials such that g has all zeros of the same sign. Then the polynomial*
$$(fSg)(x) = \sum_{k \geq 0}^{M} k! a_k b_k x^k,$$
where $M = \min(m, n)$ has only real zeros.

3.1. **Multiplier-sequences.** A multiplier-sequence is a sequence $T = \{\gamma_i\}_{i=0}^{\infty}$ of real numbers such that if a polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ has only real zeros, then the polynomial

$$T[f(x)] := a_0 \gamma_0 + a_1 \gamma_1 x + \cdots + a_n \gamma_n x^n$$

also has only real zeros. There is a characterization of multiplier-sequences due to Pólya and Schur [25, pp. 100-124], as follows.

Theorem 3.9. *Let $T = \{\gamma_i\}_{i=0}^{\infty}$ be a sequence of non-negative real numbers and let $\phi(x) = T[e^x] = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!}$ be its exponential generating function. Then T is a multiplier-sequence if and only if ϕ is a real entire function which can be written as*

$$\phi(x) = cx^n e^{\beta x} \prod_{k=1}^{\infty} (1 + \delta_k x),$$

where $c > 0$, $\beta \geq 0$, $\delta_k \geq 0$, $n \in \mathbb{N}$ and $\sum_{k=1}^{\infty} \delta_k < \infty$.

The following lemma is well known (see [25]).

Lemma 3.10. *A nonnegative multiplier-sequence has no internal zeros.*

Theorem 3.11. *Let $\{\lambda_k\}_{k=0}^{\infty}$ be a non-negative multiplier-sequence, and let $\alpha < \beta \in \mathbb{R}$ be given. Define two $\mathbb{R}$-bilinear forms $\mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}[x]$ by*

$$f \cdot g \;:=\; \sum_{k \geq 0} \frac{\lambda_k}{k!} f^{(k)}(x) g^{(k)}(x) (x-\alpha)^k (x-\beta)^k,$$

$$f \circ g \;:=\; \sum_{k \geq 0} \frac{\lambda_k}{k!} f^{(k)}(x) g^{(k)}(x) (x-\alpha)^k.$$

If f is real-rooted and g is $[\alpha, \beta]$-rooted, then $f \cdot g$ is real-rooted. If f is real-rooted and g is $[-\infty, \alpha]$-rooted, then $f \circ g$ is real-rooted.

Proof. We prove the statement for $\cdot$ since the case for $\circ$ is similar. We may assume that $\lambda_0 > 0$. Clearly the theorem is true if $\lambda_i = 0$ for all $i > 0$, so by Lemma 3.10 we may assume that $\lambda_1 > 0$. Let g have all zeros simple and in the interval (α, β), and let ϕ be the linear operator defined by $\phi(f) = f \cdot g$. Then $\phi = \phi_F$, where

$$F(x, z) = \sum_{k \geq 0} \lambda_k \frac{g^{(k)}(x)}{k!} (x-\alpha)^k (x-\beta)^k z^k.$$

Since $\{\lambda_k\}_{k \geq 0}$ is a multiplier sequence, $F(\xi, z)$ is real-rooted for all real choices of ξ. Now, $Q_0 = \lambda_0 g(x)$ and $Q_1 = \lambda_1 (x-\alpha)(x-\beta) g'(x)$, so Q_0 strictly interlaces Q_1. Moreover, $\deg \phi(x^k) = \deg Q_0 + k$ for all k, so all the hypothesis of Corollary 3.5 are fulfilled. Since any $[\alpha, \beta]$-rooted polynomial is the limit of polynomials which are (α, β)- and simple-rooted, the thesis follows by continuity. $\square$

There are a few bilinear forms on polynomials that occur frequently in combinatorics. Let $\# : \mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}[x]$ be defined by

$$(f \# g)(x) := \sum_{k \geq 0} f^{(k)}(x) g^{(k)}(x) \frac{x^k}{k!}.$$

This product is important when analyzing how the the zeros of σ-polynomials behave under disjoint union of graphs; see [10].

Theorem 3.12. *Let f be real-rooted and let g have only real zeros of the same sign. Then $f \# g$ is real-rooted.*

Proof. The theorem follows from Theorem 3.11, since $\{1\}_{k=0}^{\infty}$ is trivially a multiplier-sequence. $\qquad\square$

This generalizes a result of Wagner, who proved that $f \# g$ is real-rooted whenever f and g have only non-negative zeros; see [10, 35].

The *diamond product* of two polynomials f and g is given by

$$(3.2) \qquad (f \diamond g)(x) = \sum_{k \geq 0} \frac{f^{(k)}(x)}{k!} \frac{g^{(k)}(x)}{k!} x^k (x+1)^k.$$

This product is important in the theory of (P, ω)-partitions and the Neggers-Stanley conjecture and was first studied by Wagner in [36, 37]; see also Section 4 of this paper. Applying Theorem 3.11 with the multiplier-sequence $\{\frac{1}{k!}\}_{k \geq 0}$ we get

Theorem 3.13. *Let f be real-rooted and let g have all zeros in the interval $[-1, 0]$. Then $f \diamond g$ is real-rooted.*

This was first proved by Wagner [37] under the additional hypothesis that f has all zeros in $[-1, 0]$, and generalized by the present author in [5].

A sequence of real numbers $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ is called a *multiplier n-sequence* if for any real-rooted polynomial $f = a_0 + a_1 x + \cdots + a_n x^n$ of degree at most n the polynomial $\Gamma[f] := a_0 \gamma_0 + a_1 \gamma_1 x + \cdots + a_n \gamma_n x^n$ is real-rooted. There is a simple algebraic characterization of multiplier n-sequences [12], as follows.

Theorem 3.14. *Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers. Then Γ is a multiplier n-sequence if and only if $\Gamma[(x+1)^n]$ is real-rooted with all its zeros of the same sign.*

Recall the definition of the *hypergeometric function* $_2F_1$:

$$_2F_1(a, b; c; z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m z^m}{(c)_m m!},$$

where $(\alpha)_0 = 1$ and $(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1)$ when $m \geq 1$. The Jacobi polynomial $P_n^{(\alpha, \beta)}(x)$ can be expressed as follows [26, p. 254]:

$$(3.3) \qquad P_n^{(\alpha, \beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left(-n, 1+\alpha+\beta+n; 1+\alpha; \frac{1-x}{2}\right).$$

We need the following lemma.

Lemma 3.15. *Let n be a positive integer and r a non-negative real number. Then $\Gamma = \{\binom{-n-r}{k}\}_{k=0}^{\infty}$ is a multiplier n-sequence.*

Proof. Let $r > 0$. Then

$$
\begin{aligned}
\Gamma[(x+1)^n] &= \sum_{k=0}^{n} \binom{-n-r}{k} \binom{n}{k} x^k \\
&= {}_2F_1(-n, n+r; 1; x) \\
&= P_n^{(0, r-1)}(1-2x),
\end{aligned}
$$

where the last equality follows from (3.3). Since the Jacobi polynomials are known (see [26]) to have all their zeros in $[-1, 1]$ when $\alpha, \beta > -1$, we have that $\Gamma[(x+1)^n]$

has all its zeros in $[0, 1]$. The case $r = 0$ follows by continuity when we let r tend to zero from above.   □

For any real number q let $\Gamma_q := \{q + k\}_{k=0}^{\infty}$. The following corollary was already known to Laguerre.

Corollary 3.16. *Let $n > 1$ be a positive integer. Then Γ_q is a multiplier n-sequence if and only if $q \notin (-n, 0)$.*

Proof. Let $q \in \mathbb{R}$ be given. We have to determine for which $n > 1$ the zeros of $\Gamma_q[(x + 1)^n]$ are all real and of the same sign. Now,

$$\Gamma_q[(x + 1)^n] = (x + 1)^{n-1}\{(n + q)x + q\},$$

and the theorem follows.   □

4. The E-transformation

The *E-transformation* is the invertible linear operator $\mathcal{E} : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by

$$\mathcal{E}\left(\binom{x}{i}\right) = x^i,$$

for all $i \in \mathbb{N}$. The PF-preserving properties of this linear operator was first studied in [6] and later in [36, 37] and [5]. It is important in the theory of (P, ω)-partitions since it maps the order-polynomial of a labeled poset to the E-polynomial of the same labeled poset; see [6, 36]. In, [6] Brenti proved the following theorem. Let $\lambda(f)$ and $\Lambda(f)$ denote the smallest and the largest real zero of the polynomial f, respectively.

Theorem 4.1. *Suppose that $f \in \mathbb{R}[x]$ has only real zeros and that $f(n) = 0$ for all $n \in ([\lambda(f), -1] \cup [0, \Lambda(f)]) \cap \mathbb{Z}$. Then $\mathcal{E}(f)$ has all zeros real and is non-positive.*

In this section we will prove the following theorem.

Theorem 4.2. *For all $n \in \mathbb{N}$ we have*

$$PF_1[\{x^i(x + 1)^{n-i}\}_{i=0}^n] \subseteq PF\left[\binom{x}{i}\right].$$

Moreover if $f \in PF_1[\{x^i(x + 1)^{n-i}\}_{i=0}^n]$, then $\mathcal{E}(f)$ has simple zeros and

$$\mathcal{E}((x + 1)^d) \ll \mathcal{E}(f) \ll \mathcal{E}(x^d).$$

The diamond product (3.2) is intimately connected with the E-transformation. By the Vandermonde identity

$$\binom{x}{i}\binom{x}{j} = \sum_{k \geq 0} \binom{k}{k - i, i + j - k, k - j}\binom{x}{k},$$

it follows (see [37]) that

(4.1) $$(f \diamond g)(x) := \mathcal{E}(\mathcal{E}^{-1}(f)\mathcal{E}^{-1}(g)).$$

We will later need a symmetry property of $\mathcal{E}$. Let $\mathcal{R} : \mathbb{R}[x] \to \mathbb{R}[x]$ be the algebra automorphism defined by $\mathcal{R}(x) = -1 - x$.

Lemma 4.3.

$$\mathcal{R}\mathcal{E} = \mathcal{E}\mathcal{R}.$$

Proof. Let n be a non-negative integer. Using the identity

$$\binom{x+n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k}$$

and the fact that $\binom{-x-1}{n} = (-1)^n \binom{x+n}{n}$, we get

$$
\begin{aligned}
\mathcal{E}\mathcal{R}\binom{x}{n} &= (-1)^n \mathcal{E}\binom{x+n}{n} \\
&= (-1)^n \sum_{k=0}^{n} \binom{n}{k}\mathcal{E}\binom{x}{k} \\
&= (-1-x)^n \\
&= \mathcal{R}\mathcal{E}\binom{x}{n},
\end{aligned}
$$

and the lemma follows. $\qquad\square$

Lemma 4.4. *Let $\alpha \in [-1,0]$ and let f be a polynomial such that $\mathcal{E}(f)$ is $[-1,0]$-rooted. Then $\mathcal{E}((x-\alpha)f)$ is $[-1,0]$-rooted and $\mathcal{E}(f)$ interlaces $\mathcal{E}((x-\alpha)f)$. If $\mathcal{E}(f)$ in addition only has simple zeros, then so does $\mathcal{E}((x-\alpha)f)$.*

Proof. Let $g = \mathcal{E}(f)$ and let $\alpha \in [-1,0]$. By (3.2) and (4.1) we have that

$$(4.2) \qquad \mathcal{E}((x-\alpha)f) = (x-\alpha)g + x(x+1)g'.$$

Since g interlaces $(x-\alpha)g$ and $x(x+1)g'$, it also interlaces the sum by Lemma 2.2. Also, if $x \notin [-1,0]$, then the summands have the same sign, so $\mathcal{E}((x-\alpha)f)$ cannot have any zeros outside $[-1,0]$. Suppose that g has only simple zeros. Then by (4.2) the only possible common zeros of g and $\mathcal{E}((x-\alpha)f)$ are 0 and -1. If $\deg(f) \geq 1$ it also follows from (4.2) that the multiplicities of 0 and -1 of $\mathcal{E}((x-\alpha)f)$ are the same as those of g. Hence the (simple) zeros of g separate the zeros of $\mathcal{E}((x-\alpha)f)$ except possibly at $0, -1$, and we conclude that $\mathcal{E}((x-\alpha)f)$ has only simple zeros. $\qquad\square$

Lemma 4.5. *For all integers $n \geq 1$ we have*

$$(x+1)\mathcal{E}(x^n) = x\mathcal{E}((x+1)^n).$$

Proof. We may write

$$x^n = \sum_{k=1}^{n} a_k \binom{x}{k},$$

where $a_k \in \mathbb{R}$. Thus

$$
\begin{aligned}
\mathcal{E}((x+1)^n) &= \sum_{k=1}^{n} a_k \mathcal{E}\left[\binom{x}{k} + \binom{x}{k-1}\right] \\
&= \sum_{k=1}^{n} a_k (x^k + x^{k-1}) \\
&= (x+1)x^{-1}\mathcal{E}(x^n).
\end{aligned}
$$

$\qquad\square$

Let f and g be standard real-rooted polynomials of degree n and let the zeros of f and g be $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$, respectively. We write $f \leq g$ if $\alpha_i \leq \beta_i$ for $1 \leq i \leq n$.

Theorem 4.6. *Suppose that f and g are $[-1,0]$-rooted with $f \leq g$. Then $\mathcal{E}(f)$ and $\mathcal{E}(g)$ are $[-1,0]$- and simple-rooted, with $\mathcal{E}(f) \ll \mathcal{E}(g)$.*

Proof. By Lemma 4.4 and induction we only have to show that $\mathcal{E}(f) \ll \mathcal{E}(g)$. If f and g have the same zeros except for one, i.e., $f = (x-\alpha)h$ and $g = (x-\beta)h$, where $\alpha < \beta$, then

$$\mathcal{E}(g) = \mathcal{E}(f) - (\beta - \alpha)\mathcal{E}(h),$$

and since $\mathcal{E}(h)$ interlaces $\mathcal{E}(f)$ we have $\mathcal{E}(f) \ll \mathcal{E}(g)$ by Lemma 2.2.

Now, suppose that f and g are $[-1,0]$-rooted polynomials of degree n such that $f \leq g$. Then there are $[-1,0]$-rooted polynomials $\{h_i\}_{i=0}^M$ with

$$(x+1)^n = h_0 \leq h_1 \leq \cdots \leq h_M = x^n,$$

such that $f, g \in \{h_i\}_{i=0}^M$ and h_{i-1} and h_i only differ in one zero for $1 \leq i \leq n$. We therefore have

$$\mathcal{E}(h_0) \ll \mathcal{E}(h_1) \ll \cdots \ll \mathcal{E}(h_M),$$

and since $\mathcal{E}(h_0) \ll \mathcal{E}(h_M)$, by Lemma 4.5, the theorem follows from Lemma 2.3. $\square$

A consequence of Theorem 4.6 is that if $\{f_i\}_{i=1}^m$ is a sequence of standard $[-1,0]$-rooted polynomials of the same degree d, then by Lemma 2.2 and Theorem 4.6, the image under $\mathcal{E}$ of any non-negative sum $F = \sum_{i=1}^m \mu_i f_i$ will be $[-1,0]$-rooted with

$$\mathcal{E}((x+1)^d) \ll \mathcal{E}(F) \ll \mathcal{E}(x^d).$$

It is easy to see that a standard polynomial f of degree d is $[-1,0]$-rooted if and only if f can be written as

$$f(x) = (x+1)^d g\Big(\frac{x}{x+1}\Big),$$

where g is a standard and $(-\infty, 0)$-rooted. On the other hand, since $x^i(x+1)^{d-i}$ is $[-1,0]$-rooted, we have that F can be written as a non-negative sum of standard $[-1,0]$-rooted polynomials of degree d if and only if

$$F(x) = \sum_{i=0}^d a_i x^i (x+1)^{d-i},$$

where $a_i \geq 0$. This proves Theorem 4.2.

5. t-STACK SORTABLE PERMUTATIONS

For relevant definitions regarding t-stack sortable permutations, we refer the reader to [2]. Let $W_t(n,k)$ be the number of t-stack sortable permutations in the symmetric group $\mathcal{S}_n$ with k descents, and let

$$W_{n,k}(x) = \sum_{k=0}^{n-1} W_t(n,k) x^k.$$

Recently, Bóna [1, 3] showed that for fixed n and t the numbers $\{W_t(n,k)\}_{k=0}^{n-1}$ form a unimodal sequence. When $t = n-1$ and $t = 1$ we get the Eulerian and the Narayana numbers (see [34] and [31, Exercise 6.36]), respectively. These are known to be PF-sequences, and Bóna [2, 3] has raised the question of whether this is true for general t. Here we will settle the problem to the affirmative for $t = 2$ and $t = n-2$.

The numbers $W_2(n, k)$ are surprisingly hard to determine despite their compact and simple form. It was recently shown that

$$W_2(n, k) = \frac{(n + k)!(2n - k - 1)!}{(k + 1)!(n - k)!(2k + 1)!(2n - 2k - 1)!}.$$

See [4, 14, 19, 21] for proofs and more information on 2-stack sortable permutations.

From the case $r = 0$ in Lemma 3.15 and the identity

$$\sum_{k=0}^{n} \binom{2n - k - 1}{n - 1} \binom{n}{k} x^k = (-1)^n \sum_{k=0}^{n} \binom{-n}{k} \binom{n}{k} (-x)^{n-k},$$

it follows that $\binom{2n-k-1}{n-1}$ is an n-sequence.

Theorem 5.1. *For all $n \geq 0$ the sequence $\{W_2(n, k)\}_{k=0}^{n-1}$, which records 2-stack sortable permutations by descents, is PF.*

Proof. We may write $W_2(n, k)$ as

$$W_2(n, k) = \frac{\binom{2n-k-1}{n-1} \binom{n+k}{n-1} \binom{2n}{2k+1}}{n^2 \binom{2n}{n}}.$$

A simple consequence of the notion of PF-sequences reads as follows: If $\{a_i\}_{i \geq 0}$ is PF, then so is $\{a_{ki}\}_{i \geq 0}$, where k is any positive integer. Applying this to the polynomial $x(1 + x)^{2n}$ we see that $\sum_k \binom{2n}{2k+1} x^k$ is real-rooted. Therefore the polynomial

$$\sum_{k=0}^{n-1} \binom{n+k}{n-1} \binom{2n}{2k+1} x^k = \sum_{k=0}^{n-1} \binom{2n - k - 1}{n - 1} \binom{2n}{2k+1} x^{n-1-k}$$

is real-rooted. Another application of Lemma 3.15 gives that $W_{n,2}(x)$ is real-rooted. □

It is easy to see that a permutation $\pi \in \mathcal{S}_n$ is $(n - 2)$-stack sortable if and only if it is not of the form $\sigma n1$. Thus the generating function satisfies

$$xW_{n,n-2}(x) = A_n(x) - xA_{n-2}(x),$$

where $A_n(x)$ is the nth Eulerian polynomial.

Theorem 5.2. *For all real numbers $t > -2$ and integers $n > 2$, the polynomial*

$$A_n(t, x) = A_n(x) + txA_{n-2}(x)$$

is real- and simple-rooted. Moreover, $A_n(t, x)/x$ strictly interlaces $A_{n+1}(t, x)/x$ for $-2 < t \leq 3$.

Corollary 5.3. *For all $n \geq 2$ we have that $\{W_{n-2}(n, k)\}_{k=0}^{n-1}$ is PF. Moreover, $W_{n,n-2}(x)$ strictly interlaces $W_{n+1,n-1}(x)$.*

Proof of Theorem 5.2. It is well known that $A_{n-1}(x) \ll xA_{n-2}(x)$ and $A_{n-1}(x) \preceq A_n(x)$. So by Lemma 2.2 we have that $A_n(t, x)$ is real- and simple-rooted for $t \geq 0$. However, when $t < 0$ a similar argument does not apply.

Let $E_n(t, x) = A_n(t, \frac{x}{1+x})$. Then

$$E_n(t, x) = E_n(x) + tx(1 + x)E_{n-2}(x),$$

where the coefficient to x^k in $E_n(x)$ counts the number of surjections $\sigma : [n] \to [k]$; see [6, 36]. These polynomials satisfy the recursion

$$E_n(x) = x\frac{d}{dx}((1+x)E_{n-1}(x)),$$

with initial condition $E_1(x) = x$. Thus, if we let $G_n(x) = E_{n+1}(x)/x$ we have the following recursion:

(5.1) $$G_n(x) = \frac{d}{dx}(x(1+x)G_{n-1}(x)),$$

with $G_0(x) = 1$. Obviously $G_n(x)$ is real- and simple-rooted. If we apply (5.1) two times we get the equation

$$G_n(x) = (1 + 6x + 6x^2)G_{n-2}(x) + 3x(1+2x)(1+x)G'_{n-2}(x)$$
$$+ x^2(1+x)^2 G''_{n-2}(x),$$

and for $G_n(t,x) := G_n(x) + tx(1+x)G_{n-2}(x)$ we have

$$G_n(t,x) = (1 + (6+t)x + (6+t)x^2)G_{n-2}(x) + 3x(1+2x)(1+x)G'_{n-2}(x)$$
$$+ x^2(1+x)^2 G''_{n-2}(x).$$

To apply Theorem 3.4 we need show that for all $\xi \in \mathbb{R}$ and $-2 < t < 0$ the polynomial

$$F(\xi, z) := (1 + (6+t)\xi + (6+t)\xi^2) + 3\xi(1+2\xi)(1+\xi)z + \xi^2(1+\xi)^2 z^2$$

is real-rooted. The discriminant of $F(\xi, z)$,

$$\Delta(F(\xi, z)) = \xi^2(1+\xi)^2(2 + t + (3-t)(1+2\xi)^2),$$

is non-negative when $-2 \le t \le 3$, so $F(\xi, z)$ real-rooted for these t. Since all the Q_ks are standard, it is easy to see that condition (III) in the statement of Theorem 3.4 is satisfied. Moreover, $1 + (6+t)x + (6+t)x^2$ strictly interlaces $3x(1+2x)(1+x)$ when $t > -2$, so Theorem 3.4 applies. Since G_n strictly interlaces G_{n+1}, we have by Theorem 3.4 and Corollary 2.5 that $\phi_F(G_n)$ strictly interlaces $\phi_F(G_{n+1})$. Thus $A_n(t,x)$ strictly interlaces $A_{n+1}(t,x)$. $\square$

6. q-Eulerian and W-Eulerian polynomials

A q-analog of the Eulerian polynomials was introduced and studied in [16] and further studied in [9]. It is defined by

$$A_n(x; q) := \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{exc}(\pi)} q^{c(\pi)},$$

where $c(\pi)$ and $\mathrm{exc}(\pi)$ denote the number of *cycles* and *excedances* in π, respectively. These polynomials satisfy the recursion

$$A_{n+1}(x; q) = (nx + q)A_n(x; q) - x(x-1)\frac{\partial}{\partial x}A_n(x; q),$$

with initial condition $A_0(x; q) := 1$. See [9] for a proof. The following theorem appears in [9].

Theorem 6.1. *Let $q \in \mathbb{R}$, $q > 0$. Then the polynomials $A_n(x, q)$ have only real non-positive simple zeros.*

Brenti also makes the following conjecture.

Conjecture 6.2. *Let $n, m \in \mathbb{N}$. Then $A_n(x; -m)$ has only real zeros.*

In what follows we will prove this conjecture using multiplier n-sequences. For $n \in \mathbb{N}$ define the polynomials $E_n(x; q)$ by

$$E_n(x; q) := (1 + x)^n A_n\left(\frac{x}{1 + x}; q\right).$$

It is clear that $E_n(x; q)$ is real-rooted if and only if $A_n(x; q)$ is real-rooted. These polynomials satisfy a somewhat easier recursion. Namely,

$$(6.1) \qquad E_{n+1}(x; q) = (1 + x)\left\{qE_n(x; q) + x\frac{\partial}{\partial x}E_n(x; q)\right\},$$

with initial condition $E_0(x; q) = 1$. Now, for $q \in \mathbb{R}$ let $\Gamma_q : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ be the linear operator defined by $\Gamma_q(f(x)) = qf(x) + xf'(x)$. Since $\Gamma_q(x^n) = (q + n)x^n$ we may apply Corollary 3.16.

Theorem 6.3. *Let $q \in \mathbb{R}$ and $n \in \mathbb{N}$. If $q \geq 0$, $n \leq -q$ or $q \in \mathbb{Z}$, then $E_n(x; q)$ has only real zeros.*

Proof. We may write (6.1) as

$$E_{n+1}(x; q) = (x + 1)\Gamma_q[E_n(x; q)].$$

The cases $q \geq 0$ and $n \leq -q$ follow from Corollary 3.16 by induction. We may therefore assume that $q = -m$ is a negative integer. We claim that $\deg E_n(x; q) = n$ if $n \leq m$ and $\deg E_n(x; q) = m$ if $n \geq m$. From this the real-rootedness follows by Corollary 3.16 and induction. The case $n \leq m$ is clear since

$$\Gamma_q[x^{n-1}] = -(m - n + 1) < 0.$$

The case $n > m$ also follows by induction. Suppose that $n \geq m$ and that $\deg E_n(x; q) = m$. Then by the recursion we have that $\deg E_{n+1}(x; q) \leq m + 1$. Moreover, since $\Gamma_q[x^m] = 0$ we have that $\deg E_{n+1}(x; q) \leq m$. Let $a \neq 0$ be the coefficient to x^m of $E_n(x; q)$. Then the coefficient to x^m of $E_{n+1}(x; q)$ is $a\Gamma_q[x^{m-1}] = -a$, so $\deg E_{n+1}(x; q) = m$, and the thesis follows. $\square$

The Eulerian polynomial $P(W, x)$ of a finite Coxeter group W is the polynomial

$$P(W, x) = \sum_{\sigma \in W} x^{d_W(\sigma)},$$

where $d_W(\sigma)$ is the number of W-descents of σ; see [8]. This polynomial is also the generating function for the h-vector of the Coxeter complex associated to (W, S). For Coxeter groups of type A_n we have that $P(A_n, x) = A_n(x)/x$, the shifted Eulerian polynomial. Also, for Coxeter groups of type B_n it is known (see [8]) that $P(B_n, x)$, has only real zeros. It is easy to see that $P(W_1 \times W_2, x) = P(W_1, x)P(W_2, x)$ for finite Coxeter groups W_1 and W_2. Also, the real-rootedness can be checked ad hoc for the exceptional groups. Thus, by the classification of finite irreducible Coxeter groups, to prove that $P(W, x)$ has only real zeros for all finite Coxeter groups it suffices to prove that $P(D_n, x)$ is real-rooted for Coxeter groups of type D_n. The real-rootedness of $P(D_n, x)$ is conjectured by Brenti in [8]. It is known that the Eulerian polynomials of type A_n, B_n and D_n are related by (see [8, 27, 33])

$$P(D_n, x) = P(B_n, x) - n2^{n-1}xP(A_{n-1}, x).$$

This relationship was first noticed by Stembridge [33]. One step towards proving the real-rootedness of $P(D_n, x)$ is to learn more about the relationships between the zeros of $P(B_n, x)$ and $P(A_n, x)$.

Brenti [8] introduced a q-analog of $P(B_n, x)$

$$(6.2) \qquad B_n(x; q) = \sum_{\sigma \in B_n} q^{N(\sigma)} x^{d_B(\sigma)},$$

where $d_B(\sigma)$ is the number of B_n-descents of σ and $N(\sigma)$ is the number of negative entries of σ; see [8]. He proved that

$$(6.3) \qquad \sum_{i \geq 0} ((1 + q)i + 1)^n x^i = \frac{B_n(x; q)}{(1 - x)^{n+1}},$$

and that $B_n(x; q)$ is real- and simple-rooted for all $q \geq 0$. Suppose that $f(i)$ is a polynomial in i of degree d; then the polynomial $W(f)$ is defined by

$$\sum_{i \geq 0} f(i) x^i = \frac{W(f)(x)}{(1 - x)^{d+1}}.$$

One can show (see [6]) that $\mathcal{E}(f)$ and $W(f)$ are related by

$$(6.4) \qquad \mathcal{E}(f)(x) = (1 + x)^{\deg(f)} W(f)(\frac{x}{1 + x}).$$

It follows that $W(f)$ has only real non-positive roots if and only if $\mathcal{E}(f)$ is $[-1, 0]$-rooted.

Since $((1 + q)i + 1)^n$ is a $[-1, 0]$-rooted polynomial in i for any $q \geq 0$, it follows from, e.g. Theorem 4.2, that $B_n(x; q)$ is real-rooted in x for any fixed $q \geq 0$. It is natural to generalize $B_n(x; q)$ to have $n + 1$ parameters as $B_n(x; \mathbf{q}) := W\big(\prod_{i=0}^{n} ((1 + q_i)x + 1)\big)$. This polynomial has a nice combinatorial interpretation, as follows.

Theorem 6.4. *For all $n \in \mathbb{N}$ we have*

$$B_n(x, \mathbf{q}) = \sum_{\sigma \in B_n} q_1^{\chi_1(\sigma)} q_2^{\chi_2(\sigma)} \cdots q_n^{\chi_n(\sigma)} t^{d_B(\sigma)},$$

where

$$\chi_i(\sigma) = \begin{cases} 1 & \text{if } \sigma_i < 0, \\ 0 & \text{if } \sigma_i > 0. \end{cases}$$

Proof. The proof is an obvious generalization of the proof of Theorem 3.4 of [8]. □

Note that this theorem gives a semi-combinatorial interpretation of the W-transform of any $[-1, 0)$-rooted polynomial.

Corollary 6.5. *Let $n \in \mathbb{N}$ and let $q_1, q_2, \ldots, q_n$ be non-negative real numbers. Then $B_n(x; \mathbf{q})$ has only real and simple zeros.*

We need the following lemma on the degree of $W(f)$.

Lemma 6.6. *Let $f \in \mathbb{R}[x]$. Then*

$$\deg W(f) = \deg f - \mathrm{mult}(-1, \mathcal{E}(f)).$$

Moreover, $\mathrm{mult}(-1, \mathcal{E}(f))$ is equal to the maximal integer k such that

$$(x + 1)(x + 2) \cdots (x + k)$$

divides f.

Proof. Since $\deg \mathcal{E}(f) = \deg f$ for all f, we have by (6.4) that $\deg W(f) = \deg f - \text{mult}(-1, \mathcal{E}(f))$. If we expand f in the basis $\{\binom{-x-1}{i}\}$ as

$$
\begin{aligned}
f(x) &= \sum_{i \geq 0} (-1)^i a_i \binom{-x-1}{i} \\
&= \sum_{i \geq 0} \frac{a_i}{i!}(x+1)\cdots(x+i),
\end{aligned}
$$

we have by Lemma 4.3 that

$$
\mathcal{E}(f)(x) = \sum_{i \geq 0} a_i(x+1)^i,
$$

and the lemma follows. $\qquad\square$

We now have more precise knowledge of the location of the zeros of $B_n(x; q)$ for any given $q \geq 0$.

Theorem 6.7. *Let $0 < q < t \in \mathbb{R}$ and let $n > 0$ be an integer. Then*

$$
B_n(x; 0) \preceq B_n(x; t) \ll B_n(x; q) \ll x B_n(x; 0),
$$

where the first three polynomials have no common zeros.

Proof. Let $0 < r < s < 1$. Then by the proof of Lemma 4.4 we have

$$
\mathcal{E}(x^n) \ll \mathcal{E}(x(x+r)^{n-1}) \ll_{strict} \mathcal{E}((x+r)^n) \ll_{strict} \mathcal{E}((x+r)^{n-1}(x+s))
$$
$$
\ll \mathcal{E}((x+s)^n) \ll_{strict} \mathcal{E}((x+s)^{n-1}(x+1)) \ll \mathcal{E}((x+1)^n),
$$

where $\ll_{strict}$ means strictly alternating left of. Since $(x+1)\mathcal{E}(x^n) = x\mathcal{E}((x+1)^n)$, this implies

$$
\mathcal{E}(x^n) \ll_{strict} \mathcal{E}((x+r)^n) \ll_{strict} \mathcal{E}((x+s)^n) \ll_{strict} \mathcal{E}((x+1)^n).
$$

Now since

$$
B_n(x; q) = (q+1)^n W((x+\frac{1}{1+q})^n) = (q+1)^n (1-x)^n \mathcal{E}((x+\frac{1}{1+q})^n)(\frac{x}{1-x}),
$$

we see by Lemma 6.6 that $\deg B_n(x; 0) = n - 1$ and $\deg B_n(x; q) = n$ if $q \neq 0$. Moreover, the alternating property is preserved under the operation (6.4), and the theorem follows. $\qquad\square$

It follows from (6.2) that $P(B_n, x) = B_n(x; 1)$ and $P(A_n, x) = B_n(x; 0)$.

Corollary 6.8. *For all integers $n \geq 1$ we have that $P(A_n, x)$ strictly interlaces $P(B_n, x)$.*

Since $P(A_n, x) \ll xP(A_{n-1}, x)$ and $P(A_n, x) \preceq P(B_n, x)$, we have by Lemma 2.2 that for all $t \geq 0$ the polynomial $P(B_n, x) + txP(A_{n-1}, x)$ is real-rooted. Unfortunately a similar argument does not apply when $t < 0$.

One can extract more from (6.3). Brenti [8] proved that the polynomial

$$
\sum_{\sigma \in B_n, N(\sigma) \in \{k, n-k\}} x^{d_B(\sigma)}
$$

is real-rooted for all choices of $0 \leq k \leq n$. Using Theorem 4.6 we can extend this result to the following corollary.

Corollary 6.9. *Let S be any subset of $[0, n]$. Then the polynomial*

$$P(B_n, S; x) := \sum_{\sigma \in B_n, N(\sigma) \in S} x^{d_B(\sigma)}$$

has only real and simple zeros.

Proof. Comparing the coefficient of q^i in both sides of (6.3) we see that $P(B_n, S; x) = W(f_n(S; x))$, where

$$f_n(S; x) = \sum_{s \in S} \binom{n}{s} x^s (x + 1)^{n-s}.$$

So the theorem follows from Theorem 4.2. □

One instance of Corollary 6.9 is particularly interesting. Recall that a Coxeter group of type D_n is isomorphic to the subgroup

$$D_n = \{\sigma \in B_n : \quad 2 \mid N(\sigma)\}.$$

Hence, we have the following corollary.

Corollary 6.10. *For all $n \in \mathbb{N}$ the polynomial*

$$\sum_{\sigma \in D_n} x^{d_B(\sigma)}$$

has only real and simple zeros.

Note that the above polynomial is not $P(D_n, x)$, since B_n-descents and D_n-descents are not the same.

7. The h-vector of a family of simplicial complexes defined by Fomin and Zelevinsky

Fomin and Zelevinsky [17] recently associated to any finite Weyl group W a simplicial complex $\Delta_{FZ}(W)$. For the classical Weyl groups the corresponding h-vectors (polynomials) are given by

$$h(\Delta_{FZ}(A_{n-1}), x) = \frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k} \binom{n}{k+1} x^k,$$

$$h(\Delta_{FZ}(B_n), x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n}{k} x^k,$$

$$h(\Delta_{FZ}(D_n), x) = h(\Delta_{FZ}(B_n), x) - nx h(\Delta_{FZ}(A_{n-2}), x).$$

It is known that the h-polynomials corresponding to A_n and B_n have only real zeros. Here we will show that so has $h(\Delta_{FZ}(D_n), x)$.

Theorem 7.1. *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha \geq 0, 2\alpha + \beta > 0$ and let $n \geq 2$ be an integer. Then the polynomial*

$$F_n(\alpha, \beta) := \alpha h(\Delta_{FZ}(B_n), x) + \beta n x h(\Delta_{FZ}(A_{n-2}), x)$$

is real- and simple-rooted. Moreover, $h(\Delta_{FZ}(B_{n-1}), x)$ strictly interlaces $F_n(\alpha, \beta)$ if $\alpha > 0$ and strictly alternates left of $F_n(\alpha, \beta)$ if $\alpha = 0$.

Corollary 7.2. *Let W be a finite Weyl group. Then $h(\Delta_{FZ}(W), x)$ has only real and simple zeros.*

PETTER BRÄNDÉN

Proof. For the exceptional Weyl group one can check the real-rootedness ad hoc; see [28]. The other cases follows from Theorem 7.1. □

The *Hadamard product* of two polynomials

$$
\begin{aligned}
p(x) &= a_0 + a_1 x + \cdots + a_m x^m, \\
q(x) &= b_0 + b_1 x + \cdots + b_n x^n
\end{aligned}
$$

is the polynomial

$$(p \star q)(x) = a_0 b_0 + a_1 b_1 x + \cdots + a_N b_N x^N,$$

where $N = \min(m, n)$. Málo proved that if the zeros of p are real and the zeros of q are real and of the same sign, then the zeros of $p \star q$ are real as well. This also follows from Theorem 3.8 since $p \star q = \Gamma[pSq]$, where Γ is the multiplier sequence $\{\frac{1}{k!}\}_{k=0}^{\infty}$. It is known (see e.g. [18]) that if f has only real zeros, then all zeros of $\Gamma[f]$ are real and simple except for possibly at the origin.

Proof of Theorem 7.1. We may write $F_n(\alpha, \beta)$ as

$$F_n(\alpha, \beta) = \alpha(x+1)f + (2\alpha + \beta)g,$$

where $f = (x+1)^{n-1} \star (x+1)^{n-1}$ and $g = \left(x(x+1)^{n-1}\right) \star (x+1)^{n-1}$.

By the discussion before this proof we have that for all real choices of $\gamma, \delta \in \mathbb{R}$ the polynomial

$$\gamma f + \delta g = \left((\gamma + \delta x)(x+1)^{n-1}\right) \star (x+1)^{n-1}$$

is real- and simple-rooted. By the Obreschkoff theorem we infer that f strictly alternates left of g. Now, since $f \preceq (x+1)f$ and $f \ll g$ we know by Lemma 2.2 that f either interlaces or alternates left of $F_n(\alpha, \beta)$ for all $\alpha, \beta \in \mathbb{R}$ such that $\mathrm{sgn}(\alpha) = \mathrm{sgn}(2\alpha + \beta)$. Moreover, since g and f have no common zeros, neither does $F_n(\alpha, \beta)$ and f (provided that $2\alpha + \beta \neq 0$). □

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