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CAUCHY - RIEMANN EQUATIONS IN SEVERAL VARIABLES

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CONTENTS

Introduction.

Bishop [2] has stated: « It is thought that a manifold $M^{n+1} \subset C^n$ has, in general, the property that holomorphic functions in a neighborhood of M extend to be holomorphic in some fixed open set ».

Historically the problem of extending holomorphic functions from a neighborhood of a submanifold of C^n has been considered by Levi, Hartogs, and Bochner in the case of a hypersurface. More recently work has been done by Lewy, Bishop, Weinstock, and Wells. (See [6] for a general introduction and discussion).

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Certain submanifolds of C^n are geometrically well-placed, and inherit a C-R structure from C^n (see below). This makes it possible to define a complex of differential operators on them (the $\bar\partial$ complex on $(0, p)$ forms) which has been used by J. J. Kohn [14] as a prototype in a study of sub elliptic complexes, so these C-R structures are important objects to study in themselves. But it turns out that extendibility is closely related to simple invariants of the C-R structure.

Essentially this thesis is devoted to a discussion of Bishop's statement. Chapter I contains some necessary linear algebra. In II, we define the concept of a C-R *manifold*, which is a pair $(M, H(M))$, where $H(M)$ is a subbundle of $T(M) \otimes C$ (here and in the following, all tensor products are over R) so that $H(M) \cap \overline{H(M)} = 0$, and $H(M)$ is involutive. Some examples are given. A complex manifold M is a C-R manifold when $H(M)$ is taken to be its holomorphic tangent bundle. If N is a real submanifold of M, $(N, T(N) \otimes C \cap H(M))$ is a C-R *submanifold* when the fiber dimension of $T(N) \otimes C \cap H(M)$ is constant. When the dimension of that intersection in minimal (as a function of the dimensions of M and N) N is called *generic.* « Most » C-R submanifolds are generic. An important invariant of the C-R manifold $(H, H(M))$ is the *Levi algebra* of M, $\mathcal{L}(M)$, the sub-algebra of vector fields generated by $H(M)$ and $\overline{H(M)}$. We always assume that $\mathcal{L}(M)$ is constant dimensional, and define the *excess dimension* of $\mathcal{L}(M)$, ex dim $\mathcal{L}(M)$, to be the codimension of $H(M) + \overline{H(M)}$ in the bundle whose sections are $\mathcal{L}(M)$.

In III we analyze the concepts of extendibility and holomorphic hull for generic C-R submanifolds of C^n, and in particular we prove a theorem suggested by H. Rossi about local triviality for such submanifolds when ex dim $\mathcal{L}(M) = 0$. Later we give a general example (Reinhardt submanifolds) of embedded C-R submanifolds of C^n whose hull can be exactly constructed. This leads to examples of M^{n+1} in C^n having the property described by Bishop.

Let M be a C-R submanifold of C^n. If ex dim $\mathcal{L}(M) > 0$, we show in IV that there is a non-trivial family of analytic discs with boundaries on M. We use this in V to show that if M is a generic C-R submanifold of C^n, and if $e =$ ex dim $\mathcal{L}(M) > 0$, then M is extendible (in the sense of Bishop) to a subset of C^n containing a manifold N with $\dim N = \dim M + e$. We also show that if M is compact, it is always extendible to a manifold N with $\dim N = \dim M + 1$. The same result is true if M is a submanifold containing no complex submanifolds.

The following is substantially the text of a doctoral dissertation written at Brandeis University under the direction of Professor Hugo Rossi. I would like to thank Professor Rossi for his help and constant encouragement.

I. Real Subspaces of a Complex Vector Space

A. Complex Structure.

Let W be a finite-dimensional complex vector space. There is a real linear map $J : W \longrightarrow W$ so that $J^2 = - I_W$. J is given by multiplication by i.

If V is a real vector space with a linear map J so that $J^2 = - I_V$ the V has the structure of a complex vector space V_J, if for any $v \in V$ $(a + bi)$ is defined to be $av + bjv$. Then $\dim_C V_J = \frac{1}{2} \dim_R V$. J called a *complex structure* on V.

If V is a real vector space, then $V \otimes C$ is a complex vector space, called the *complexification of* V, obtained by defining J on an element $v \otimes c$ of $V \otimes C$ as:

$J(v \otimes c) = v \otimes ic$, and extending J linearly to all of $V \otimes C$. Then $J^2 = - I_{V \otimes C}$, and $\dim_C V \otimes C = \dim_R V$. $V \otimes C$ has an important automorphism of period two, $-$, defined by requiring that $\overline{v \otimes c} = v \otimes \bar{c}$ ($\bar{c}$ is the complex conjugate of c).

There are maps re: $V \otimes C \longrightarrow V$ and im: $V \otimes C \longrightarrow V$ defined by $\operatorname{re}(a) = \frac{a + \bar{a}}{2}$ (an element of $V \otimes 1$, indentified to V) and by $im(a) = \frac{a - \bar{a}}{2i}$ (again in $V \otimes 1$, identified to V).

(Another way of obtaining the complexification of V is to consider the vector space $V \times V$, and define a J by $J(v, w) = (- w, v)$. The complex vector space so obtained is isomorphic to $V \otimes C$, and the isomorphism $I : V \otimes C \longrightarrow V \times V$ is just $I(a) = (\operatorname{re}(a), im(a))$. The important $-$ automorphism becomes $\overline{(v, w)} = (v, - w)$).

If V already has a complex structure given by a linear map K with $K^2 = - I_V$, then $V \otimes C$ splits naturally into the sum of two complex subspaces, $H_K(V) + A_K(V)$ with $\overline{H_K(V)} = A_K(V)$. $H_K(V)$ (resp. $A_K(V)$) is called the *space of holomorphic vectors* (depending on K) (resp. the *space of antiholomorphic vectors* (depending on K)). $H_K(V)$ is generated by vectors of the form $v \otimes 1 - (Kv) \otimes i$ (which can be read $v - ikv$), so $A_K(V)$ consists of vectors of the form $v + ikv, v \in V$. If the linear map K is extended to $V \otimes C$ by requiring that $K(v \otimes e) = (Kv) \otimes c$, it is not hard to see that $H_K(V)$ is the $(+ i)$ eigenspace of K and $A_K(V)$ is the $(- i)$ eigenspace of K (and these are the only eigenspaces of K).

On the other hand, if $V \otimes C$ is written as the direct sum of two complex subspaces $H + A$ and $\overline{H} = A$, this splitting induces a linear map K on V with $K^2 = - I_V$, and H (resp. A) is just $H_K(V)$ (resp. $A_K(V)$). If $v \otimes 1 = a + h$, with $a \in A$ and $h \in H$, then $\overline{a - h} = h - a$, so $a - h = i\,(im\,(a - h))$. And Kv is just $im\,(a - h)$.

V_K is naturally isomorphic (as a complex vector space) to $H_K(V)$ (correspond an element $v \in V$ with $v - iKv \in H_K(V)$).

If W is a complex vector space, $H(W)$ (resp. $A(W)$) will denote $H_K(W)$ (resp. $A_K(W)$) where K in the « complex structure » of W.

B. Subspaces and Generic Subspaces.

Let W be a complex vector space of complex dimension n, and V a real subspace of W of real dimension k.

1. DEFINITION : $m(V)$ is the maximal complex subspace of W contained in V.

$$m(V) \text{ is just} = \{x \in V \mid ix \in V\}.$$

$V \otimes C$ is canonically imbedded in $W \otimes C$. Define $H(V) = H(W) \cap (V \otimes C)$ and $A(V) = A(W) \cap (V \otimes C)$. Then $\overline{H(V)} = A(V)$ (in $V \otimes C$) and $H(V) \cap A(V) = 0$. Using A we get:

2. THEOREM : *$H(V) + A(V) = T \otimes C$ for a subspace T of V, and T is a complex subspace of W; T is $m(V)$*

$m(V)$ and $H(V)$ are naturally isomorphic.

3. THEOREM : $\max(0, k - n) \leq \dim_C(m(V)) \leq \frac{k}{2}$.

Proof: $\dim_C(m(V)) \leq \frac{1}{2} \dim_R V = \frac{k}{2}$. And : $\dim_R W \geq \dim_R V + \dim_R iV - \dim_R(V \cap iV)$, so that $\dim_R(V \cap iV) \geq 2k - 2n$, and $\dim_C(m(V)) \geq k - n$ (for $V \cap iV = m(V)$). #

Let $G_R^p(W)$ (resp. $G_C^p(W)$) be the collection of p dimensional real (resp. complex) vector subspaces of W. Then $G_R^p(W)$ (resp. $G_C^p(W)$) has the structure of a compact C^∞ (resp. complex-analytic) manifold (Steenrod [29], p. 35). Put $G_C(W) = G_C^0(W) + G_C^1(W) + \dots + G_C^n(W)$. If $V \in G_R^k(W)$ then $m(V) \in G_C(W)$.

4. THEOREM: *If* $p = \max(0, k - n)$, *then* $m : G_R^k(W) \to G_C(W)$ *has these properties :*

1) $m^{-1}(G_C^p(W))$ *is a dense open subset of* $G_R^k(W)$.

2) $m \mid m^{-1}(G_C^p(W))$ *is a* C^∞ *map.*

Proof: A simple argument based on rank. The work of Sommer [28], §§ 1-3, can be used to show this result. #

(Note that $m^{-1}(G_C^p(W))$ is the complement of a lower dimensional algebraic set in $G_R^k(W)$, and m is, in fact, a fibration).

5. DEFINITION: Elements of $m^{-1}(G_C^p(W))$ are called generic subspaces of dimension k, and other elements of $G_R^k(W)$ are called *exceptional.*

The concept of generic subspace is not really satisfactory categorically, for the inverse image of a generic subspace V by a complex linear map is generic if $\dim H(V) > 0$ but need not be in other cases, and the image of a generic subspace by a complex linear map need not be generic. Genericity is not preserved well by taking products; a proper complex subspace of a complex vector space is not generic.

C. The *C-R* Vector Space Category.

We define the *C-R vector space category* by giving its objects and maps. An *object* in the category is a pair (V, W), where V is a real vector space, W is a subspace of V, and W has a complex vector space structure compatible with its real structure. (Equivalently we can give a subspace W and linear map $J_W : W \to W$ with $J_W^2 = -I_W$. Or a subspace H of $V \otimes C$ can be given so that $H \cap \bar{H} = 0$. Then W is obtained by requiring that $W \otimes C = H + \bar{H}$ in VC). A *map* of the category is a pair of real linear maps $(f_0, f_1) : (V, M) \to (V', W')$ so that:

1) $f_0 : V \to V'$ and $f_1 : W \to W'$.

2) $f_1 = f_{0 \mid W}$.

3) $f_1 : W \to W'$ is a complex linear map. (Other conditions equivalent to (3) are: $f_1 \circ J_W = J_{W'} \circ f_1$, or $f_0 \otimes 1 : V \otimes C \to V' \otimes C$ takes H into H'.

1. REMARK: An example of an object in the *C-R* vector space category is provided by $(V, m(V))$ where V is a real subspace of a complex vector space. This is, in a sense, the most general example. If (V, W) is an object, there is a complex vector space $\tilde{V}$ and an injection $j : V \to \tilde{V}$ so that $j(V)$ is a generic subspace of $\tilde{V}$, and $(V, W) \approx (j(V), m(j(V))$. And any map

from (V, W) to (V', W') can be realized as the restriction of an appropriate complex linear map from $\widetilde{V}$ to $\widetilde{V}'$.

2. DEFINITION: The *C-R codimension* of (V, W) is $\dim_R (V/W)$. An interpretation of this is apparent:

3. THEOREM: *If V is a generic subspace of a complex vector space W, and if $m(V) \neq 0$, then*

$$C\text{-}R \operatorname{codim}(V, m(V)) = \operatorname{codim}_R \text{ of } V \text{ in } W.$$

Proof. Examine B4 and B5. #

II. C R MANIFOLDS AND THE LEVI ALGEBRA.

A. Objects and Maps.

C-R manifolds are designed to look « tangentially » like the C-R vector space category. There are many examples of such manifolds.

If V is a vector bundle, let $\Gamma(V)$ be the collection of C^∞ sections of V.

1. DEFINITION: A *C-R manifold* is a pair $(M, H(M))$ where M is a real differentiable manifold of dimension $n+k$ $(n \geq k)$ and $H(M)$ is a k-dimensional complex subbundle of $(T(M) \otimes C)$. The following two conditions are satisfied:

a) If $A(M) = \overline{H(M)}$, then $H(M) \cap A(M) = \{0\}$ (the zero-section).

b) $H(M)$ is involutive. That is, if $\alpha, \beta \in \Gamma(H(M))$, so is $[\alpha, \beta] \in \Gamma(H(M))$.

2. THEOREM: *If M is a real differentiable manifold of dimension $n+k$, then (a) of 1 is equivalent to either of the following:*

1) There is a $(2k)$-dimensional subbundle R of $T(M)$ so that R is a complex vector bundle (that is, there is a real bundle map: $J: R \rightarrow R$ with $J^2 = -I_R$).

2) There is a reduction of the group to $T(M)$ from $GL(n+k, R)$ to a linear group whose elements are of the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where $A \in GL(k, C)$, B is a $(2k)$ by $(n-k)$ matrix and $C \in GL(n-k, R)$.

Proof: *a*) $\rightarrow$ 1) It is clear that $R = \operatorname{re}(H(M) + A(M))$, and J is obtained as in *I*.

1) $\to$ 2) We obtain this reduction by taking a covering of M by charts which exhibit (R, J) as a complex subbundle.

2) $\to$ *a*) It is clear from I how to obtain $H(M)$ from a knowledge of R and its complex structure. #

REMARK: There is a reduction of the group of $T(M)$ from $GL(n+k, R)$ to $U(k) \times 0(n-k)$ (which is a linear group whose elements are of the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ with $A \in U(k)$, $B \in 0(n-k)$). This is accomplished in the usual way with a Riemannian metric (Nomizu [22], § 8).

3. THEOREM: *If M is a real differentiable manifold of dimension $n+k$ satisfying (a) of 1, then (b) of 1 is equivalent to either of the following:*

1) *If η is any differential form annihilating $\Gamma(H(M))$ then $d\eta(\alpha, \beta) = 0$ (for any α, β both in $\Gamma(H(M))$.*

2) *If $R = \mathrm{re}(H(M) + A(M))$, and J is as in (2-1), then $[\alpha, \beta] + J[J\alpha, \beta] + J[\alpha, J\beta] - [J\alpha, J\beta] = 0$ for $\alpha, \beta \in \Gamma(R)$ (the Nijenhuis tensor for C-R manifolds).*

Proof: *b*) $\to$ 1) and 1) $\to$ *b*) are simple uses of the formula

$$2\, d\eta(\alpha, \beta) = \alpha(\eta(\beta)) - \beta(\eta(\alpha)) + \eta([\alpha, \beta]):$$

b) $\to$ 2) Suppose $\alpha, \beta \in \Gamma(R)$. Then $[\alpha - iJ\alpha, \beta - iJ\beta] = K - iJK$ (for some $K \in \Gamma(R)$). But then $K = [\alpha, \beta] - [J\alpha, J\beta]$, and $-JK = -[J\alpha, \beta] - i[\alpha, J\beta]$. And (2) follows.

2) $\to$ *b*) as above. #

REMARKS: If $(M, H(M))$ is a C-R manifold, we shall often say « M is a C-R manifold ».

Note that $(T(M)_p, R_p)$ $(p \in M)$ is an object in the C-R vector space category.

(We follow Sweeney [30] in the following presentation). Consider the exact sequence

$$0 \to H(M) + A(M) \to T(M) \otimes C \to T(M) \otimes C/H(M) + A(M) \to 0.$$

Taking duals we get:

$$(*) \qquad 0 \to (H(M) + A(M))^0 \to (T(M) \otimes C)^* \underset{\longleftarrow\;\;\;\prime}{\to} (H(M)) + A(M))^* \to 0$$

where $(H(M)) + A(M))^0$ is the dual of $T(M) \otimes C / H(M) + A(M)$, and consists of all linear functionals which are 0 on $H(M) + A(M)$ in $T(M) \otimes C$. Taking the m-th exterior product we obtain:

$$0 \to K \to \Lambda^m (T(M)\, C)^* \to \Lambda^m (H(M)^* + A(M)^*) = \\ = \sum_{p+q=m} \Lambda^p H(M)^* \otimes \Lambda^q A(M)^* \to 0$$

where K consists of all linear combination $\eta_1 \wedge \dots \wedge \eta_m$ where at least one $\eta_i \in (H(M) + A(M))^0$.

If we choose a splitting map $r: H(M)^* + A(M)^* \to (T(M) \otimes C)^*$ of the sequence $(*)$, then $\Lambda^m r$ splits the sequence of m-th exterior products. If we define $D^{p,q} = \Gamma(\Lambda^p H(M)^* \otimes \Lambda^q A(M)^*)$, we obtain a map $\bar\partial : D^{p,q} \to \to D^{p,q+1}$ by composing the following sequence:

$$\Gamma(\Lambda^q H(M) \otimes \Lambda^q A(M)^*) \xrightarrow{\text{natural } inj} \Gamma(\Lambda^m (H(M)^*) \xrightarrow{\Lambda^m r}$$

$$\to \Gamma(\Lambda^m (T(N) \otimes C)) \xrightarrow{\text{exterior deriv}} (\Lambda^{m+1} (T(M) \otimes C)) \xrightarrow{\text{restriction}}$$

$$\to \Gamma(\Lambda^{m+1} (H(M)^* + A(M)^*)) \xrightarrow{\text{projection}} \Gamma(\Lambda^p H(M)^* \otimes \Lambda^{q+1} A(M)^*).$$

So if $\Phi \in D^{p,q}$, $\bar\partial \Phi$ is essentially the part of $(d\Lambda^m r)\,\Phi$ in $D^{p,q+1}$.

4. THEOREM: *$\bar\partial : D^{0,q} \to D^{0,q+1}$ is well-defined.*

Proof: If we choose another splitting $r': H(M)^* + A(M)^* \to (T(M) \otimes C)$ of the sequence $(*)$, then $r' = r + k$, with $k: H(M)^* + A(M)^* \to (H(M) + A(M))^0$. $\bar\partial : D^{p,q} \to D^{p,q+1}$ will be well defined if $d(\Lambda^m r)\,\Phi - d(\Lambda^m (r+k))\,\Phi$ has no $(p, q+1)$ part. If $\Phi = \zeta_1 \wedge \dots \wedge \zeta_p \otimes \tau_1 \wedge \dots \wedge \tau_p$, $\zeta_i \in \Gamma(H(M)^*)$, $\tau_j \in \Gamma(A(M)^*)$. then $(\Lambda^m r)\,\Phi - (\Lambda^m (r+k))\,\Phi$ consists of terms of the form $L = \pm r\zeta_1 \wedge \dots \dots \wedge k\zeta_j \wedge \dots \wedge r\zeta_p \wedge r\tau_1 \dots \wedge r\tau_q$. k must occur at least once on a term of ζ_i or τ_i (since we are taking the difference with $(\Lambda^m r)$, there are no « pure » $\Lambda^m r$ terms).

Then dL has terms like $dr\,\zeta_1 \wedge \dots \wedge k\,\zeta_j \dots \wedge r\tau_q$ (which clearly has no part in $D^{p,q+1}$ because of the $k\zeta_j$) and $r\zeta_1 \wedge \dots \wedge dk\,\zeta_j \wedge \dots \wedge r\tau_q$. But this has no $D^{p,q+1}$ part by 2-1). But we must assume also that $p = 0$ (for $\bar\partial$ to be well-defined), otherwise dk could have (does in certain examples we must include) some (1,1) terms. #

5. THEOREM: *$\bar\partial : D^{0,q} \to D^{0,q+1}$ is a complex (that is, $\bar\partial^2 = 0$) and this statement can be added to the equivalences in theorem 2.*

Proof: The fact that $\bar{\partial}^2 = 0$ is equivalent to the Nijenhuis tensor $= 0$ (in 2-2)) is a routine computation. We just must examine the effects of $\bar{\partial}^2$ on functions ($D^{0,0}$), then the coefficients of the $(0, 2)$ forms involved consist exactly of terms like the Nijenhuis tensor. #

REMARK: The cohomology groups defined by theorem 4 have been investigated by Kohn [14], in the case of M compact and C-R codim $= 1$, and are important in themselves.

EXAMPLES: *a*) S^4 has no non-trivial C-R structure. For: S^4 has no complex structure (eliminating $n = 2$, $k = 2$) and also its tangent bundle has no two dimensional subbundle (eliminating $n = 3$, $k = 1$).

b) It is not necessarily true that an object satisfying (1-*a*) can be expanded to satisfy both requirements of 1. That is, given an $H(M)$ satsfying (1-*a*), there need be no $\widetilde{H(M)}$ with $\widetilde{H(M)} \supseteq H(M)$, and $\widetilde{H(M)}$ satisfying all of axiom 1. Take $M = R^5$, and an H which has global sections

$$\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_5} = \alpha \qquad \text{(so } k = 2,\ h = 3\text{)}.$$

$$\frac{\partial}{\partial x_3} + i \frac{\partial}{\partial x_4} = \beta$$

Then $[\alpha, \beta] = \frac{\partial}{\partial x_5} \notin \Gamma(H)$, and $[\alpha, \beta]$ is real (so any $\widetilde{H}$ containing H and involutive would have $\frac{\partial}{\partial x_5} \in (\widetilde{H} \cap \widetilde{\bar{H}})$).

c) Contact manifolds (studied by Gray [7] and Sasaki and Tanno [27], etc.) have $n = k + 1$. An almost contact manifold satisfies (1-*a*) only.

d) Let G be a Lie algebra, and G_C its complexification. Let $\mathcal{H}$ be a Lie subalgebra of G_C so that $\mathcal{H} \cap \bar{\mathcal{H}} = 0$. If G is a Lie group with Lie algebra G, then the collection of left-invariant vector fields generated by at the identity form a basis at each point of a homogeneous subbundle of $T(G) \otimes C$ which satisfies 1.

e) Complex manifolds have subbundles satisfying 1, their *holomorphic tangent bundles*. (If only 1 *a*) is true, then the manifold is « almost complex », and (1-*b*), (3-1), (3-2), 5 are wellknown conditions for the integrability of an almost complex manifold.)

f) At the opposite extreme from (*e*) is the following situation: consider a partial differential operator $P = \sum_1^n a_j \frac{\partial}{\partial x}$ on R^n where the a_j are C^∞

complex-valued functions. If, at each point of R^n, the span of the vectors $a_1, \dots, a_n$ of R^2 is two-dimensional, then P determines in the obvious way a subbundle H of $T(R^n) \otimes C$ satisfying 1 (with $k = 1$). (Note that if the span of the vectors $a_1, \dots, a_n$ of R^2 is always one dimensional, the situation (solutions, etc.) is essentially completely handled by the Frobenius theorem).

g) Certain fibre bundles $\pi : M \longrightarrow N$ used in the study of deformations of complex structure have fiber a complex manifold (as in Kodaira and Spencer [3]). Then the R of (2-1) is all of the vertical tangent bundle (a subbundle of $T(M)$) and J is defined by using the complex structure of the fiber. M is a C-R manifold equipped with this (R, J). (If π, the fibering map, is a global product, we have the important example $N \times T$, a real manifold product with a complex manifold. This is the « flat » case).

If $(M, H(M))$ is a C-R manifold, and N is a differentiable submanifold of M, then (provided that the dimension of the fiber of $(T(N) \otimes C) \cap H(M)|_N$ is constant over N) defining $H(N) = (T(N) \otimes C) \cap H(M)|_N$ makes $(N, H(N))$ into a C-R manifold. This remark applied to example *e*), complex manifolds, provides the objects which are the main source of our interest:

h) (Much of what is said here will be true also for real submanifolds of Stein manifolds). C^n is, of course, a C-R manifold. $H(C^n)_p$ is generated by tangent vectors of the form $\Sigma\, a_j \left(\dfrac{\partial}{\partial \bar{z}_j}\right)_p$ (where $z_j = x_j + iy_j$), and $A(C^n)_p$ consists of tangent vectors of the form $\Sigma\, a_j \left(\dfrac{\partial}{\partial z_j}\right)_p$ If N is a real submanifold of C^n, then $H(N) = T(N) \otimes C) \cap H(C^n)|_N$ is, at $p \in N$, all linear combinations $\Sigma\, a_j \left(\dfrac{\partial}{\partial \bar{z}_j}\right)_p$ which are also « tangent » to N. If the fiber dimension is constant, $(N, H(N))$ is a C-R manifold (and a *C-R submanifold* of $(C^n, H(C^k))$).

Such N are also called *embedded C-R* manifolds. Not every C-R manifold is embeddable. Example: in (*g*), consider the product $N \times T$, with N any real monifold and T any compact complex manifold. It is an open question whether any C-R manifold is locally embeddable. (A real analytic manifold with real analytic C-R structure is locally embeddable. See VB).

If M is a real C^∞ submanifold of C^n, then $p \in M$ is a *generic point of* M if $T(M)_p$ is a generic subspace of $T(C^n)_p$ (in the sense of IB5, for $T(M)_p$ is naturally a real subvectorspace of $T(C^n)_p$, which, by affine translstion of C^n, has a complex structure). If p is a generic point of M, then there is an open neighborhood N_p of p in M so that if $q \in N_p$, then q is also a generic point of M (just IB4). Then $(N_p, H(N_p))$ is a C-R manifold, called a *generic submanifold of* C^n. Any hypersurface of C^n is a generic submanifold.

If $(M, H(M))$ is a C-R manifold, let $R = \text{re}\,(HM) + A(M))$, a subbundle of $T(M)$. If $p \in M$, we define the *C-R codimension of M at p* to be $\dim_R((TM)_p/R_p)$

(see $IC2$). If this number is the same for all of M (as when M is connected) it is called the *C-R codimension of M*. Using $IC3$ we know: if M is a generic submanifold of C^n with non-trivial holomorphic tangent bundle, then $C\text{-}R$ codim $(M, H(M)) = \text{codim}_R$ of M in C^n.

We can give a general example af non-trivial generic submanifolds of C^n with $\dim_R = n + k$. Let $\varrho_1 \ldots \varrho_{n-k}$ be C^∞ real-valued functions on C^n. Suppose $p \in \bigcap_j \varrho_j^{-1}(0)$, and $d\varrho_1(p) \wedge \ldots \wedge d\varrho_{n-k}(p) \neq 0$. Then there is a neighborhood N of p in C^n so that $N \cap \bigcap_j \varrho_j^{-1}(0) = M$ is a C^∞ submanifold of C_n of codimension $n - k$. If, in addition, $\bar\partial\varrho_1(p) \wedge \ldots \wedge \bar\partial\varrho_{n-k}(p) \neq 0$ (the ϱ_j are *holomorphically transverse* at p) then N can be chosen so that M is a generic submanifold of C^n. ($\varrho_1 = x_1$, $\varrho_2 = y_1$ in C^n is a submanifold which is not generic, for $\bar\partial\varrho_1 \wedge \bar\partial\varrho_2 = 0$).

6. Definition: Let $(M, H(M))$, $(N, H(N))$ be $C\text{-}R$ manifolds. Put $R = \text{re}(H(M) + A(M))$ and $Q = \text{re}(H(N) + A(N))$. A differentiable map $f: M \to N$ is a *C-R mapping* if, for any $p \in M$, $(df_p, df_p \mid R_p): (T(M)_p, R_p) \to (T(N)_{f(p)}, Q_{f(p)})$ is a map in the $C\text{-}R$ vector space category.

When M and N are complex manifolds, such an f is a complex analytic map - and the condition in 6 is equivalent to requiring that f satisfy the Cauchy-Riemann equations.

We can read off from IC equivalent forms of the definition 6.

7. Theorem: *If $f: (M, H(M)) \to (N, H(N))$ is a differentiable map, the following are equivalent:*

1). *f is a C-R mapping.*

2). *$df \circ J_R = J_Q \circ df$ (J_R, J_Q are the complex structures on R and Q of 2-1).*

3). *$df \otimes 1_C: T(M) \otimes C \to T(N) \otimes C$ does this: $df \otimes 1_C(H(M)) \subseteq H(N)$.*

4). *If $\Lambda f: \Lambda N \to \Lambda M$ is the map naturally induced by f on the exterior algebras, then $\bar\partial_M \circ \Lambda f = \Lambda f \circ \bar\partial_N$ (where $\bar\partial_M$, $\bar\partial_N$ are the $\bar\partial$ maps of 4 on M and N).*

Proof: 1), 2), 3) are equivalences from IC. That (4) is equivalent is a usual linear algebra argument. #

If M is a $C\text{-}R$ submanifold of C^n, and $M \cap \bigcap_j \varrho_j^{-1}(0)$ (as before) we have the following further equivalences:

8. Theorem: *$f: M \to C$ is a C-R map only when there are C^∞ functions a_j defined on M so that $\bar\partial f \wedge \sum_j a_j \bar\partial\varrho_j = 0$. If further the ϱ_j are holomorphically transverse (so M is a generic submanifold of C^n) this is the same as requiring that $\bar\partial f \wedge \bar\partial\varrho_1 \wedge \ldots \wedge \bar\partial\varrho_{n-k} = 0$.*

Proof : Trivial. #

Traditionally f is said to be « relatively holomorphic » (note that the restriction to M of any function holomorphic in neighborhood of M is a C-R map), and the partial differential equations of theorem 8 are the « induced » or « tangential » Cauchy-Riemann equations.

B. 0-Complex Manifolds.

1. THEOREM : Let $(M, H(M))$ *be a C-R manifold. M is a complex manifold and* $H(M)$ *its holomorphic tangent bundle only when* $T(M) = \mathrm{re}\,(H(M) + A(M))$.

Proof: If M is a complex manifold, then $H(M)$ clearly has the desired property. On the other hand, if $T(M)$ is $\mathrm{re}\,(H(M) + A(M))$, we have exactly the hypotheses of the Newlander-Nirenberg theorem [19] so M is a complex manifold. #

If $(M, H(M))$ is a C-R manifold, a submanifold N of M is a *complex submanifold* if N is a C-R submanifold of M and C-R codim $N = 0$. Such an N is, by 1, a complex manifold.

Certain C-R manifolds are very far away from having complex submanifolds :

2. DEFINITION : A (non-trivial) C-R manifold $(M, H(M))$ is *0-complex* if no open subset of M as a complex submanifold.

Examples : Any strictly pseudo-convex hypersurface of C^n is 0-complex. The intersection in C^n of transverse, holomorphically transverse strictly pseudo-convex hyper-surfaces provides examples of 0 complex manifolds of any C-R codimens. (A sphere $|z - a| = r$ is the simplest example of a strictly pseudo convex hypersurface). (See C 12).

$(M, H(M))$ is 0-complex only when : if N is any connected complex manifold, $f \colon N \longrightarrow M$ a non-constant C-R map, then dim $N = 0$. An algebraic interpretation is provided by :

3. THEOREM : *If* $(M, H(M))$ *is a C-R manifold, the following are equivalent :*

1.) *There is an open subset U of M possessing a complex submanifold.*

2.) *There is an open subset U of M, a submanifold N of U, and a* $\overline{}$*-invariant subalgebra* ζ *of* $\Gamma(T(M)\,C)$ *so that* $\zeta|_N \subseteq \Gamma(H(N) + A(N))$.

3.) *There is an open subset U of M, a submanifold N of U, and an element* $u \in \Gamma(H(M))$ *(with* $u \in T(N) \otimes C$*) so that* $[u, \bar{u}]_N = 0$.

Proof: 1) $\rightarrow$ 2) If N is a complex submanifold of U, we can select ζ so that $\zeta|_N = (H(N) + A(N))$ (by extending $H(N)$ as a subbundle of $H(M)$ over M, for example, and taking ζ to be the algebra of sections of the extended bundle).

2) $\rightarrow$ 1) On N, ζ is a $\overline{}$-invariant subalgebra of $\Gamma(H(N) + A(N))$. All we must show is that there is a complex submanifold in some open set of N. Let σ be that open subset of N where the distribution ζ has maximal rank. So there $\zeta = \Gamma(V)$, for some bundle V. Then (by A and IC), $\zeta' = \Gamma(\text{re } V)$ is an involutive subalgebra of $\Gamma(T(N)|_\sigma)$. Hence, by the Frobenius theorem, there is a maximal integral submanifold S of ζ', and $T(S) \otimes C = V = (V \cap H(N)|_\sigma) + (V \cap A(N)|_\sigma)$. Then (1) S is the desired complex submanifold.

1) $\rightarrow$ 3) Take a coordinate z on the complex submanifold S of U. Then extend the element $\frac{\partial}{\partial z}$ of $\Gamma(H(S))$ to any element u of $\Gamma(H(U))$. Since $\left[\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right] = 0$, we have $[u, \bar{u}]|_S = 0$.

3) $\rightarrow$ 2) Let ζ be the subalgebra generated by u and $\bar{u}$. #

The following result generalizes an interesting theorem of Bochner and Martin ([4], Chap. 3,5) on analytic mappings carrying spherical surfaces onto each other.

4. THEOREM: *Let $(M, H(M))$ be a 0-complex C-R manifold, $(N, H(N))$ a C-R manifold, and suppose $f\colon (M, H(M)) \rightarrow (N, H(N))$ is a surjective C-R map. If C-R codim $M =$ C-R codim N then* $\dim M = \dim N$, *and a dense open subset of N is 0 complex.*

Proof: By Sard's theorem (Milnor [18], 2) we can find a regular value $q \in N$ of f with $f(p) = q$. Then df_p is a surjective linear map from $C^a \times R^b$ to $C^c \times R^d$, where $2a + b = \dim M$, $2c + d = \dim N$, C-R codim $M = b$, and C-R codim $N = d$. Since T is onto, $2a + b \geq 2c + d$. Since $b = d$, $2a \geq 2c$. Then $f^{-1}(q)$ is a submanifold of M (since q is a regular value) and if $2a > 2c$ we see that $f^{-1}(q)$ is a non-trivial complex submanifold of M. So $2a = 2c$. By further use of Sard's theorem, df_p is a C-R isomorphism for a dense open subset of $f(p)$'s. Since 0-complexity is local (by 3) this dense open subset is 0-complex.

In particular, 4 states that if a strictly pseudo convex hypersurface H of C^m is mapped onto a hypersurface H' of C^n by a C-R map, then $m = n$, and a dense open subset of H' is 0-complex (but there may be complex submanifolds of H').

REMARK: We can also define q-complex C-R manifolds, and prove a result similar to 4.

C. The Levi Algebra and Levi Forms.

Let $(M, H(M))$ be a C-R manifold. We know by A1 that $[\Gamma(H(M)), \Gamma(H(M))] \subseteq \Gamma(H(M))$, and similarly for $A(M)$. An important invariant of $(M, H(M))$ is the Lie subalgebra of vector fields generated by sections of $H(M) + A(M)$, and how much it differs from just $\Gamma(H(M) + A(M))$. We have already essentially worked with this in B (B3). So :

1. DEFINITION : $\mathcal{L}(M)$, the *Levi algebra of* $(M, H(M))$, is the subalgebra of $\Gamma(T(M) \otimes C)$ generated by $\Gamma(H(M) + A(M))$.

Note that $\mathcal{L}(M)$ is $\overline{}$-invariant. There is a dense open subset σ of M which is the union of finitely many open sets Q_j (plus perhaps a lower dimensional set) so that $\mathcal{L}(M)|_{Q_j} = \Gamma(V_j)$, where V_j is a $\overline{}$-invariant complex subbundle of $T(M) \otimes C$. From here on we make the *assumption* that M is one of the Q_j. So $\mathcal{L}(M) = \Gamma(V)$, for V a $\overline{}$-invariant subbundle of $T(M) \otimes C$ containing $H(M) + A(M)$.

2. DEFINITION : The *excess dimension of* $\mathcal{L}(M)$ (*ex dim* $\mathcal{L}(M)$) is $\dim_C (V/(H(M) + A(M))$.

Ex dim $\mathcal{L}(M)$ is an important invariant for the local study of embedded C-R manifols.

There is also in hierarchy of Levi forms (suggested by H. Rossi and the work of Hermann [10]) which expose the structure of the Levi algebra — the first form is related to the classical Levi form of a hypersurface. But first we need a lemma showing that a « relative » second fundamental form for subbundles of $T(M) \otimes C$ is well-defined. (We will state and prove this for $T(M)$; the lemma and proof remain valid for $T(M) \otimes C$).

3. LEMMA : Let $V \subseteq W_1 \subseteq W_2$ be subbundles of $T(M)$. Suppose $A : T(M) \to T(M)/V$, $B : T(M) \to T(M)/W_1$, and $C : T(M) \to T(M)/W_2$ are the natural projections, and in addition that $B[\Gamma(V), \Gamma(W_1)] \subseteq B(\Gamma(W_2))$. Then there is a natural bilinear bundle map $S_{V, W_1, W_2} : V \times B(W_2) \to T(M)/W_2$ given by the following : if $\alpha \in \Gamma(V)$, $\beta \in \Gamma(W_2)$, then $S_{V, W_1, W_2}((\alpha) p, B(\beta) p) = C([\alpha, \beta]) p$.

Proof : This in a local question, so we shall suppose that near p $\Gamma(V)$ is generated by sections v_i, $\Gamma(W_1)$ is generated by sections w^1_j and v_k, $\Gamma(W_2)$ is generated by sections w^2_l, w^1_j, and v_i, and finally $\Gamma(T(M))$ is generated by sections t_k, w^2_l, w^1_j, v_i. Then any element $\gamma \in \Gamma(T(M))$ can be written $\gamma = \Sigma a_i v_i + \Sigma b_j w^1_j + \Sigma c_l w^2_l + \Sigma d_k t_k$. Locally (up to isomorphism), $A(\gamma) = \Sigma b_j w^1_j + \Sigma c_l w^2_l + \Sigma d_k t_k$, and

$$B(\gamma) = \Sigma c_l w^2_l + \Sigma d_k t_k ,$$

and

$$C(\gamma) = \Sigma\, d_k\, t_k .$$

Using bilinearity of the bracket, it will suffice to check the proposition on $\alpha = av_{i_1}$, and $\beta = bv_{i_2} + cw^1{}_j + dw^2\, l$. Then computation shows that $C([\alpha, \beta])(p)$ is just $a(p)\, d(p)\, [v_{i_1}, w^2{}_l](p)$, which is a bilinear function of $\alpha(p) = a(p)\, v_{i_1}(p)$ and $B(\beta)(p) = d(p)\, w^2{}_j(p)$ #.

We shall also need the following perhaps more familiar « second fundamental form » lemma (for $T(M)$ — but again the result and proof are good for $T(M) \otimes C$).

4. Lemma: Let V be a subbundle of $T(M)$, and $A: T(M) \to T(M)/V$ the natural projection. Then there is a natural skewsymmetric bilinear bundle map $S_V: V \times V \to T(M)/V$ given by the following: if $\alpha, \beta \in \Gamma(V)$, then $S_V: (\alpha(p), \beta(p))$ is just $A([\alpha, \beta])(p)$.

Proof: In the same spirit as 3 #.

5. Definition: L_1, the *first Levi form* of $(M, H(M))$, is a bilinear map $L_1: H(M) \times \overline{H}(M) \times T(M) \otimes C/(H(M) + A(M))$ given by $L_1(a, b) = S^{H(M)+A(M)}(a, \overline{b})$ (where $S^{H(M)+A(M)}$ is the map of 4, with $V = H(M) + A(M)$).

6. Remarks: Note that L_1 is skew-hermitian, so that $L_1(c_1 a, c_2 b) = -c_1 \overline{c_2}\, L_1(a, b)$ for complex numbers c_1, c_2. The information contained in the map $S = S^{H(M)+A(M)}$ of (4) is completely given by L_1, for, since $\Gamma(H(M))$ and $\Gamma(A(M))$ are involutive, $S|_{H(M) \times H(M)} = S|_{A(M) \times A(M)} = 0$. If η is any non-vanishing purely imaginary 1-form annihilating $H(M) + A(M)$, then $(\eta \circ L_1)$ is a bilinear hermitean map to C, more clearly recognized as a « Levi form » (especially in the case of C-R codim $= 1$, see e. g. 11).

It is not necessarily true that the image of L_1 is a subbundle, but when we use Levi forms in what follows, we will make the *assumption* (without further remark) that the image of L_1 (and of other Levi forms when used) will be subbundles.

We will define the k^{th} Levi form in terms of the $(k-1)^{st}$ Levi form, $k \geq 2$. L_{k-1}, the $(k-1)^{st}$ Levi form, is a bilinear bundle map, and $(*)$ $L_{k-1}: (H(M) + A(M)) \times imL_{k-2} \to T(M) \otimes C/(imL_{k-1})^*$ where $(imL_{k+2})^*$ is a subbundle of a quotient bundle $T(M) \otimes C/Q_{k-2}$ of $T(M) \otimes C$, and, if $D: T(M) \otimes C \to T(M) \otimes C/Q_{k-2}$ is the usual projection, $D^{-1}(imL_{k-2}) = (imL_{k-2})^*$. (If $k = 2$, put $imL_{k-2} = H(M) + A(M)$, and $L_{k-1} = S^{H(M)+A(M)}$ of 4). And $D([\Gamma(H(M) + A(M), \Gamma((imL_{k-2})^*)]) \subseteq \Gamma(imL_{k-1})$ Put $(imL_{k-1})^* = D^{-1}(imL_{k-1})$. Then:

7. Definition: $L_k = S_{H(M)+A(M);(imL_{k-2})^*,(imL_{k-1})^*}$ (in 3).

8. Remarks: $L_k : (H(M) + A(M)) \times imL_{k-1} \rightarrow T(M) \otimes C/(imL_{k-1})^*$ is a bilinear bundle map, imL_{k-1} is a subbundle of a quotient bundle $(T(M) \otimes C/(imL_{k-2})^*)$, and $D'([\Gamma(H(M) + A(M), \Gamma((imL_{k-1})^*)]) \subseteq (imL_k)$ if $D' : T(M) \otimes C \rightarrow T(M) \otimes C/(imL_{k-1})^*$ is the natural quotient map (by 3). These remarks show that the inductive hypotheses (*) for 7 are satisfied, and therefore the definition can be «continued» to $k+1$.

$L_k : (H(M) + A(M)) \times imL_{k-1} \rightarrow T(M) \otimes C/(imL_{k-1})^*$ is essentially the elements of $\mathcal{L}(M)$ which are obtained by bracketing k times projected on the collection of elements of $\mathcal{L}(M)$ (assumed to be sections of a bundle) obtained by bracketing fewer than k times. (For proof, examine 3, 4, 7.)

So L_k provides a rough idea of the structure of $\mathcal{L}(M)$, indeed:

9. Theorem: 1) *If* $L_k = 0$, *then* $L_{k+j} = 0$ (*all* $j > 0$).
2) *If* ex dim $\mathcal{L}(M) = t$, *then* $L_{t+1} = 0$.
3) *If* $L_1 = 0$, *then* ex dim $\mathcal{L}(M) = 0$.
4) *If* $\nu = C\text{-}R$ codim M, *then* ex dim $\mathcal{L}(M) \leq \nu$ (*so, by* 1), $L_{\nu+1} = 0$.

Proof: Trivial from 1, 2, 8. #

10. Remarks: In M is 0-complex, then we see from 8,9, and B3 that ex dim $\mathcal{L}(M) > 0$, and $L_1 \neq 0$ (but not conversely!)

Note that the converse of (9-2) is not true: ex dim $\mathcal{L}(M)$ can be large even though $L_2 = 0$.

The k^{th} Levi farm of Hermann [10] is essentially a $(k+1)$-linear map $L_k : H(M) \times ... \times H(M) \rightarrow T(M) \otimes C/N_{k-1}$ where $L_k(a_1, ..., a_{k+1})$ is projection into $T(M) \otimes C/N_{k-1}$ of $[a_1 [a_2 [... [a_k, a_{k+1}]...]]]$, and N_{k-1} is a suitably chosen subbundle (just the images of $L_0, L_1, ..., L_{k-1}$).

When M is a hypersurface in C^n (hence a generic submanifold, see example (h) of A) then historically the Levi form L_1 (by 9-4) the only one which is possibly non-trivial in this case) has long been known, and has a number of interpretations. Suppose $0 \in M$, then:

11. Theorem: *The following hermitean maps on* $H(M)_0$ *are the same, that is, they have the same number of non-zero eigenvalues and the same absolute value of signature*:

1) $[i L_1] : H(M)_0 \times H(M)_0 \rightarrow T(M)_0/(H(M)_0 + A(M)_0)$.
2) *If* η *is a purely imaginary* 1-*form annihilating* $H(M) + A(M)$ and *if* $\eta_0 \neq 0$, *we consider the map* $(a(0), b(0)) \rightarrow (d\eta)(a, \overline{b})(0)$ *where* $a, b \in \Gamma(H(M))$ *from* $H(M)_0 \times H(M)_0 \rightarrow C$.
3) *If* $f : C^n \rightarrow R$, *and* 0 *is a regular value of* f, *and* $M = f^{-1}(0)$, *let* $(CHf)_0$ *be the complex hessian matrix of* f *at* 0 (*that is, the entry*

in the (i,j) place is $\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}|_0$). *Then* $(S, T) \rightarrow S (CHf)_0 \bar{T}$ *is the map considered (for* $S, T \in H(M)_0$, *contained in* $T(C^n)_0 \otimes C$ *canonically).*

4) *If* ζ^i *are linearly independent* 1*-forms generating* $\Lambda^1 (H(M))$ *at* 0, *the Levi form is a map whose matrix* (c_{ij}) *is given by:*

$$f_{\bar{\zeta}_j \zeta_i} - f_{\zeta_i \bar{\zeta}_j} = \Sigma a_{ij}^k f_{\zeta k} + \Sigma b_{ij}^k f_{\bar{\zeta} k} + c_{ij} f_\eta ,$$

where η *is any purely imaginary* 1*-form annihilating* $H(M) + A(M)$.

Proof. 1) $\rightarrow$ 2) is easy, using the formula $2d\eta(\alpha, \beta) = \alpha(\eta(\beta)) - \beta(\eta(\alpha)) + \eta([\alpha, \beta])$ and the fact that if $T(M) \otimes C$ is split near 0 into $H(M) + A(M) + K$, then K_0 is naturally isomorphic to $T(M_0) \otimes C / (H(M)_0 + A(M)_0)$, and η_0 is just a multiplication by a non-zero constant on a suitable K_0.

2) $\rightarrow$ 3) $H(M)$ is just (ker ∂f) $\cap$ $H(C^n)$. Put $Q = \partial f$. This is a purely imaginary 1-form on $T(M)$ (for there it is $\frac{1}{2}(\partial f - \bar{\partial} f)$, since $\partial f = - \bar{\partial} f$ on $T(M) = \ker df$) which is non-vanishing and annihilates $H(M) + A(M)$. So $dQ(S, \bar{T})$ by 2) « is » the Levi form, but $dQ = (\partial + \bar{\partial})(\partial f) = \partial \bar{\partial} f$, and the coefficients of the 2-form $\bar{\partial} \partial f$ in standard coordinates are just (CHf).

2) $\rightarrow$ 4) This method was given by Kohn [14] who also gave the characterization 2). If ω is a 1-form, then f_ω is defined as $\langle df, \omega \rangle$ for a suitable (Hermitean) $\langle , \rangle$ An investigation in local coordinates on M easily shows that $f_{\bar{\zeta}_j \zeta_i} - f_{\zeta_i \bar{\zeta}_j}$ is a first order operator, and so can be expressed by $\Sigma a_{ij}^k f_{\zeta k} + \Sigma b_{ij}^k f_{\bar{\zeta} k} + c_{ij} f_\eta$. But if η is purely imaginary, $\overline{c_{ij}} = c_{ij}$. The hermitean form c_{ij} is the Levi form, merely by using the proof of (2) again. #

REMARK: Hermann [9] has given still another characterization of the Levi form, as the second fundamental form of M relative to a suitable (complex structure-invariant) connection.

12. COROLLARY: M is 0 complex if $\partial \bar{\partial} f$ has all positive or all negative eigenvalues.

Proof: B3 and 11.

If $\partial \bar{\partial} f$ has all positive or all negative eigenvalues, M is called *strictly pseudo-convex*. But note that not every 0-complex hypersurface is strictly pseudo-convex.

D. ex dim $\mathcal{L}(M) = 0$.

In the case that $(M, H(M))$ is the product of a real manifold N and a complex manifold T (so that $H(M)$ merely consists of the holomorphic tangent vectors of T), we see that ex dim $\mathcal{L}(M) = 0$. The converse is true locally :

1. **Theorem** : *Let* $(M, H(M)$ *be a C-R manifold with* ex dim $\mathcal{L}(M) = 0$. *Then if* $p \in M$, *there is an open neighborhood of p in M which is C-R isomorphic to an open neighborhood of* 0 *in* $R^p \times C^q$, *with the natural C-R structure on* $M^p \times C^q$ *(and with* $p =$ *C-R codim M).*

Proof. Perhaps one way to prove this would be to assert that re $\mathcal{L}(M)$ is an involutive subalgebra of $\Gamma(T(M))$ (in fact, it is $\Gamma(\text{re}\,(H(M) + A(M)))$) and use the Frobenius theorem to find maximal integral submanifolds of M for the distribution, each of which will be naturally equipped with a complex structure by the splitting of re $(H(M) + A(M)) \otimes C$. However one problem remains : how to guarantee that the complex structure varies nicely along neighboring maximal integral manifolds. This is the content, of L. Nirenberg's complex Frobenius theorem, with his $k = \frac{1}{2} n$, in Nirenberg [20]. See also Hörmander [12]. #

In case M is an embedded C-R manifold, somewhat more can be said (see III A).

1 shows that in some sense $\mathcal{L}(M)$ measures how different $(M, H(M))$ is from a product structure.

III. Hulls and Reinhardt Submanifolds.

A. Local flatness for generic C-R submanifolds.

Let K be a subset of C^n.

1. Definition : $f \in H(K)$ if there is an open set U of C^n containing K so that $f : U \to C$ is a holomorphic function.

2. Definition : K is *extendible* to a connected subset K' of C^n if $K \subsetneq K'$, and res : $H(K') \to H(K)$ is onto (where res is the natural restriction map).

3. Definition : An open subset U of C^n is a *domain of holomorphy* if U is not extendible in any complex manifold.

4. Definition: If K is a subset of U, an open set in C^n, then $\widehat{K}_U = \{p \in U; |f(p)| \leq \sup_k |f| \text{ all } f \in H(U)\}$. $\widehat{K}_U$ is the $H(U)$ *hull of k.*

Note that if K is extendible to K', then $K' \subseteq \widehat{K}_{C^n}$.

5. Theorem: *U is a domain of holomorphy only when K compact subset of* $U \to \widehat{K}_U$ *compact subset of U.*

Proof: Hormander [11], 2.5. #

6. Definition: A subset L of an open set U is $H(U)$ *convex* if: K compact subset of $L \to \widehat{K}_U$ compact subset of L.

The important concepts of extendibility and convexity are measured (at least locally, for generic C-R submanifolds of C^n) by ex dim of the Levi algebra. First we show that if ex dim $= 0$, then the submanifold is locally holomorphically convex, in a number of ways.

7. Theorem: *Let M be a generic C-R submanifold of* C^n. *The following are equivalent for* $p \in M$:

a.) For sufficiently small open neighborhoods U of $p \in M$, ex dim $\mathcal{L}(U) = 0$.

b.) For sufficiently small open neighborhoods U of $p \in M$, *U is C-R isomorphic to an open set in* $R^p \times C^q$, *with* $p = C$-R codim M.

c.) There is a fundamental sequence of open neighborhoods U of $p \in M$ *so that* $U = \bigcap_{j \in Z} S_j$, S_j *domains of holomorphy in* C^n

d.) There is a fundamental sequence of open neighborhoods U of $p \in M$ *so that U is not extendible.*

e.) For sufficiently small open balls B of C^n *with center p*, $B \cap M$ *is* $H(B)$ *convex.*

Proof: We have already seen a) $\to$ b) (IID1). It remains to remark that another proof can be given for embedded C-R manifolds, not using the powerful complex Frobenius theorem of Nirenberg. For this, see Rossi [26].

b) $\to$ c) Wells (see [33] and [35]) gives an explicit construction of T_ε, normal tubes over U of height ε. These are then shown to be domain of holomorphy. Take $S_j = T_{1/j}$.

In the language of Rossi [24], U is called an S_δ.

e) $\longrightarrow$ d) is obvious.

the results d) $\longrightarrow$ a), c) $\longrightarrow$ a) will be proven by the theorems of V A.

b) $\longrightarrow$ e) is all that remains. First, we observe that by choosing B small enough in C^n we can (since M is a submanifold) obtain $B \cap M = \cup K_i$, where K_i are increasing compact subsets of M. (This is an important step in the example $M = \{|z| = 1\} \subset C$, one must choose B small enough so that $B \cap M \neq M$, for if $B \cap M = M$, $B \cap M$ would not be $H(B)$ convex). It is enough to show that $\widehat{K}_{iB}$ is a compact subset of $B \cap M$, or (since B is a domain of holomorphy) that $\widehat{K}_{iB} \subseteq B \cap M$.

We can define T_ε and S_j as in b) $\longrightarrow$ c). (In fact, T_ε can be defined as $\{z \mid \Phi(z) < \varepsilon\}$ for a plurisubharmonic function Φ.) T_ε is a family of domains of holomorphy decreasing to S_j as $\varepsilon \longrightarrow \frac{1}{j}$. Then $(B, B \cap S_j)$ is a *Runge pair* in the sense of Behnke [1] (see also Bremermann [5]). Then $\widehat{K}_{iB} = \widehat{K}_{iB \cap S_j}$ (since $(B, B \cap S_j)$ is a Runge pair) but $\cap S_j = B \cap M$ (take $U = B \cap M$ in b)). So finally we obtain $\widehat{K}_{iB} \subseteq B \cap M$ as desired. #

REMARK: If M is not generic, all implications of 7 are true (with the same proofs) except d) $\longrightarrow$ a), and c) $\longrightarrow$ a), and e) $\longrightarrow$ a).

7 above is a fairly full characterization of the case ex dim $\mathcal{L}(M) = 0$. When ex dim $\mathcal{L}(M) > 0$, we will obtain some information about the local hull in IV and V. We will indicate a suggestion of this general result by computing a particular class of examples in what follows.

B. Reinhardt Submanifolds.

If K is a subset of C^n, $K^* = \{z \in K \mid$ none of the coordinates of z is $0\} = K \cap \{z \mid z_1 z_2 \dots z_n \neq 0\}$. We define a differentiable map of maximal rank at each point $L : C^{n*} \longrightarrow R^n$ by $L((z_1, \dots, z_n)) = (\log|z_1|, \dots, \log|z_n|)$.

If $p \in R^n$, $L^{-1}(p)$ is a generic submanifold of C^n of dimension n, and hence has no complex tangent vectors. (Also note that $L^{-1}(p)$ is the minimal boundary of a polydisc, and is thus an n-torus, $S^1 \times \dots \times S^1$ (n times)).

1. LEMMA: If M is a submanifold of R^n of dimension k, $L^{-1}(M)$ is a generic $C-R$ submanifold of C^n, of $C-R$ codimension $n-k$.

Proof: Computation. #

2. Definition: A subset K of C^n is a *Reinhardt set* if for any $k = (k_1, \ldots, k_n) \in K$, then $(e^{i\theta_1} k_1, \ldots, e^{i\theta_n} k_n) \in K$ (for any $(\theta_1, \ldots, \theta_n) \in R^n$. K is in *general position* if $K = K^*$.

K is a Reinhardt set in general position only when $L^{-1}(L(K^*)) = K^*$.

If $N \subset R$, let chN be the convex hull of N in R^n.

The remainder of this investigation depends on the following theorem:

3. Theorem: *Let U be an open connected Reinhardt subset of C^n in general position. Then U is extendible to $L^{-1}(ch(L(U)))$. And if U is extendible to V, then $V \subset L^{-1}(ch(L(U)))$.)*

Proof. If U is extendible to V, then $V \subseteq L^{-1}(ch(L(U)))$ (see Rossi [25]). We need two lemmas to prove the reverse inclusion:

4. Definition: A map $A : R^k \times (C - \{0\}) \to C^n$ is a *continuous family of annuli* if A is continuous, and $A_p(z) = A(p, z)$ is a complex analytic map for $I_p < |z| < 0_p$ (I_p, 0_p real positive numbers varying continuously with p). A is degenerate at p if $A_p(I_p \leq |z| \leq 0_p)$ is constant.

5. Lemma: Let N be a connected, simply connected neighborhood of 0 in R^k, and suppose that $A : N \times (C - \{0\}) \to C^n$ is a continuous family of annuli, with A degenerate at O. Then $\bigcup_{p \in N} A_p(|z| = I_p \text{ or } |z| = O_p)$ is extendible to $\bigcup_{p \in N} A_p(0_p \leq |z| \leq I_p)$.

Proof: This is the annulus Kontinuitätssatz. The proof for discs (Wells [33]) goes through almost without change #.

6. Lemma: Let $\gamma : [0,1] \to R^n$ be a continuous curve. Suppose that the straight line from $\gamma(0)$ to $\gamma(1)$ has relatively rational slope. Then $L^{-1}(\gamma([0, 1]))$ is extendible to $L^{-1}(ch(\gamma([0, 1])))$.

Proof: First we consider an example in R^2. Define $\gamma(t) = (0, 2t)$, $t \leq \frac{1}{2}$, and $\gamma(t) = (2t - 1), 2 - 2t)$, $t > \frac{1}{2}$. If H_t is a horizontal line going through $(0, t)$, then L^{-1} of the segment of H_t bounded by γ is just an annulus product with a circle in C^2. When $t = 1$, the annulus is degenerate. By applying 5, we obtain the conclusion of this lemma for this γ.

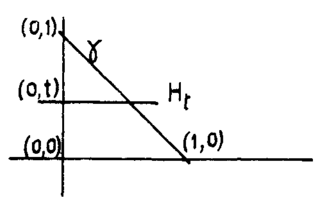

Fig. 1

Now consider the general case, $\gamma : [0, 1] \longrightarrow R^n$, and let the straight line connecting $\gamma(0)$ and $\gamma(1)$ be given by $t \xrightarrow{A_1} (a_1 t + b_1, \dots, a_n t + b_n)$. Then hypothesis that the slopes be relatively rational means that either $a_j = 0$, or $\frac{a_k}{a_j}$ is rational. We will assume that there is a continuous family of stright lines A_q, $0 \leq q \leq 1$, given, with A_q parallel to A_1, and the endpoints of A_q on γ. We assume also that A_0 is a point. So A_q is given by : $t \xrightarrow{A_q} (a_1 t + b_1(q), \dots, a_n t + b_n(q))$.

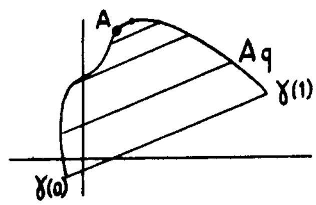

Fig. 2

Consider $T = L^{-1}(A_q([0, 1]))$. Computation shows that T is an $(n-1)$-parameter family of Riemann surfaces, each with two boundary curves on an n-torus ($L^{-1}(A_q(0))$ and $L^{-1}(A_q(1))$). Because of the « relatively rational » assumption, these curves are just circles and T is an $(n-1)$-parameter family of annuli, and gives the conclusion of the lemma. #

Remark : We can also prove 6 directly without the assumption « relatively rational slope ». If $\frac{a_k}{a_j}$ is not rational, the Riemann surfaces are strips, with two boundary lines, and the functions we must extend are almost periodic on the strip. But this can be done with a suitable version of 5, since a Cauchy formula holds for almost periodic functions on a strip.

To return to the proof of 3; we apply 6, and note two things: each point in $ch\,L(U)$ can be obtained in the image of a finite number of constructions of the type of 6 (since U is open), and the extension by means of the constructions of 6 is consistent, that is, whatever finite sequence is used to obtain a point of $chL(U)$ leaves independent the values of the functions extended. #

Consider a curve A in R^n, a l-dimensional submanifold given by $t \longrightarrow (a_1(t), \dots, a_n(t)) \in R^n$. $A^{(j)}$ is the j^{th} derivative of A, and is the map $t \longrightarrow (a_1^{(j)}(t), \dots, a_n^{(j)}(t))$. We will make the *assumption* that for values of t near any value under discussion, the dimension of the linear span of the vectors $A^{(1)}(t), \dots, A^{(j)}(t), \dots$, (all j) remains constant and that if $a^{(j)}(t) = 0$ for some t, $a^{(j)}(t) \equiv 0$. (This last requirement appears rather strong. But it is equivalent to domanding that the images of certain Levi forms be vector bundles, as we always require. The reader can state the results corresponding to 7, 8, and 10 if the last assumption is not made. Examine $x = t^2$, $y = t^3$ in R^2).

7. THEOREM: *The subset* $\}A(t) \mid |t - t_0| < c\{$ *has exactly a k-dimensional convex hull (for small enough c) only when* $A^{(k)}(t), A^{(k-1)}(t), \dots, A^{(1)}(t)$ *are linearly independent for t near* t_0, *and* $A^{(k+1)}(t)$ *is linearly dependent on* $A^{(k)}(t), \dots, A^{(1)}(t)$ *(so that* $A^{(k+1)}(t) = \sum_{q=1}^{k} v_q(t) A^{(q)}(t)$ *for some differentiable functions* v_q*).*

Proof: Taylor's theorem. ♯

The formula for $A^{(k+1)}$ appearing in 7 is, when suitably normalized, called the Frenet formulas.

Consider now $L^{-1}(A) = M$ is a generic $C - R$ submanifold of C^n, of $C - R$ codimension $n - 1$. A coordinate map for this $(n+1)$ dimensional manifold is provided by:

$$R^{n+1} \ni (t, \theta_1, \dots, \theta_n) \xrightarrow{R} \begin{cases} z_1 = e^{a_1(t)} e^{i\theta_1}, \dots, z_n = e^{a_n(t)} e^{i\theta_n} \\ \bar{z}_1 = e^{a_1(t)} e^{-i\theta_1}, \dots, \bar{z}_n = e^{a_n(t)} e^{-i\theta} . \end{cases}$$

Then

$$dR\left(\frac{\partial}{\partial t}\right) = \sum_j \left(a_j' e^{a_j} e^{i\theta_j} \frac{\partial}{\partial z_j} + a_j' e^{a_j} e^{-i\theta_j} \frac{\partial}{\partial \bar{z}_j}\right),$$

and

$$dR\left(\frac{\partial}{\partial \theta_j}\right) = i e^{a_j} e^{i\theta_j} \frac{\partial}{\partial z_j} - i e^{a_j} e^{-i\theta_j} \frac{\partial}{\partial \bar{z}_j} .$$

We can explicitly find a single non-zero tangent vector field S generating $\Gamma(H(M))$. It is:

$$S = dR\left(\frac{\partial}{\partial t}\right) + \sum_j \frac{1}{i} a_j' \, dR\left(\frac{\partial}{\partial \theta_j}\right) = 2 \sum_j a_j' e^{a_j} e^{i\theta_j} \frac{\partial}{\partial z_j} ,$$

and $\bar{S}$ generates $\Gamma(A(M))$.

8. THEOREM: 1.) *If for small evough c,* $A_c = \}A(T) \mid |t - t_0| < c\{$ *has exactly a k-dimensional convex hull, then ex dim* $\mathcal{L}(M_c) \geq k$ *(where* $M_c = L^{-1}(A_c)$*).*

2.) *If we suppose in addition that*

$$A^{(k+1)}(t) = \sum_{q=1}^{k} v_q(t) A^{(q)}(t) \quad \text{for} \quad |t - t_0| < c$$

(as in 7), then if we put $\mathcal{L}_q(S, \bar{S}) = [S[\ldots \overset{q \text{ times}}{[S[S, \bar{S}]]} \ldots]]$, *we have* : $S, \bar{S}, \mathcal{L}_1(S, \bar{S}), \ldots, \mathcal{L}_{k-1}(S, \bar{S})$ *are linearly independent, and*

$$\mathcal{L}_k(S, \bar{S}) = v_1(t)(S - \bar{S}) + \sum_{q=2}^{k} v_p(t)\,\mathcal{L}_{q-1}(S, \bar{S})$$

(over $p \in L^{-1}(A(t))$*).* $\left(\textit{Explicitly, } \mathcal{L}_{q-1} = -2i dR\left(\Sigma a_j^{(q)} \frac{\partial}{\partial \theta_j}\right).\right)$

Proof: A computation with dR gives the form of $\mathcal{L}_{q-1}$ of (2). The other conclusions follow immediately. #

We might call the formula of (8·2) (in analogy with (7)) the *complex Frenet formulas.*

We will need the following:

9. LEMMA: Let $\mathcal{H}$ be a Lie algebra generated by a and b, Define $\mathcal{L}_k$ by $\mathcal{L}_1 = [a, b]$, and $\mathcal{L}_k = [a, \mathcal{L}_{k-1}]$. Suppose there is a Lie algebra map $\overline{}$ on $\mathcal{H}$ so that $\overline{}^2$ is the identity, and $\bar{a} = b$ and $\bar{\mathcal{L}}_k = -\mathcal{L}_k$. Then $\mathcal{H}$ is generated as a vector space by a, b, and all the $\mathcal{L}_k$.

Proof: An induction argument based on the «length» (in brackets) of a term. #

Assume (as before 7) that A has constant dimensional convex hull at each point.

10. THEOREM: *Let A be a curve in R^n, and $M = L^{-1}(A)$. The following are equivalent for $p = A(t_0) \in R^n$.*

1. *For sufficiently small open neighborhoods of U of $L^{-1}(p)$ in M,* ex $\dim \mathcal{L}(U) = k$.

2. *For sufficiently small open neighborhoods of U of $L^{-1}(p)$ in M, U is extendible to a subset containing a manifold $\widehat{U}$, with* $\dim \widehat{U} = \dim U + k$. *When k is maximal with respect to this property, $\widehat{U}$ can be chosen to be not extendible.*

3. *The k^{th} Levi form, L_k (of* IIC*), is non-zero, and L_{k+1} is zero.*

Proof: 1) $\rightarrow$ 3) We must notice two facts: first, that $\mathcal{L}(U)$ is generated by $S, \bar{S}, \mathcal{L}_1(S, \bar{S}), \ldots, \mathcal{L}_k(S, \bar{S})$ as a vector space (by using 9 and 8·2)), and that $L_k = 0$ only when $\mathcal{L}_k(S, \bar{S})$ is linearly dependent on $S, \bar{S}, \mathcal{L}_1(S, \bar{S}), \ldots, \mathcal{L}_{k-1}(S, \bar{S})$.

1) $\to$ 2 Use 9 and the explicit form for $\mathcal{L}_{q-1}(S, \bar{S})$ given in (8-2) to conclude that $A^{(1)}(t), \ldots, A^{(k)}(t)$ are linearly independent (but no more), then use 7 and 3 applied to a neighborhood U. The $\widehat{U}$ desired contains the $(n+k+1)$-dimensional part of $L^{-1}(chL(U))$. Then $L(\widehat{U})$ is convex, so $\widehat{U}$ is not extendible.

2) $\to$ 1) Suppose that U is extendible to at most a set containing a manifold $\widehat{U}$, with $\dim \widehat{U} = \dim U + k$, k maximal, and that ex dim $\mathcal{L}(U) = j$. If $j > k$, by 1) $\to$ 2), k is not maximal, So $j \leq k$.

If $j < k$, the convex hull of A just contains a $j+1$ dimensional set by 8 and 7. Then $L(\widehat{U})$ is not in the convex hull of $L(U)$, since dim $L(\widehat{U}) = \dim L(U) + k = 1 + k$ (if dim $L(U) = 0$, 2) $\to$ 1) is trivial). So by 3 we have a contradiction U cannot be extendible to $\widehat{U}$.

Therefore $j = k$. #

A theorem completely analogous to 10 can be stated for higher dimensional Reinhardt submanifolds:

11. THEOREM: *Let A be a submanifold of R^n, $p \in A$. Suppose that the convex hull of open subsets of A has constant dimension near p. Put $M = L^{-1}(A)$. The following are equivalent:*

1.) *For sufficiently small open neighborhoods U of $L^{-1}(p)$ in M*, ex dim $\mathcal{L}(U) = k$.

2.) *For sufficiently small open neighborhoods U of $L^{-1}(p)$ in M, U is extendible to a subset containing a manifold $\widehat{U}$, with* $\dim \widehat{U} = \dim U + k$. *When k is maximal with respect to this property, $\widehat{U}$ can be chosen to be not extendible.*

Proof: It is not difficult to state and prove results corresponding to 7, 8, 9 in this more general case. Then the proof goes as in 10. (Or we can just study families of curves on A, and apply 10.) #

12. REMARKS: Note that there is no (3) in 11. Such a result would be false. For L_2 could be zero, but the first brackets (corresponding to second derivatives of the coordinates of A) could be non-zero, and bring more than one new dimension to $\mathcal{L}(U)$.

Even (10-3) is not true for general C-R submanifolds of C^n of dimension $n+1$, because it is essential (9) that $\bar{\mathcal{L}}_k = -\mathcal{L}_k$. This is not always true — it's true here only because of the form of $\mathcal{L}_k$ given in (8-2).

We can find curves A in R^n whose hulls contain an open subset of R^n. This gives rise to generic $(n+1)$-dimensional submanifolds M of C^n so that M is extendible to an open subset of C^n — we can even obtain M extendible to all of C^{n^*} by choosing A correctly. (Example: $n = 2, A$ given by $x = t$, $y = e^t \sin t$, has convex hull all of R^2. The $\bar{\partial}$ equation induced on R^3 (from M) is

$$\left(i \frac{\partial f}{\partial u_1} - \frac{\partial f}{\partial u_2} - \frac{d\,(e^{u_1} \sin u_1)}{d u_1} \frac{\partial f}{\partial u_3} = 0 \right).$$

10 forces us also to say that ex dim is only a local invariant for the study of extension: in general, there may be $p \in A$ so that any small enough neighborhood of $L^{-1}(p)$ in M will have ex dim $= 0$, but $M = L^{-1}(A)$ may still be extendible to an open set. (Example: $A(t) = (\cos t, \sin t), 0 \leq t \leq \pi$, $A(t) = (\cos t, 0), \pi < t \leq 2\pi$, in R^2. Let p be any point with $\pi < t < 2\pi$).

C. Osculating Manifolds.

Let M be a C-R manifold, and $i: M \rightarrow C^n$ a C-R map which embeds M generically in C^n. We wish to construct a C-R manifold $\widetilde{M}$ « approximating » M to first order, so that $\widetilde{L}_1$ (the first Levi form for $\widetilde{M}$) is nicely related to L_2, the second Levi form for M. Let $\nu(i) \rightarrow M$ be the normal bundle of the embedding i. There is a « natural » map, the exponential map given by the usual straight-line metric in C^n, $E: \nu(i) \rightarrow C^n$, so that $E\,|$ zero-section is the identity onto $i(M)$, and $E \mid U$ is a C^∞ isomorphism onto an open neighborhood σ of $i(M)$ in C^n, for some neighborhood U of the zero-section of $\nu(i)$.

Consider now $L_1: H(M) \times H(M) \rightarrow T(M) \otimes C/(H(M) + A(M))$. Let $p: T(M) \otimes C \rightarrow T(M) \otimes C/(H(M) + A(M))$ be the canonical projection, and $(imL_1)^* = p^{-1}(imL_1)$. Put $H^* = (H(M) + (imL_1)^* + i\,(imL_1)^*) \cap H(C^n)$. If $L_1 \neq 0$, then $H^* \neq H(M)$, and H^* and $A^* = \overline{H^*}$ are subbundles of $T(C^n) \otimes C$ over M.

If $H^* \neq H(M)$, then $\operatorname{re}(H^* + A^*)$ is not a subbundle of $T(M)$ in-fact, $\dim_C (imL_1)$ is $\dim_R j\,(\operatorname{re}(H^* + A^*))$, where j is part of the sequence: $0 \rightarrow T(M) \xrightarrow{di} T(C^n)|_M \xrightarrow{j} (M) \rightarrow 0$, all bundles over M. In fact, $j(\operatorname{re}(H^* + A^*))$ will determine $\widetilde{M}$.

1. THEOREM: *There is a generic C-R submanifold $\widetilde{M}$ of C^n having $i(M)$ as submanifold, with $H(\widetilde{M})|_{i(M)} = H^*$, so that* $\dim_R \widetilde{M} = \dim_R M + \dim_C imL_1$.

Let $\widetilde{L}_1: H(\widetilde{M}) \times H(\widetilde{M}) \rightarrow T(\widetilde{M}) \otimes C/(H(\widetilde{M}) + A(\widetilde{M}))$ be the first Levi form of M, and L_2: $(H(M) + A(M)) \times (imL_1) \rightarrow T(M) \otimes C/(imL_1)^*$ be the second Levi form of M. If $a \in H(M)$, and $b \in (imL_1)^* \cap H(C^n)$ with $p(b) = b'$, then

$L_1(a, b) = L_2(a, b')$ (by recognizing the isomorphism between

$T(M) \otimes C/(imL_1)^*$ and $T(\widetilde{M})|_{i(M)} \otimes C/(H(\widetilde{M})|_{i(M)} + A(\widetilde{M})|_{i(M)}))$.

Proof: Put $\widetilde{M} = E(U \cap j(\text{re}(H^* + A^*)))$. Since $i(M)$ is generic, we can select U so that $\widetilde{M}$ will be a generic submanifold of C^n. Then $H(\widetilde{M})|_{i(M)} = H^*$, and the other properties are easily checked merely from the definitions of the Levi forms. #

We call $\widetilde{M}$ the *first order osculating C-R manifold to M.* It (locally) does not depend on i:

2. **Theorem**: *The germ of $\widetilde{M}$ at M is independent of the embedding: that is, let $i: M \to C^n$, $k: M \to C^n$ be two generic embeddings of M, $\widetilde{M}_i$, $\widetilde{M}_k$ the manifolds derived from i and k respectively by applying 2. Then the natural isomorphism $\tau: i(M) \to k(M)$ extends to a C-R isomorphism $\bar{\tau}: U \to U'$ where U, U' are neighborhoods of $i(M)$ and $k(M)$ in $\widetilde{M}_i$ and $\widetilde{M}_k$, respectively.*

Proof: Let $E: \nu(i) \to C^n$, $E': \nu(j) \to C^m$ be respective exponential maps. Then we observe that H_i^* and H_k^* are naturally isomorphic (both are just $H(M) + H$ (some complexification of imL_1)).

Then $\text{re}(H_i^* + A_i^*)$ and $\text{re}(H_k^* + A_k^*)$ are isomorphic, as C-R manifolds. That is, if $\sigma: M \to M$ is the isomorphism of zero sections, σ can be extended to $\bar{\sigma}: \text{re}(H_i^* + A_i^*) \to (H_k^* + A_k^*)$. (We may have to change $\bar{\sigma}$ slightly to get a C-R isomorphism). But $\widetilde{M}_i = E(\theta \cap j(\text{re}(H_i^* + A_i^*))$ and $\widetilde{M}_k = E'$ $(\theta' \cap l(\text{re}(H_k^* + A_k^*))$ (if j, l are the usual maps onto the normal bundle). We can select θ and θ' to be $\bar{\sigma}$-invariant, so that $E' \circ \bar{\sigma} \circ E^{-1}$ is the desired $\bar{\tau}$ extending τ and U (respectively U') is just the image of $\theta \cap j(\text{re}$ $(H_i^* + A_i^*)$ (respectively the image of $\theta' \cap l(\text{re}(H_k^* + A_k^*)))$. #

3. **Remark**: By iteration, 1 reduces the study of the higher order Levi forms to the first form — but the relationship is not entirely clear.

Now consider an embedding $I: R \to R^n$ of R in R^n. $I(R)$ is a curve, A, in R^n. As above we have a sequence $0 \to T(R) \xrightarrow{dI} T(R^n)|_A \xrightarrow{J} N(A) \to 0$, where $N(A)$ is the normal bundle over A. Then there is an exponential map $E: N(A) \to R^n$ which is the identity on the zero-section and an isomorphism is a neighborhood U' of the zero-section. Then $I''(p) \in T(R^n)_{I(p)}$ so I'' determines, by composition with J, a subbundle R of dimension one of $N(A)$. We define $\widetilde{A}$, the *osculating strip with center A*, to be $E(R \cap U')$.

4. THEOREM: $H(\widetilde{L^{-1}(A)})|_{L^{-1}(A)} = H(L^{-1}(\widetilde{A}))|_{L^{-1}(A)}$.

Proof: We merely must check (see B 8) that $[S, \bar{S}]$ and $d(L^{-1})(I'')$ generate the same complex vector space in $T(C^n)|_{L^{-1}(A)} \otimes C$. By (B 8·2)) this is true. #

Note that $L^{-1}(\widetilde{A})$ and $\widetilde{L^{-1}(A)}$ are *not* the same, for any choice of U and U' in the above, but they agree to first order at M. We will see later that if M extends to a manifold N with $M \subseteq \partial N$, then $H(\bar{N})|_M \supseteq H(\widetilde{M})|_M$, but M need not be extendible to $\widetilde{M}$ itself.

So $M \rightarrow \widetilde{M}$ is a transition which reduces the orders of the Levi forms by one (1), is natural (3,4), and helps to diagnose the first order direction of extension.

Hermann [10] has also studied « holomorphic osculating spaces », and essentially has the following result, which may help explain the role of $\widetilde{M}$ in extension:

5. THEOREM: *Let f be a holomorphic function defined in a neighborhood of M. Then the derivatives of f on M in any direction tangent to $\widetilde{M}$ are determined by the derivatives of f on M in directions tangent to M.*

Proof: We use the Cauchy Riemann equations and the fact that $\widetilde{M}$-directions are brackets of M directions.

This is exactly the computation in Hermann [10], theorem 2.1. #

IV. GOING UP ONE DIMENSION

The purpose of this section is to prove the following:

1. THEOREM: *Let M be a generic C^q submanifold of C^n of real dimension $n + k$ (q finite, q sufficiently large). Suppose that ex dim $\mathcal{L}(M) \geq t > 0$.*

If $p \in M$, then for a sufficiently small ball B in C^n whith center at p, there is a C^q generic submanifold M of B_p of real dimension $n + k + 1$, with ex dim $\mathcal{L}(N) \geq t - 1$ and $M \cap B_p \subset \partial N$. And N will be given as a subset of the regular set of a map $F\colon R^{n+k-1} \times \{|z| \leq 1\} \rightarrow C^n$ so that

1.) *F is analytic in the second factor.*

2.) *$F(N^{n+k-1} \times \{|z| = 1\}) \subseteq M$.*

3.) *F contains degenerate discs.*

Proof: First put $p = 0$. Since ex dim $\mathcal{L}(M) > 0$, we can find $u \in \Gamma(H(M))$ so that $[u, \bar{u}]_0 \notin (H(M) + A(M))_0$. Then by arguments used by Wells [33], Weinstock [32], also Hörmander [11] (using $z_k = x_k + iy_k$) we can assume there is a function ϱ and vector $w \in H(M)_0$ so that: $M \subset \varrho^{-1}(0)$, $d\varrho(0) \neq 0$, $\partial p(0)(w) = 0$, and $\sum_{j,l=1}^{n} \frac{\partial^2 \varrho(0)}{\partial z_j \partial \bar{z}_l} w_j \bar{w}_l < 0$ by a linear change of variables, $\varrho(q) = y_n + A(q) + 0(3)$ ($0(3)$ means terms vanishing to order 3 at 0), and $A(q)$ is the second-order Taylor series terms. $T(M)_0 \otimes C$ is a linear subspace of $T(C^n)_0 \otimes C$ generated by

$$\left(\frac{\partial}{\partial z_1}\right)_0, \dots, \left(\frac{\partial}{\partial z_k}\right)_0, \dots, \left(\frac{\partial}{\partial \bar{z}_1}\right)_0, \dots, \left(\frac{\partial}{\partial \bar{z}_k}\right)_0, \text{ and } \left(\frac{\partial}{\partial x_{k+1}}\right)_0, \dots, \left(\frac{\partial}{\partial x_n}\right)_0 .$$

$$A(q) = \sum_{\substack{1 \le j \le k,\, j = n \\ 1 \le l \le k,\, l = n}} \frac{\partial^2 \varrho(0)}{\partial z_j \partial \bar{z}_l} z_j \bar{z}_l + \mathrm{re} \sum_{\substack{1 \le j,\, l \le k \\ j,\, l = n}} \frac{\partial^2 \varrho(0)}{\partial z_j \partial z_l} z_j z_l +$$

$$(1) \qquad \sum_{j,\, l=k+1}^{n-1} \frac{\partial^2 \varrho(0)}{\partial x_j \partial x_l} x_j x_l + \frac{\partial^2 \varrho(0)}{\partial x_j \partial y_l} x_j y_l + \frac{\partial^2 \varrho(0)}{\partial y_j \partial y_l} x_j y_l +$$

$$(2) \qquad \sum_{1 \le l \le k,\, l = n} \sum_{j=k+1}^{n-1} \frac{\partial^2 \varrho(0)}{\partial x_j \partial x_l} x_l x_j + \frac{\partial^2 \varrho(0)}{\partial x_l \partial y_j} x_l y_j + \frac{\partial^2 \varrho(0)}{\partial y_l \partial x_j} y_l x_j + \frac{\partial^2 \varrho(0)}{\partial y_l \partial y_j} y_l y_j .$$

Analysis of $A(q)$: Put $z_j' = z_j$ for $j \le k$, and $z_n' = z_n + i \sum_{1 \le j,\, l \le k} \frac{\partial^2 \varrho(0)}{\partial z_j \partial z_l} z_j z_l$ (substitution of Hörmander). Then $A(q) = \sum_{\substack{1 \le j,\, l \le k \\ j,\, l = n}} A_{jl} z_j' \bar{z}_l + (1) + (2)$. By hypothesis, (A_{jl}) is not positive semidefinite, so using a linear change of coordinates we can obtain $A_{11} < 0$. If we follow this by a suitable unitary coordinate change, we can even assume (A_{jl}) diagonal.

Term (1) is $0(|x|^2) + 0'(|t|^2)$, where x is a direction normal to M and t is a direction tangent to M — but not a term involving z_1.

Term (2) is: $\sum_{l=k+1}^{n-1} (s_l x_1 + t_l y_1) + 0'(|z|\,|x|)$, where s_l, t_l are real, independent of z_1 (they are $0(|x|)$, x normal to M), and $0'(|z|\,|x|)$ contains no term of the form '$|z_1|\,|x|$'. (We will combine the terms $0'(|t|^2)$ into $0'(|z|\,|x|)$ — their treatment will be te same in what follows).

Then by using the inverse function theorem, M is given in a suitable small neighborhood of 0 by:

$y_{k+1} = f_{k+1}(z, x)$

$y_{n-1} = f_{n-1}(z, x)$

$$y_n = |z_1|^2 + \sum_{2\leq j\leq k} \lambda_j |z_j|^2 + 0'(|z_{,}|x|) + 0(|x|^2) + 0(3) + \sum_{l=k+1}^{n-1} (s_l x_1 + t_l y_1).$$

The functions $(z_1, \dots, z_k, x_{k+1}, \dots, x_n)$ are coordinates for M near 0.

We will now describe a family of discs with boundaries on M, whose interiors will « fill up » a manifold N of dimension $n + k + 1$.

Let $\bar{D} = \{z \mid |z| \leq 1\} \subset C$. Consider the map $W: R \times D \times C^{k-1} \to C_k$ given by $W(\varrho, t, w) = (\varrho t, w)$. W is a C^∞ function, and for ϱ, w constant, the partial map of $W: \bar{D} \to C^k$ is analytic. Note that for $\varrho = 0$, the disc is degenerate, i.e., just a point. Under these conditions Bishop and Weinstock (using genericity of M) show that there is a map F defined in a neighborhood of 0 in $R^{n-k} \times R \times \bar{D} \times C^{k-1}$ with values in C^n having the following properites:

(i) $F(a, \varrho, -, w)$ is C^q on $\bar{D}$ and analytic on D.

(ii) F is C^q (Actually, Weinstock asserts that F is C^q on $R^{n-k} \times \times R \times \partial D \times C^{k-1}$ but by (i) F is analytic in the third variable — hence we can use Cauchy's formula and differentation under the integral to establish as much differentiability in D as on ∂D).

(iii) $F(a, \varrho, t, w) \subset M$ if $|t| = 1$.

(iv) $x_{k+j} \circ F(a, \varrho, 0, w) = a_j$ for any $a, \varrho, w, 1 \leq j \leq n - k$.

(v) $z_j \circ F(a, \varrho, t, w) = z_j \circ W(\varrho, t, w)$, $1 \leq j \leq k$.

Note that the domain space of F has exactly $(n + k + 1)$ parameters. We will show that for certain points in a neighborhood of 0, the Jacobian has rank $(n + k + 1)$ — and this will display the desired N as a subset of the image of F.

Let $F_j = z_j \circ F$, so $F = (F_1, \dots, F_n) \in C^n$. By (v), $\dfrac{\partial(F_j, \bar{F})}{\partial(z_l, \bar{z})}$, $2 \leq l, j \leq k$ has rank $(2k - 2)$. Applying (iv) at $(a, \varrho, 0, w)$, $\dfrac{\partial(\mathrm{re}\, F_j)}{\partial x_l}$, $\left\{\begin{matrix} 1 \leq l \leq n-k \\ k+1 \leq j \leq n \end{matrix}\right\}$ has rank $n - k$ (it is the $(n - k) \times (n - k)$ identity matrix). Note that at $(a,0,t,w)$. $\dfrac{\partial(\mathrm{re}\, F_j)}{\partial x_l}$ is also the identity matrix (this uses the degeneracy of the discs at 0).

The Jacobian matrix is in upper triangular block form, with two of the diagonal blocks the ones investigated above. So in any sufficiently small neighborhood of 0, the Jacobian of F has renk at least $(n-k) + (2k-2) = = n + k - 2$.

We now consider the map $(F_1, F_n): R^{n-k} \times R \times \bar{D} \times C^{k-1} \to C^2$, which is given by the first and last components of F, with all the variables except ϱ, t, and c_n (the center of F_n) held constant. We will show that this map is injective in a suitably small neighborhood of 0.

(In the following we imitate closely the methods of Bishop and Weinstock.)

In C^2 described by (z_1, x_n, y_n), « M » is given by

$$y_n = |z_1|^2 + \sum_{l=k+1}^{n-1} s_l x_1 + t_l y_1 + \underbrace{\sum_{2\leq j\leq k} \lambda_j |z_j|^2 + 0'(|z||x|) + 0(|x|^2)}_{T(2)} + 0(3).$$

Let $T(2)$ be the order 2 terms indicated. Relative to z_1, x_n, y_n, $T(2)$ is nothing but a constant + further $0(3)$ terms $\left(\text{because the functions appearing in } T(2) \text{ have first derivatives perpendicular to } \frac{\partial}{\partial z_1}, \frac{\partial}{\partial x_n}, \text{ and } \frac{\partial}{\partial y_n}\right)$. So $y_n = |z_1|^2 + \sum_{l=k+1}^{n-1} s_l x_1 + t_l y_1 +$ const (depending on the nbd) $+ 0(3)$.

Let $y_n = h_n(x, z)$, and let $g(x, z)$ be the $0(3)$ term. Put $h'(x, z) = h(x, z) - g(x, z)$, so h' is just the terms of order less than 3. Using property (iii) of F, we can consider the functions on ∂D which have image in M: (If $f: \partial D \to R^p$, we define $Tf: \partial D \to R^p$ to be the trace on ∂D of the harmonic conjugate of f which vanishes at the origin). Then $x_n = c_n - Th(x, w)$ (c_n is the real coordinate of the center of the disc). And $x_n = c_n - Th'(x,w) - Tg(x, w)$, so $z_n = c_n - Th'(x, w) + ih(x, w) - Tg(x, w) + ig(x, w)$. Define x', z' by

$$\begin{cases} x_n' = c_n - Th'(x, w) \\ z_n' = x' + ih'(x, w) \end{cases}$$

(As Bishop remarks, these discs approximate the given ones — we have thrown away $0(3)$ terms). Then ($\| \ \|$ is the Sobolev 1-norm on the boundary curve) $\|x\| \leq \|x'\| + \|Tg(x, w)\| \leq \|x'\| + \|g(x, w)\|$.

(The last inequality follows from Lemma 3.2 of Weinstock, essentially $\|Tf\| \leq \|f\|$.) $\leq \|x'\| + R\|(x, w)\|^2$ (by the corollary to Lemma 3.4 of Weinstock — essentially because g is $0(3)$ at 0 and we are taking the 1-norm.)

Take x, w small enough, so that $\|x\| + \|w\| \leq (9R)^{-1}$. Then the last inequality gives $\|x\| \leq \|x'\| + \frac{1}{9}\|x\| + \frac{1}{9}\|w\|$, so ($\alpha$) $\|x\| \leq 9\|(x', w)\|$. Let us take another element (c_n^*, ϱ^*, t^*) of the domain space, with y, y', Z, Z' the functions corresponding to x, x', z, z'. Then:

(β) $\|(z_n - z_n') - (Z_n - Z_n')\| \leq C\|g_n(x, w) - g_n(y, w)\| \leq C\|(x - y, 0)\|$ $(\|(x, w)\| + \|(y', w)\|)^2$ (by the corollary to Weinstock's Lemma 3.4) $\leq C$ $\|(x' - y')\|(\|(x', w)\| + \|(z, w)\|)^2$ by (α) for some C.

Let us consider F defined on the open set U_{r_0} given by $|c_n| + |L| < \varrho < r_0$ (any $t \in \bar{D}$) (where $|L|$ consists of the other variables which we have « held constant »). Then we will show that for sufficiently small r_0, F is 1-1 on U_{r_0} with the « L » variables held constant.

So we suppose $z_n = Z_n$. The following is the essential reasoning: using the above (β), the Sobolev lemma (which shows that the 1-norm along the 1-dimensional curves forming the boundary dominate the uniform norm), and the maximum modulus principle applied to our analytic discs, (if the uniform norm is dominated on the boundary, it is dominated on the whole disc) we have:

$$(\gamma)\ |z_n' - Z_n'| \leq c\,(\varrho + \varrho^*)^2\, \|x' - y'\|.$$

(We have reduced the problem to the approximating quadratic model!) We can compute h' explicitly (pp. 41-43 of *Weinstock*) to get

$$z_n' = c_n - (\Sigma s_l\, y_1 + t_l\, x_1) + i\varrho^2 + (\Sigma s_l\, x_1 + t_l\, y_1) + \text{Const}\,(L)$$

$$z_1 = t\varrho.$$

The « Const » term is constant relative to T (there are no first order z_n, y_1, x_1 terms in $0'(|z||x|)$ or $0(|x^2|)$). Note also that s_l, t_l are bounded by $c\,|\varrho|$ since they are, up to a constant, elements of L.

The same formulas hold true for Z_n', Z_1', with the variables c_n, ϱ, t having $*$'s on the right hand side (but the Const (L) term is the same). If we show $c_n^* = c_n$, and $\varrho = \varrho^*$, then (since $z_1 = \varrho t$) we'll have $t = t^*$. We note that $\|x' - y'\| \leq CB$, where $B = |c_n - c_n^*| + (\varrho + \varrho^*)\,|\varrho - \varrho^*|$ (here we use the character of s_l, t_l terms).

Then apply (γ) to see $|\varrho^2 - \varrho^{*2}| \leq C\,(\varrho + \varrho^*)^2 B$. Apply ($\gamma$) to the real parts; we obtain similarly $|c_n - c_n^*| \leq C\,(\varrho + \varrho^*)^2 B$. So $B = |c_n - c_n^*| + (\varrho + \varrho^*)\,|\varrho - \varrho^*| \leq C\,(\varrho + \varrho^*)^2 B \leq 4r_0^2\, CB$. This is impossible if r_0 is sufficiently small, unless $B = 0$. Therefore $c_n = c_n^*$, $\varrho = \varrho^*$, and $t = t^*$, and we are done.

The injectivity combined with the implicit function theorem gives the additional three dimensions we need in the rank of the Jacobian, so we now know that $F|\,U_{r_0}$ is a nontrivial family of analytic discs. When $\varrho = 0$, the C^q family contains degenerate discs. $F(U_{r_0})$ consists of discs whose boundaries lie on M and whose interiors are not on M.

By the openness of the genericity condition (see IIA, example (h), also IB4) there will be a simply connected open subset of U_{r_0}, call it U_{r_0}', so that $F(U_{r_0}')$ is generic. Put $N = F(U_{r_0}')$. Note that $M \cap B^p \subset N$ for small enough balls B_p.

We must show ex dim $\mathcal{L}(N) \geq t-1$.

Near 0, $\Gamma(T(M) \otimes C)$ is generated by $\zeta_1, \ldots, \zeta_k \in \Gamma(H(M))$, and $\bar{\zeta}_1, \ldots \bar{\zeta}_k \in \Gamma(A(M))$, and $x_{k+1}, \ldots, x_n$, the « other » vector fields on M. Since ex dim $\mathcal{L}(M) \geq t$, we assume that t of the x_1's are obtained by Lie operations from the ζ_i's and $\bar{\zeta}_j$'s: say $x_{k+1}, \ldots, x_{k+t}$. Because $M \cap B_p \subset N$, we can extend all vector fields over N, preserving their linear independence and Lie relations; a field v is extended to $\tilde{v}$. Then $\tilde{x}_{k+1} \in \mathcal{L}(N)$, $1 \leq j \leq t$. But at most one of the $\tilde{x}_{k+j}$ can be in $\Gamma(H(N) + A(N))$, for it is generated by $\tilde{\zeta}_1, \ldots \ldots, \tilde{\zeta}_k, \tilde{\bar{\zeta}}_1, \ldots, \tilde{\bar{\zeta}}_k$, and one other pair of vector fields, d and $\bar{d}$ (holomorphic vectors fields along the disc structure of N). But only a 1-dimensional combination of red and imd can be tangent to M (by the non-triviality of the discs) so only a 1-dimensional linear combination of the $\tilde{x}_{k+j}$ can be in the vector fields generated by red and imd. (Say it is the field generated by $\tilde{x}_{k+1}$).

Then $\tilde{x}_{k+2}, \ldots, \tilde{x}_{k+t}$ are in $\mathcal{L}(N)$ but not in $\Gamma(H(N) + A(N))$, so ex dim $\mathcal{L}(N) \geq t-1$. #

(We do not know whether ex dim $\mathcal{L}(M) = t$ implies ex dim $\mathcal{L}(N) = t-1$).

2. Example: An informative example to examine is given by the 3-dimensional hypersurface of C^2 defined by the equation $y_2 = |z_1|^2$, where $z_k = x_k + iy_k$. In this simple case a disc whose boundary is on M can be given explicitly (for fixed x_2). Then the boundary is $|z_1|^2 = y_2 > 0$, and the full map f (from R^4 to C^2) is:

$$(x_2, y_2, R, \theta) \rightarrow \begin{cases} z_1 = R\sqrt{y_2}\, e^{i\theta} \\ z_2 = x_2 + iy_2. \end{cases}$$

It is easy to see that F is the type of map satisfying (i)-(v) above. In this case it is even simple to see that F is injective, for its Jacobian is $2y_2 R$.

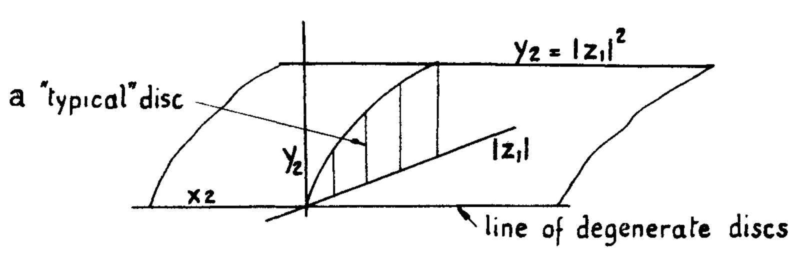

Fig. 3

(N = the union of the discs, the open set below the paraboloid.) This section's work was an $0(3)$ perturbation of this.

V. CONSEQUENCES AND CONJECTURES

A. Principal Results.

The following Kontinuitätssatz is proven by Wells [33].

1. THEOREM: *Let U be a simply connected domain in R^d and let F: $\bar{D} \times U \longrightarrow C^n$ be a continuous family of analytic discs parameterized by U with the property that for some point u_0 in U, $F(D \times \{u_0\})$ is a point (a degenerate disc). Then if f is holomorphic in a domain B containing $F(\partial D \times U)$, there is a domain $A \supset B$ such that A contains $F(\bar{D} \times U)$, and there is a holomorphic function g in A such that $g \mid B = f$.*

As in the work of Wells and Weinstock, these hypotheses ase fulfilled for a suitable U (see *IV*1 and *IV*2), and we obtain the following, which are our main new results:

2. THEOREM: *If M is a generic C-R submanifold of C^n, and $L_1 \neq 0$ then M is extendible to a set containing a C-R submanifold N with* $\dim N = \dim M + 1$.

3. THEOREM: *If M is a generic C-R submanifald of C^n, and $e =$* ex dim $\mathcal{L}(M) > 0$, *then M is extendible to a set containing a generic submanifold N with* $\dim N = \dim M + e$.

Proof: Apply IV1 and 1 inductively. #

We note a maximum principle: the maximum modulus of a function on N is dominated by its maximum modulus on M.

The following result is analagous to the Bochner-Hartogs result that the boundary of a C^∞ relatively compact open subset of C^n must be extendible to the interior of the set.

4. THEOREM: *Let M be a compact non-trivial generic C-R submanifold of C^n (so $\dim_C H(M) > 0$). Then M is extendible to a set containing a submanifold N with* $\dim N = \dim M + 1$.

Proof: If for some open set U of M, ex dim $\mathcal{L}(U) > 0$, then (2) M is extendible to such an N.

So we may assume ex dim $\mathcal{L}(M) = 0$. Then computation shows that the distribution $\Gamma(\mathrm{re}\,(H(M) + A(M)))$ is completely integrable and its maximal integral submanifolds possess the structure of a complex submanifold of C^n. Then the result desired follows from:

5. LEMMA: No compact subset of C^n is the union of the images of non-constant analytic maps from complex submanifolds of $\dim_C > 0$.

Proof: Suppose K is compact, and $K = \bigcup_{\lambda \in L} U_\lambda$, where $f_\lambda : M_\lambda \to U_\lambda$ is a non-constant complex analytic map. There are three proofs now:

1.) Consider the function algebra $D(K)$ given as the uniform closure on K of functions on K which are analytic on each U_λ. Then $D(K)$ is a closed, separating subalgebra of $C(K)$, the continuous complex-valued function on K. By a theorem of Bishop [3], a point p of K is a ***peak point*** for $D(K)$: i. e., $|g(p)| > |g(q)|$ for $q \in K, q \neq p$ and some $g \in D(K)$. But $p \in U_{\lambda_0}$ for some λ_0. So consider now an $f \in H(K)$ close enough to g. Then by the maximum modulus principle, $f(U_{\lambda_0}) = f(p)$. Since f_{λ_0} is non-constant, this is a contradiction, for in the above take any $q \in U_{\lambda_0}$.

2.) Put $t = \inf_\lambda \dim U_\lambda$. Consider the function $z_1 : K \to C$ (the first coordinate in C^n). Since K is compact, $|z_1|$ attains a maximum on K, say when $z_1 = t$.

If $U_{\lambda_0} \cap \{z_1 = t\} \neq \Phi$, then $U_{\lambda_0} \subset \{z_1 = t\}$. This is also a consequence of the maximum principle, for $|z_1|$ must be constant on U_{λ_0}. Then put $K' = \bigcup_{U_{\lambda_0} \subset \{z_1 = t\}} U_{\lambda_0} \subset \{z_1 = t\}$. K' is a compact subset of $z_1 = t$, an affine translate of C^{n-1}.

Continue the argument to obtain K' as a compact subset of C^t. But then K' is the union of open sets in C^t, a contradiction.

3.) The function $\sum_{i=1}^{n} |z_i|^2 = g(z)$ is plurisubharmonic and attains a maximum value t on $p \in K$. If $p \in U_\lambda$, then by the maximum principle for plurisubharmonic functions (Gunning and Rossi [3], IXC 3) used as above, $g^{-1}(t) \supset U_\lambda$. Then $U_\lambda \subset \left\{ \sum_{i=1}^{n} |z_i|^2 = t \right\}$. But hyperspheres are 0-complex (see IIB and IIC 12) hence $\dim U_\lambda = 0$. This is false. #

6. REMARK: If M is compact as in 4, then M is probably extendible to a manifold N with $\partial N \supset M$ and $\dim N = \dim M + 1$. (There is (at least) cobordism obstruction to having $\partial N = M$).

7. THEOREM: *If M is a non-trivial 0-complex generic C-R submanifold of C^n, then M is extendible to a set containing a submanifold N, with* $\dim N = \dim M + 1$.

Proof: Since M is 0-complex, $L_1 \neq 0$ (IIB 10), and so the result by 1. #

B. Conjectures.

If M is real analytic with a real analytic C-R structure, then certain results are known (obtained by considering the complexification of M).

1.) M is locally embeddable as a generic C-R submanifold of C^n (Rossi [26]).

2.) If M is a trivial C-R submanifold of C^n (so dim $H(M)$ is 0), then M is a holomorphically convex subset of C^n.

3.) If M is a C-R submanifold of C^n, every C-R map $f\colon M \to C$ is the restriction of an element of $H(M)$ (see Tomassini [31]).

(As a result of the above, we see in particular that the differential equation of IIIB 12 $i\dfrac{\partial f}{\partial u_1} - \dfrac{\partial f}{\partial u_2} - \dfrac{d(e^{u_1}\sin u_1)}{du_1}\dfrac{\partial f}{\partial u_3} = 0$ has as « solution » all functions analytic in C^{2*}).

If M is compact, Wells [34] has shown that if M is analytic and is a trivial C-R manifold of C^n, then M is an S_δ. See also [21] for phrasing of (2) for C^∞ manifolds.

We conjecture that (1) is true without change for C^∞ manifolds, and that certain statements close to (3) should be true for C^∞ manifolds. We will describe the situation as it is now understood.

We would like to suggest a version of (3) for C^∞. (Let $CR(M)$ be the collection of complex-valued C-R functions on a C-R manifold M).

If M is a compact generic C-R submanifold of C^n, then there is a generic C-R submanifold N with $\dim N = \dim M + 1$, and ∂N contains an open subset of M, and each function $f \in CR(M)$ determines a function $F \in CR(N)$ in the sense that the « boundary values » of F on $\partial N \cap M$ are just f on $\partial N \cap M$.

To attempt to prove this we would recall how we first obtained N in IV as a union of disjoint discs in C^n whose boundaries are in M. So if $v \in N$, v is in a certain disc D_v with $\partial D_v \subset M$. The obvious way to try to extend f is:

$$F(v) = \frac{1}{2\pi i}\int_{\partial D_v} \frac{f(z)}{z - v}\,.$$

Then F so constructed is analytic in the disc direction, but two results remain to be proved:

i) $F \in CR(N)$

ii) F assumes the correct « boundary values » in M.

We were not able to prove either of these, but the following lemma would probably be useful in proving (1):

Lemma: Let Q be an $(n+p)$-dimensional generic submanifold of C^n, $p > 0$. Then $f \in CR(Q)$ iff $\int_Q f\,\bar{\partial}\tau = 0$, for all compactly supported $(n, p-1)$ forms τ in C^n.

Proof: A Stoke's theorem computation, using the fact that $f \in CR(Q)$ only when $\bar{\partial} f = 0$. #

To understand an « approximate » version of (3) suggested by R. Nirenberg, we begin with the description of some function algebras.

If K is a compact subset of C^n, then $H(K)$ is, as before, the collection of functions holomorphic in a neighborhood of K, and $C(K)$ is the uniform algebra of continuous complex-valued functions of K. $A(K)$ is the closed subalgebra of $C(K)$ generated by restrictions of functions in $H(K)$. Finally, let $\mathscr{A}(K)$ be the topological algebra of germs of functions holomorphic about K.

The natural restriction map $\mathscr{A}(K) \xrightarrow{i} A(K)$ is a continuous 1-1 map onto a dense subset of $A(K)$, so that the induced map $\mathcal{S}(\mathscr{A}(K)) \xrightarrow{i^*} \mathcal{S}(A(K))$ of maximal ideal spaces (with the usual topology) is a homeomorphism, for $\mathcal{S}(A(K))$ is compact. (If K is a point in C^n, we are asserting here that the topological rings $C_c[[x_1, \dots, x_n]]$ (convergent power series) and C have isomorphic duals).

If U_j is a neighborhood of K, then there is a Riemann domain $\pi_j: \widehat{U}_j \to C^n$ and $\widehat{U}_j \approx \mathcal{S}(A(U_j))$. But $\mathcal{S}(\mathscr{A}(K)) = \varprojlim \widehat{U}_j$ if U_j is a sequence of neighborhoods decreasing to K, where the inverse limit maps are the natural restrictions. The π_j commute with these maps, so there is induced a continuous projection $\mathcal{S}(A(K) \xrightarrow{\pi} C^n$ by following the isomorphism to $\mathcal{S}(\mathscr{A}(K))$ above by $\varprojlim \pi_j$. (This projection agrees with the one given by Rossi [24] and Wells [35]: if $A(K, C^n)$ is the uniform subalgebra of $C(K)$ generated by restrictions to K of $H(C^n)$, there is an injection $A(K, C^n) \to$ $\to A(K)$, so $\mathcal{S}(A(K)) \to \mathcal{S}(A(K, C^n))$, but $\mathcal{S}(A(K, C^n)) \approx \widehat{K}_{C^n} \subset C^n$. The map from $\mathcal{S}(A(K))$ to C^n obtained agrees with the one defined above). The inverse limit description above has been used by Harvey and Wells [23] to obtain information about sheaf cohomology on holomorphically convex subsets of C^n.

Rename $\mathcal{S}(A(K))$ the *envelope of holomorphy of* K, $E(K)$, and think of it as a limit of Riemann domains over C^n.

$E(K) \xrightarrow{\pi} C^n$. Then $\pi(E(K)) = \widehat{K}$, the *hull of* K; Wells [35] has shown that if K is extendible to K', then $K' \subset \widehat{K}$.

Return to the case of a compact C-R manifold M. Let $D(M)$ be the uniform closure of $CR(M)$. R. Nirenberg's version of (3) for C^∞ is:

If M is a compact C-R submanifold of C^n, $f \in CR(M)$, then there is F holomorphic in a neighborhood of M so that $\|f - F\|_M < \varepsilon$.

If this is true, we can conclude that $A(M) = D(M)$. (When M is a trivial C-R submanifold $(\dim H(M) = 0)$ we would expect $C(M) = A(M)$. This has been shown — R. Nirenberg and Wells [21].)

Then we have the following situation: $E(M) \xrightarrow{\pi} C^n$ with $M \subset \widehat{M} \subset \widehat{M}_{C^n}$. $E(M)$ is the space of maximal ideals of $A(M)$ or $D(M)$ or $\mathscr{A}(M)$. And this result is probably true:

$$\widehat{M} = \bigcup_{U \text{ domain of holo.},\ U \supset M}, \quad \text{and} \quad \widehat{M} = \bigcup_{M \text{ extendible to } N} N.$$

We conjecture: $E(M)$ possesses the structure of a C^∞ space, and so does M. π is a C^∞ space map, and $A(E(M)) = \pi^*(A(C^n))$ (anti-holomorphic tangent bundle) is a « bundle » over $E(M)$, and each $f \in CR(M)$ extends to an $\bar{f} \in CR(E(M))$: that is, on any pure-dimensional part of $E(M)$, $d\bar{f}(A(E(M))) = 0$. (And $\dim E(M) = \dim M + \operatorname{ex}\dim \mathscr{L}(M)$.)

We would hope that (1) would go over without change in the C^∞ case, and this fact would enable us to reduce the local analysis of « nice » systems of first order partial differential equations with complex coefficients to the local study of the inhomogeneous Cauchy-Riemann equations.

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