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ON LOCAL CONVEXITY OF BOUNDED WEAK TOPOLOGIES ON BANACH SPACES

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In this paper we prove that the bw topology on a Banach space E, i.e. the finest topology which agrees with the weak topology on bounded sets of E, is a locally convex topology if and only if the Banach space E is reflexive.

1. Introduction. If E is a Banach space, the bounded weak (bw) topology is the finest topology which agrees with the weak topology on bounded sets. Wheeler in [**7**, p. 251] proves that the bw topology on c_0 is not locally convex. This result gives a counterexample to a remark of Day [**2**, p. 42] which said that the bw topology is locally convex always. This fact suggests a question: Under what conditions on E is it true that bw is a locally convex topology?. The theorem of Banach and Dieudonné (2.2) shows that reflexivity is a sufficient condition. In this paper we obtain that reflexivity is also a necessary condition.

2. Notations, definitions and preliminary results. The notations for topological vector spaces are taken primarily from [**6**], but we employ the definition of polarity found in [**4**].

DEFINITION 2.1. If E is a locally convex space (lcs), the equicontinuous weak* (ew*) topology on E' is the finest topology on E' which coincides with the weak* topology $\sigma(E', E)$ on equicontinuous sets of E'.

The following result characterizes this topology when E is a metrizable lcs.

THEOREM 2.2. (*Banach-Dieudonné.*) *Let E be a metrizable locally convex space and E' its dual. The ew* topology on E' is the topology of the uniform convergence on precompact subsets of E.*

For a demonstration of this theorem we refer the reader to [**4**] or [**6**].

As an immediate consequence of this theorem we have that if E is a metrizable lcs, the ew* topology on E' is locally convex.

Other results about ew* may be found in [**1**] and [**2**].

Definition 2.3. If E is a locally convex space, the bounded weak (bw) topology on E is the finest topology on E which agrees with the weak topology $\sigma(E, E')$ on bounded sets.

This definition is equivalent to Day's [**2**, p. 41]:

"The bw topology is the collection of all subsets U of E satisfying: for each bounded set B of E, there is a $\sigma(E, E')$-open V with $U \cap B = V \cap B$."

Obviously the last definition is not changed if we choose the bounded sets on a fundamental family of bounded sets in E.

As follows from [**1**, p. 265], the bw topology is semi-linear, i.e. addition and scalar multiplication functions are separately continuous. Moreover, if E is a Banach space, it can be shown [**3**, p. 21] that bw is a vectorial topology if and only if it is a locally convex one.

A general result of Collins [**1**, p. 266], which can be extended to the complex case, makes the following definition valid.

Definition 2.4. The convex bw (cbw) topology on a locally convex space E is the unique locally convex topology with a base of all convex neighborhoods of 0 in the bw topology.

It is easy to see that the cbw topology is the finest locally convex topology which agrees with the weak topology on bounded sets. In [**7**, p. 251] may be found the following result which characterizes the cbw topology:

Theorem 2.5. *If E is a lcs, the cbw topology on E is that of uniform convergence on compact subsets of the completion of* $(E', \beta(E', E))$.

As consequence of this result and Theorem 2.2 we obtain:

Corollary 2.6. *If E is a Banach space, the cbw topology is the restriction to E of the ew* topology on E''.*

In particular if E is reflexive we have:

Corollary 2.7. *If E is a reflexive Banach space, the bw topology on E is a locally convex topology.*

A different introduction and other results about this topology may be seen in [**3**].

If E is a Banach space, we denote by B, B'', S the closed unit ball of E and E'' and the unit sphere of E, respectively, and we will write, for each $n \in \mathbf{N}$, $B_n = nB$, $B''_n = nB''$ and $S_n = nS$.

3. The bw topology and local convexity.

LEMMA 3.1. *Let E be a separable, non-reflexive Banach space. If E contains no subspace isomorphic to l^1, there exists a subset A of E which is bw-closed but is not closed in the restriction to E of the ew* topology on E''.*

Proof. It is well known that for each $n \in \mathbf{N}$ the sphere S_n is $\sigma(E'', E')$-dense in the closed ball B''_n. As E is a separable Banach space that contains no subspace isomorphic to l^1 it follows from Rosenthal ([**5**], Theorem 3) that S_n is $\sigma(E'', E')$-sequentially dense in the ball B''_n, i.e. each $z \in B''_n$ can be approximated in $\sigma(E'', E')$ by a sequence contained in S_n. Hence if $\phi \in E'' \setminus E$ and $\|\phi\| = 1$, there exists for each $n \in \mathbf{N}$ a sequence $(x_{k,n})_{k \in \mathbf{N}}$ contained in S_n and converging to $n^{-1}\phi$ in $\sigma(E'', E')$.

We define $A = \{x_{k,n}\colon k, n \in \mathbf{N}\}$. For each $m \in \mathbf{N}$ we have

$$\begin{aligned} A \cap B_m &= \{x_{k,n}\colon k \in \mathbf{N}, n \le m\} \\ &= \left(\{x_{k,n}\colon k \in \mathbf{N}, n \le m\} \cup \{n^{-1}\phi\colon n \le m\}\right) \cap B_m, \end{aligned}$$

and since the set $\{x_{k,n}\colon k \in \mathbf{N}, n \le m\} \cup \{n^{-1}\phi\colon n \le m\}$ is $\sigma(E'', E')$-compact, it is $\sigma(E'', E')$-closed; then the set $A \cap B_m$ is closed in the restriction of $\sigma(E'', E')$ to B_m, but this topology is the same as the restriction of $\sigma(E, E')$ to B_m. This proves that A is bw-closed.

On the other hand, let U be a neighborhood of 0 in the ew* topology; there exists W, ew*-neighborhood of 0 such that $W + W \subset U$, and as W is absorbent, there exists $n_0 \in \mathbf{N}$ such that $n_0^{-1}\phi \in W$. By the definition of ew* topology we know there exists a V, $\sigma(E'', E')$-neighborhood of 0 satisfying

$$W \cap B''_{2n_0} = V \cap B''_{2n_0}.$$

As $(x_{k,n_0})_{k \in \mathbf{N}}$ converges to $n^{-1}\phi$ in the $\sigma(E'', E')$-topology there exists $k_0 \in \mathbf{N}$ such that $x_{k_0,n_0} - n_0^{-1}\phi \in V$ and then

$$x_{k_0,n_0} = x_{k_0,n_0} - n_0^{-1}\phi + n_0^{-1}\phi \in \left(V \cap B''_{2n_0}\right) + W \subset W + W \subset U.$$

This proves that 0 belongs to the closure of A in the ew* topology, and since $0 \in E$, 0 is in the closure of A in the restriction of ew* to E (we denote this topology rew*). Thus A is not closed in rew*. □

PROPOSITION 3.2. *Let E be a separable Banach space that contains no subspace isomorphic to l^1. The bw topology on E is locally convex if and only if E is reflexive.*

Proof. If E is reflexive we saw in (2.7) that bw is a locally convex topology. Conversely, if E is not reflexive, (3.1) and (2.6) prove that the bw topology is not locally convex. □

LEMMA 3.3. *Let E be a Banach space and F a closed linear subspace of E. The bw topology of F is the restriction to F of the bw topology on E.*

Proof. We denote bw(E) and bw(F) the bw topology on E and F respectively. It is clear that the restriction of bw(E) to F is coarser than bw(F).

On the other hand, if U is bw(F)-open, let V be the union of U and $E \setminus F$. It is sufficient to prove that V is bw(E)-open. If B is a bounded subset of E, as $\sigma(F, F')$ coincides with the restriction of $\sigma(E, E')$ to F, there exists W, $\sigma(E, E')$-open, such that

$$U \cap B = U \cap (B \cap F) = (W \cap F) \cap (B \cap F) = (W \cap F) \cap B,$$

and then

$$V \cap B = (W \cup (E \setminus F)) \cap B,$$

and since $W \cup (E \setminus F)$ is $\sigma(E, E')$-open, V is bw(E)-open. □

PROPOSITION 3.4. *Let E be a Banach space that contains no subspace isomorphic to l^1. The bw topology on E is locally convex if and only if E is reflexive.*

Proof. If E is not reflexive, there exists a separable nonreflexive subspace F of E. Obviously F contains no subspace isomorphic to l^1. From (3.2) it follows that the bw topology on F is not locally convex and then (3.3) shows that bw is not a locally convex topology on E. This fact and (2.7) prove the theorem. □

LEMMA 3.5. *There exists a subset A of l^1 which is bw-closed but is not closed in the restriction to l^1 of the ew*-topology of $(l^1)''$.*

Proof. For each $n \in \mathbf{N}$, we denote by e_n the sequence of l^1 $(0, 0, \dots, 1, 0, \dots)$ where the one is in the nth place.

Let A_0 be the set $A_0 = \{e_n\colon n \in \mathbf{N}\}$. For each $n \in \mathbf{N}$, e_n is a $\sigma(l^1, l^\infty)$-isolated point of A_0, and then A_0 is a $\sigma(l^1, l^\infty)$-closed set which is not $\sigma(l^1, l^\infty)$-compact. Consequently A_0 is not $\sigma((l^1)'', l^\infty)$-closed. Thus there exists a linear form ϕ that belongs to the $\sigma((l^1)'', l^\infty)$-closure of A_0 and $\phi \notin A_0$. Obviously $\|\phi\| \leq 1$ and $\phi \in (l^1)''\backslash l^1$.

Now, for each $n \in \mathbf{N}$, we define

$$A_n = \{n(e_p - e_q) + e_k/n\colon p, q, k \in \mathbf{N}, p \neq q, p < k, q < k\}.$$

A_n is contained in the sphere of radius $2n + 1/n$ of l^1 and if $n \in \mathbf{N}$, $n \geq 2$, it is not hard to check that A_n is $\sigma(l^1, l^\infty)$-closed.

Let V be a balanced, convex $\sigma((l^1)'', l^\infty)$-neighborhood of 0. As ϕ is an accumulation point of A_0, $\phi + V/3n$ contains an infinite number of points of A_0. If $e_p, e_q, e_k \in (\phi + V/3n) \cap A_0$ with $p < q < k$, we have

$$\left[n(e_p - e_q) + \frac{1}{n}e_k\right] - \frac{1}{n}\phi = n(e_p - \phi) + n(\phi - e_q) + \frac{1}{n}(e_k - \phi)$$
$$\in \frac{1}{3}V + \frac{1}{3}V + \frac{1}{3n^2}V \subset V.$$

Thus $(\phi/n + V) \cap A_n$ is a nonempty set. This proves that ϕ/n belongs to the $\sigma((l^1)'', l^\infty)$-closure of A_n.

Now, we define $A = \bigcup_{n=2}^{\infty} A_n$. For each $m \in \mathbf{N}$, we have

$$A \cap B_m = \bigcup \{A_n\colon n \in \mathbf{N}, 5 \leq 2n^2 + 1 \leq mn\}.$$

This set is obviously $\sigma(l^1, l^\infty)$-closed, and thus A is a bw-closed set.

On the other hand, as for each $n \in \mathbf{N}$, ϕ/n belongs to the $\sigma((l^1)'', l^\infty)$-closure of A_n; reasoning as in the last part of the proof of Lemma 3.1 proves that 0 belongs to the closure on ew* of A, and as 0 does not belong to A, we see that A is not closed in the topology restriction to l^1 of ew* on $(l^1)''$. □

PROPOSITION 3.6. *Let E be a Banach space that contains a subspace isomorphic to l^1. Then bw is not a locally convex topology on E.*

Proof. From (3.5) we get that the bw topology on l^1 is not locally convex. Hence if E contains a subspace isomorphic to l^1, Lemma 3.3 and the conservation of bw topologies under isomorphisms prove that bw is not a locally convex topology on E. □

THEOREM 3.7. *Let E be a Banach space. bw is a locally convex topology on E if and only if E is reflexive.*

Proof. If E is reflexive, (2.7) gives us the result. Conversely, if bw is a locally convex topology on E, from (3.6) it follows that E contains no subspace isomorphic to l^1 and (3.4) shows that E must be reflexive. □

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