J. Math. Kyoto Univ. (JMKYAZ)
19-2 (1979) 269-284

On local isometric immersions of the spaces of negative constant curvature into the euclidean spaces

By

Eiji KANEDA

(Received Feb. 21, 1978)

Introduction

In his paper [9], T. Otsuki obtained an estimate for the lower bound of the dimensions of the euclidean spaces in which a space of negative constant curvature can be locally isometrically immersed. He proved that any space of negative constant curvature of dimension n cannot be isometrically immersed into the euclidean space $\boldsymbol{R}^{2n-2}$.

The main purpose of this paper is to show the following

Theorem 5.2. *Any space of negative constant curvature of dimension n can be locally isometrically immersed into the euclidean space* $\boldsymbol{R}^{2n-1}$.

We wish to prove this theorem by a method based on the theory of non-linear partial differential equations established by M. Kuranishi [8] and H. Goldschmidt [4].

Let (M, g) be an n-dimensional Riemannian manifold. Let P be the differential equation of isometric immersions of (M, g) into the euclidean space $\boldsymbol{R}^m$ with $m \geqq n$, which is a system of non-linear parital differential equations of order 1. Adjoining to this system P the equation of Gauss-Weingarten which is obtained by differentiating the equation of isometric immersions, we obtain the system $P^{(1)}$ of order 2, the first prolongation of P. Similarly adjoining to the system $P^{(1)}$ the equation which is obtained by differentiating the equation of Gauss-Weingarten, we obtain the system of order 3, the second prolongation of P. A formal solution of order 1 can be always be extended to a formal solution of order 2, while a formal solution of order 2 cannot be necessarily extended to a formal solution of order 3. There exists an obstruction to extending the formal solutions of order 2 to the formal solutions of order 3, which is called the equation of Gauss.

Recently J. Gasqui [3] gave a new proof of the famous theorem of Janet-Cartan, showing that the system Q which is obtained by adjoining the equation of Gauss to the system $P^{(1)}$ forms an involutive system under the assumption $m \geqq \frac{1}{2}n(n+1)$. Another proof of the theorem of Janet-Cartan from a somewhat different viewpoint was delivered by N. Tanaka in his lecture at Kyoto University before

Gasqui's paper was published (cf. [6]). He proved that the symbol $\mathfrak{q}$ of the system Q is isomorphic to the second prolongation $\mathfrak{h}^{(2)}$ of the symbol $\mathfrak{h}$ of the linear operator L which was first introduced by himself (see [10]), and proved that if $m \geqq \frac{1}{2}n(n+1)$, then at each generic point of Q the symbol $\mathfrak{h}$ is involutive. Here we remark that the vanishing of the Spencer cohomology group of the symbol $\mathfrak{h}$ plays an important role in his proof of the theorem of Janet-Cartan.

In this paper we need to investigate the system Q under the assumption $m < \frac{1}{2}n(n+1)$. Unfortunately the symbol $\mathfrak{h}$ cannot be involutive in this case. Hence the method developed by N. Tanaka cannot be applied to our problem. However, by letting (M, g) be a space of negative constant curvature, we can prove the following

Theorem 5.1. *If $m=2n-1$, then there exists an open fibered submanifold $\pi_1^2\colon P_\#^{(1)} \to P$ of the vector bundle $\pi_1^2\colon P^{(1)} \to P$ such that the intersection $Q_\# = Q \cap P^{(1)}$ forms an involutive differential equation.*

Theorem 5.2 cited above now follows from Theorem 5.1.

Following the formulations given by N. Tanaka, we recall in § 1 the differential equations $P, P^{(1)}$, etc. In § 2, we define the formal Gaussian variety with respect to a curvature like tensor. § 3 and § 4 are devoted to the proof of Theorem 3.1 that describes the properties of the formal Gaussian variety with respect to a curvature like tensor of negative constant curvature. Finally in § 5 we prove Theorem 5.1.

The author would like to express his gratitude to Professor N. Tanaka for his kind advices and constant encouragements.

§ 1. The differential equations associated with isometric immersions

Let M be an n-dimensional differentiable manifold(*) and let $T=T(M)$ (resp. $T^*=T^*(M)$) be the tangent (resp. cotangent) bundle of M. By $\otimes^k T^*$ (resp. $S^k T^*$), we mean the bundle of k-tensors (resp. symmetric k-tensors) on M. Let g be a Riemannian metric on M. We denote by ∇ (resp. R) the covariant differentiation (resp. the curvature tensor) associated with the Riemannian connectoin on M induced from g.

Let $\boldsymbol{R}^m$ be the m-dimensional euclidean space with $m \geqq n$ and $\langle\ ,\ \rangle$ be the standard inner product of $\boldsymbol{R}^m$. We denote by $\bar{g}$ the cannonical Riemannian metric of $\boldsymbol{R}^m$ induced from $\langle\ ,\ \rangle$.

By definition an isometric immersion f of the Riemannian manifold (M, g) into the euclidean space $\boldsymbol{R}^m$ is an immersion of M into $\boldsymbol{R}^m$ which is a solution of the equation

$$f^*\bar{g}=g,$$

where f^*g stands for the Riemannian metric on M induced from g by f.

Let f be an isometric immersion of (M, g) into $\boldsymbol{R}^m$. Then at each $p \in M$, we have the following equalities (cf. Proposition 2 of Appendix in [6]):

(*) Throughout this paper we shall assume the differentiability of class C^∞.

$$\langle \nabla_x f, \nabla_y f \rangle = g(x, y), \tag{1.1}$$

$$\langle \nabla_z \nabla_x f, \nabla_y f \rangle = 0, \tag{1.2}$$

$$\langle \nabla_u \nabla_z \nabla_x f, \nabla_y f \rangle + \langle \nabla_z \nabla_x f, \nabla_u \nabla_y f \rangle = 0, \tag{1.3}$$

$$\langle \nabla_z \nabla_x f, \nabla_u \nabla_y f \rangle - \langle \nabla_u \nabla_x f, \nabla_z \nabla_y f \rangle = -g(R(z, u)x, y), \tag{1.4}$$

$$\begin{aligned} &\langle \nabla_v \nabla_z \nabla_x f, \nabla_u \nabla_y f \rangle + \langle \nabla_z \nabla_x f, \nabla_v \nabla_u \nabla_y f \rangle - \langle \nabla_v \nabla_u \nabla_x f, \nabla_z \nabla_y f \rangle \\ &\quad - \langle \nabla_u \nabla_x f, \nabla_v \nabla_z \nabla_y f \rangle = -g(\nabla_v R(z, u)x, y) \qquad \text{for } x, y, z, u, v \in T_p. \end{aligned} \tag{1.5}$$

Classically the equation (1.2) (resp. (1.4)) is called the equation of *Gauss-Weingarten* (resp. the equation of *Gauss*).

Let $J^k(M, m)$ be the vector bundle of all k-jets of local differentiable maps of M into $\boldsymbol{R}^m$. By π^k_{k-1} we mean the canonical projection of $J^k(M, m)$ onto $J^{k-1}(M, m)$ and by π^k_{-1} the source map of $J^k(M, m)$ onto M. As usual the set of formal solutions of order k are represented by a subvariety of $J^k(M, m)$. We denote by P the subvariety of $J^1(M, m)$ composed of all 1-jets satisfying the equation (1.1) and by $P^{(1)}$ the subvariety of $J^2(M, m)$ composed of all 2-jets satisfying the system of equations (1.1) and (1.2). We also denote by Q the subvariety of $J^2(M, m)$ composed of all 2-jets satisfying the system of equations (1.1), (1.2) and (1.4) and by $Q^{(1)}$ the subvariety of $J^3(M, m)$ composed of all 3-jets satisfying the system of equations (1.1)$\sim$(1.5). Note that $P^{(1)}$ (resp. $Q^{(1)}$) is the first prolongation of P (resp. Q) in the usual sense.

We now give the explicit expressions of the varieties P, $P^{(1)}$, Q and $Q^{(1)}$.

Let $\otimes^k T^* \otimes \boldsymbol{R}^m$ be the vector bundle of all $\boldsymbol{R}^m$-valued k-tensors on M. Let us set $T^k(M, m) = \sum_{i=0}^k \otimes^i T^* \otimes \boldsymbol{R}^m$. We shall represent every element $\omega \in T^k(M, m)$ by $\omega = (p; \omega_0, \omega_1, \cdots, \omega_k)$, where p is the origin of ω and $\omega_i \in \otimes^i T_p^* \otimes \boldsymbol{R}^m$ $(i = 0, 1, \cdots, k)$. As in Appendix in [6], we shall consider the vector bundle $J^k(M, m)$ as a subbundle of the vector bundle $T^k(M, m)$. We have $J^0(M, m) = T^0(M, m)$ and $J^1(M, m) = T^1(M, m)$. The bundles $J^2(M, m)$ and $J^3(M, m)$ can be characterized as follows: $J^2(M, m)$ consists of all $(p; \omega_0, \omega_1, \omega_2) \in T^2(M, m)$ such that

$$\omega_2(x, y) = \omega_2(y, x) \qquad \text{for any } x, y \in T_p.$$

$J^3(M, m)$ consists of all $(p; \omega_0, \omega_1, \omega_2, \omega_3) \in T^3(M, m)$ such that

$$\begin{aligned} &(p; \omega_0, \omega_1, \omega_2) \in J^2(M, m); \\ &\omega_3(x, y, z) = \omega_3(y, x, z) - \omega_1(R(x, y)z), \\ &\omega_3(x, y, z) = \omega_3(x, z, y) \qquad \text{for any } x, y, z \in T_p. \end{aligned}$$

These being prepared, we give the explicit expressions of the varieties P, $P^{(1)}$, Q and $Q^{(1)}$.

The variety P is composed of all $(p; \omega_0, \omega_1) \in J^1(M, m)$ satisfying

$$\langle \omega_1(x), \omega_1(y) \rangle = g(x, y) \qquad \text{for any } x, y \in T_p. \tag{1.6}$$

It is easily observed that P is a fibered submanifold of $\pi^1_{-1}\colon J^1(M, m)\to M$. Let $\alpha=(p;\omega_0,\omega_1)\in P$. We mean by N_α the orthogonal complement of the subspace $\omega_1(T_p)$ in $\boldsymbol{R}^m$. Then the union $N=\bigcup_{\alpha\in P}N_\alpha$ forms a vector bundle over P.

The variety $P^{(1)}$ is composed of all $(P;\omega_0,\omega_1,\omega_2)\in J^2(M, m)$ satisfying $(p;\omega_0,\omega_1)\in P$ and

$$\langle\omega_2(z, x), \omega_1(y)\rangle=0 \qquad \text{for any } x, y, z\in T_p. \tag{1.7}$$

We can easily see that $P^{(1)}$ forms a vector bundle over P which is isomorphic to the vector bundle $S^2T^*\otimes_P N$.

Analogously the variety Q consists of all $(p;\omega_0,\omega_1,\omega_2)\in P^{(1)}$ satisfying

$$\langle\omega_2(z, x), \omega_2(u, y)\rangle-\langle\omega_2(u, x), \omega_2(z, y)\rangle=-g(R(z, u)x, y) \qquad \text{for any } x, y, z, u\in T_p, \tag{1.8}$$

and the variety $Q^{(1)}$ is composed of all $(p;\omega_0,\omega_1,\omega_2,\omega_3)\in J^3(M, m)$ satisfying $(p;\omega_0,\omega_1,\omega_2)\in Q$ and

$$\langle\omega_3(u, z, x), \omega_1(y)\rangle+\langle\omega_2(z, x), \omega_2(u, y)\rangle=0, \tag{1.9}$$

$$\begin{aligned}&\langle\omega_3(v, z, x), \omega_2(u, y)\rangle+\langle\omega_2(z, x), \omega_3(v, u, y)\rangle-\langle\omega_3(v, u, x), \omega_2(z, y)\rangle\\ &\quad-\langle\omega_2(u, x), \omega_3(v, z, y)\rangle=-g(\nabla_vR(z, u)x, y) \qquad \text{for any } x, y, z, u, v\in T_p.\end{aligned} \tag{1.10}$$

In the subsequent sections we shall mainly concerned with the variety Q. In connection with the variety Q we make some definitions.

Let $\beta=(p;\omega_0,\omega_1,\omega_2)\in Q$. By definition the *symbol* $\mathfrak{q}_\beta$ of the variety Q at β is the subspace of $S^2T_p^*\otimes N_\alpha$ consisting of all $\xi\in S^2T_p^*\otimes N_\alpha$ such that

$$\begin{aligned}&\langle\xi(z, x), \omega_2(u, y)\rangle+\langle\omega_2(z, x), \xi(u, y)\rangle-\langle\xi(u, x), \omega_2(z, y)\rangle\\ &\quad-\langle\omega_2(u, x), \xi(z, y)\rangle=0 \qquad \text{for any } x, y, z, u\in T_p,\end{aligned} \tag{1.11}$$

where we set $\alpha=\pi^2_1(\beta)$. We also denote by $\mathfrak{q}^{(1)}_\beta$ the *first prolongation* of the symbol $\mathfrak{q}_\beta$, i.e., $\mathfrak{q}^{(1)}_\beta=T^*\otimes\mathfrak{q}_\beta\cap S^3T^*_p\otimes N_\alpha$.

§ 2. Formal Gaussian varieties

Let $\boldsymbol{T}$ be a finite dimensional real vector space and $\boldsymbol{T}^*$ be the dual vector space of $\boldsymbol{T}$. By $\otimes^k\boldsymbol{T}^*$(resp. $S^k\boldsymbol{T}^*$) we mean the vector space of covariant k-tensors (resp. symmetric k-tensors) of $\boldsymbol{T}$.

By definition an element $C\in\otimes^4\boldsymbol{T}^*$ is called *curvature like* if it satisfies the following:

$$\begin{aligned}&C(x, y, z, w)=-C(y, x, z, w)=-C(x, y, w, z),\\ &C(x, y, z, w)+C(y, z, x, w)+C(z, x, y, w)=0 \qquad \text{for } x, y, z, w\in\boldsymbol{T}.\end{aligned}$$

We denote by $K(\boldsymbol{T})$ the vector space of all curvature like tensors.

Let N be another finite dimensional real vector space with an inner product $\langle\ ,\ \rangle$. For each $\alpha \in S^2T^*\otimes N$, we denote by $\Omega(\alpha)$ the element in $K(T)$ defined by

$$\Omega(\alpha)(x, y, z, w)=\langle\alpha(x, z), \alpha(y, w)\rangle-\langle\alpha(x, w), \alpha(y, z)\rangle \qquad \text{for } x, y, z, w \in T.$$

Let C be any element in $K(T)$. By $\mathscr{G}(C)$ we mean the inverse image of C by the map $S^2T^*\otimes N \ni \alpha\to\Omega(\alpha) \in K(T)$. $\mathscr{G}(C)$ is called the *formal Gaussian variety* with respect to C.

Let $\alpha \in \mathscr{G}(C)$. Define a linear map $\Omega_{*\alpha}: S^2T^*\otimes N\to K(T)$ by setting

$$\begin{aligned}\Omega_{*\alpha}(\beta)(x, y, z, w)=&\langle\beta(x, z), \alpha(y, w)\rangle+\langle\alpha(x, z), \beta(y, w)\rangle\\ &-\langle\beta(x, w), \alpha(y, z)\rangle-\langle\alpha(x, w), \beta(y, z)\rangle\\ &\qquad \text{for } \beta \in S^2T^*\otimes N,\ x, y, z, w \in T.\end{aligned}$$

We denote by $\mathfrak{g}_\alpha$ the kernel of the map Ω_α, which may be identified with the tangent space to the variety $\mathscr{G}(\Omega(\alpha))$ at α. We also denote by $\mathfrak{g}_\alpha^{(1)}$ the first prolongation of the subspace $\mathfrak{g}_\alpha$ of $S^2T^*\otimes N$, i.e., $\mathfrak{g}_\alpha^{(1)}=T^*\otimes\mathfrak{g}_\alpha\cap S^3T^*\otimes N$.

Let $\langle\ ,\ \rangle$ be an inner product of T. For each $k \in R$ let us define $C_k \in K(T)$ by

$$C_k(x, y, z, w)=k(\langle x, z\rangle\langle y, w\rangle-\langle x, w\rangle\langle y, z\rangle) \qquad \text{for } x, y, z, w \in T.$$

We say that C_k is the curvature like tensor of constant curvature with sectional curvature k.

Let $O(T)$ (resp. $O(N)$) be the orthogonal group of T (resp. N) with respect to the given inner product $\langle\ ,\ \rangle$ of T (resp. N) and $\mathfrak{o}(T)$ (resp. $\mathfrak{o}(N)$) be the Lie algebra of $O(T)$ (resp. $O(N)$).

Let $a \in O(N)$, $t \in O(T)$, $\chi \in S^lT^*\otimes N$ and $C \in K(T)$. Let us define $a\chi^t \in S^lT^*\otimes N$ and $C^t \in K(T)$ by setting

$$\begin{aligned}&a\chi^t(x_1, \cdots, x_l)=a(\chi(tx_1, \cdots, tx_l))\\ &C^t(x_1, x_2, x_3, x_4)=C(tx_1, tx_2, tx_3, tx_4) \qquad \text{for } x_1, \cdots, x_4, \cdots, x_l \in T.\end{aligned}$$

It is straightforward to see that if $\alpha \in \mathscr{G}(C)$, then $a\alpha^t \in \mathscr{G}(C^t)$. Similarly if $\beta \in \mathfrak{g}_\alpha$ (resp. $\gamma \in \mathfrak{g}_\alpha^{(1)}$), then $a\beta^t \in \mathfrak{g}_{a\alpha^t}$ (resp. $a\gamma^t \in \mathfrak{g}_{a\alpha^t}^{(1)}$).

Let us consider the case $C=C_k$. Since $C_k^t=C_k$ holds for any $t \in O(T)$, we have

Proposition 2.1. *The formal Gaussian variety $\mathscr{G}(C_k)$ is invariant under the action of the product group $O(N)\times O(T)$ on $S^2T^*\otimes N$ defined by*

$$O(N)\times O(T)\times S^2T^*\otimes N \ni ((a, t), \alpha)\to a\alpha^t \in S^2T^*\otimes N.$$

Proposition 2.1 is useful in the consideration of the formal Gaussian variety $\mathscr{G}(C_k)$.

§ 3. The formal Gaussian variety $\mathscr{G}(C_k)$ with $k<0$

In this and the next sections we shall investigate the formal Gaussian variety

$\mathscr{G}(C_k)$ with $k<0$. Our main aim is to show

Theorem 3.1. *Assume that* $\dim N=\dim T-1=n$ *and* $k<0$. *Then there exists an open set* O *in* $S^2T^*\otimes N$ *such that*:

(1) $\mathscr{G}(C_k)\cap O$ *is an* $n(n+1)$*-dimensional submanifold of* $S^2T^*\otimes N$.

(2) *For each* $\alpha\in\mathscr{G}(C_k)\cap O$, *there exists a vector* $e\in T$ *such that*

$$e \rfloor \beta\neq 0 \qquad \text{for any } \beta\in\mathfrak{g}_\alpha,\ \beta\neq 0.$$

(3) $\dim \mathfrak{g}_\alpha^{(1)}=n(n+1)$ *for any* $\alpha\in\mathscr{G}(C_k)\cap O$.

Remark. In order to prove the theorem we have only to prove it in the case where $k=-1$. In fact consider the linear endomorphism of $S^2T^*\otimes N$ given by $S^2T^*\otimes N\ni\alpha\to\sqrt{-k}\alpha\in S^2T^*\otimes N$. It is clear that this endomorphism maps $\mathscr{G}(C_{-1})$ onto $\mathscr{G}(C_k)$. Moreover we have $\mathfrak{g}_\alpha=\mathfrak{g}_{\sqrt{-k}\alpha}$ and hence $\mathfrak{g}_\alpha^{(1)}=\mathfrak{g}_{\sqrt{-k}\alpha}^{(1)}$.

In the following we shall simply write $\mathscr{G}$ instead of $\mathscr{G}(C_{-1})$.

Let $\{e_a\}_{0\leqq a\leqq n}$ (resp. $\{v_i\}_{1\leqq i\leqq n}$) be an orthonormal basis of T (resp. N). Making use of these basis, we shall express N-valued covariant tensors of T in terms of their coefficients. Let $\chi\in\otimes^l T^*\otimes N$. Define an element $X=(X^k_{a_1,\cdots,a_l})_{\substack{0\leqq a_1,\cdots,a_l\leqq n\\ 1\leqq k\leqq n}}\in R^{n(n+1)^l}$ by

$$X^k_{a_1,\cdots,a_l}=\langle\chi(e_{a_1},\cdots,e_{a_l}),v_k\rangle \qquad \text{for}\quad 0\leqq a_1,\cdots,a_l\leqq n,\ 1\leqq k\leqq n.$$

Let $\alpha\in\mathscr{G}$ and let $A=(A^k_{ab})_{\substack{0\leqq a,b\leqq n\\ 1\leqq k\leqq n}}\in R^{n(n+1)^2}$ be the coefficients of α. Then we have

$$A_{ab}{}^k=A_{ba}{}^k, \tag{3.1}$$

$$\sum_{p=1}^n (A_{ac}{}^pA_{bd}{}^p-A_{ad}{}^pA_{bc}{}^p)=-(\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc}) \quad \text{for}\quad 0\leqq a,b,c,d\leqq n,\ 1\leqq k\leqq n, \tag{3.2}$$

where δ means the Kronecker's delta. Conversely, it is clear that any $\alpha\in\otimes^2T^*\otimes N$ whose coefficients $A=(A_{ab}{}^k)_{\substack{0\leqq n,b\leqq n\\ 1\leqq k\leqq n}}$ satisfy (3.1) and (3.2) is contained in $\mathscr{G}$.

Let $\alpha\in\mathscr{G}$. Assume that α satisfies the following

$$\alpha(e_0,e_0)=0,\quad \alpha(e_0,e_i)=v_i \qquad \text{for}\quad 1\leqq i\leqq n. \tag{$\sharp$}$$

Then we have $A_{00}{}^k=0$, $A_{0i}{}^k=\delta_{ik}$ for $1\leqq i,k\leqq n$. Hence by (3.1) and (3.2) we obtain

$$A_{ij}{}^k=A_{ji}{}^k=A_{ik}{}^j, \tag{3.3}$$

$$\sum_{p=1}^n (A_{ik}{}^pA_{jlp}-A_{il}{}^pA_{jkp})=-(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}) \quad \text{for}\quad 1\leqq i,j,k,l\leqq n. \tag{3.4}$$

Let us denote by $\mathring{\mathscr{G}}$ the set of all α satisfying ($\sharp$) and by $\overline{\mathscr{G}}_n$ the subvariety of R^{n^3} of all $A=(A_{ij}{}^k)_{1\leqq i,j,k\leqq n}\in R^{n^3}$ satisfying the system of equations (3.3) and (3.4). Clearly we can identify $\mathring{\mathscr{G}}$ and $\overline{\mathscr{G}}_n$ in a natural way.

Let $A=(A_{ij}{}^k)_{1\leq i,j,k\leq n}\in \bar{\mathcal{G}}_n$. By $(\bar{\mathfrak{g}}_n)_A$, 'we mean the vector space of all $B=(B_{ij}{}^k)_{1\leq i,j,k\leq n}\in \boldsymbol{R}^{n^3}$ such that

$$(3.5)\qquad B_{ij}{}^k=B_{ji}{}^k=B_{ik}{}^j,$$

$$(3.6)\qquad \sum_{p=1}^{n}(B_{ik}{}^pA_{jl}{}^p+A_{ik}{}^pB_{jl}{}^p-B_{il}{}^pA_{jk}{}^p-A_{il}{}^pB_{jk}{}^p)=0 \quad \text{for}\quad 1\leqq i,j,k,l\leqq n.$$

Naturally $(\bar{\mathfrak{g}}_n)_A$ may be idetified with the tangent space to the subvariety $\bar{\mathcal{G}}_n$ at A. We also denote by $(\bar{\mathfrak{g}}_n^{(1)})_A$ the vector space of all $C=({}^lC_{ij}{}^k)_{1\leq i,j,k,l\leq n}\in \boldsymbol{R}^{n^4}$ such that

$$(3.7)\qquad {}^lC_{ij}{}^k={}^iC_{lj}{}^k$$

$$(3.8)\qquad {}^lC_{ij}{}^k={}^lC_{ji}{}^k={}^lC_{ik}{}^j,$$

$$(3.9)\qquad \sum_{p=1}^{n}({}^qC_{ik}{}^pA_{jl}{}^p+A_{ik}{}^p\,{}^qC_{jl}{}^p-{}^qC_{il}{}^pA_{jk}{}^p-A_{il}{}^p\,{}^qC_{jk}{}^p)=0 \quad \text{for}\quad 1\leqq i,j,k,l\leqq n.$$

We prove

Proposition 3.2. *There exists an open set U of $\boldsymbol{R}^{n^3}$ such that:*

(1) *$\bar{\mathcal{G}}_n\cap U$ is a $\frac{1}{2}n(n+1)$-dimensional submanifold of $\boldsymbol{R}^{n^3}$.*

(2) *For each $A\in\bar{\mathcal{G}}_n\cap U$, there exists a vector $x=(x_1,\cdots,x_n)\in\boldsymbol{R}^n$ having the following property:*

If $B=(B_{ij}{}^k)_{1\leq i,j,k\leq n}\in(\bar{\mathfrak{g}}_n)_A$ and $B\neq 0$, then

$$\sum_{p=1}^{n}x_kB_{ij}{}^k\neq 0 \qquad \textit{for some } (i,j)\ (1\leqq i,j\leqq n).$$

(3) $\dim(\bar{\mathfrak{g}}_n^{(1)})_A=\frac{1}{2}n(n+1)$ *for any $A\in\bar{\mathcal{G}}_n\cap U$.*

Before proceeding to the proof of Proposition 3.2, we first note the following. Let $a=(a_i{}^j)_{1\leq i,j\leq n}$ be an orthogonal matrix, i.e., $\sum_{k=1}^n a_k{}^ia_k{}^j=\delta_{ij}$ and let $X=(X_{ij}{}^k)_{1\leq i,j,k\leq n}\in\boldsymbol{R}^{n^3}$ and $Y=({}^lY_{ij}{}^k)_{1\leq i,j,k,l\leq n}\in\boldsymbol{R}^{n^4}$. Define $X^a=((X^a)_{ij}{}^k)_{1\leq i,j,k\leq n}\in\boldsymbol{R}^{n^3}$ and $Y^a=({}^l(Y^a)_{ij}{}^k)_{1\leq i,j,k,l\leq n}\in\boldsymbol{R}^{n^4}$ by setting

$$(X^a)_{ij}{}^k=\sum_{p,q,r=1}^{n}a_i{}^pa_j{}^qa_k{}^rX_{pq}{}^r,$$

$${}^l(Y^a)_{ij}{}^k=\sum_{p,q,r,s=1}^{n}a_l{}^sa_i{}^pa_j{}^qa_k{}^r\,{}^sY_{pq}{}^r \qquad \text{for}\quad 1\leqq i,j,k,l\leqq n.$$

By simple calculations we know that if $A\in\bar{\mathcal{G}}_n$ then $A^a\in\bar{\mathcal{G}}_n$ and that if $B\in(\bar{\mathfrak{g}}_n)_A$ (resp. $C\in(\bar{\mathfrak{g}}_n^{(1)})_A$), then $B^a\in(\bar{\mathfrak{g}}_n)_{A^a}$ (resp. $C^a\in(\bar{\mathfrak{g}}_n^{(1)})_{A^a}$).

From now on let us assume that $n\geqq 2$ and $\bar{\mathcal{G}}_{n-1}\neq\phi$. Let $\lambda\in\boldsymbol{R}_*=\boldsymbol{R}-\{0\}$ and let $\hat{A}=(\hat{A}_{ij}{}^k)\in\bar{\mathcal{G}}_{n-1}$. Define an element $\varphi(\lambda,\hat{A})=(\varphi(\lambda,\hat{A})_{ij}{}^k)_{1\leq i,j,k\leq n}\in\boldsymbol{R}^{n^3}$ by

$$\varphi(\lambda,\hat{A})_{ij}{}^k=\sqrt{1+\lambda^2}\hat{A}_{ij}{}^k,$$

$$(*)\qquad \varphi(\lambda,\hat{A})_{ij}{}^{n}=\varphi(\lambda,\hat{A})_{in}{}^{j}=\varphi(\lambda,\hat{A})_{ni}{}^{j}=\lambda\delta_{ij},$$

$$\varphi(\lambda,\hat{A})_{nn}{}^{i}=\varphi(\lambda,\hat{A})_{ni}{}^{n}=\varphi(\lambda,\hat{A})_{in}{}^{n}=0,\qquad \varphi(\lambda,\hat{A})_{nn}{}^{n}=\lambda-1/\lambda$$
$$\text{for}\quad 1\leqq i,j,k\leqq n-1.$$

It is easily observed that $\varphi(\lambda,\hat{A})\in\bar{\mathscr{G}}_n$. We now define a map Φ: $\boldsymbol{R}_*\times\bar{\mathscr{G}}_{n-1}\times\boldsymbol{R}^{n-1}\to\boldsymbol{R}^{n^3}$ by

$$\Phi(\lambda,\hat{A},\xi)=\varphi(\lambda,\hat{A})^{\exp\tilde{\xi}}\qquad\text{for } (\lambda,\hat{A},\xi)\in\boldsymbol{R}_*\times\bar{\mathscr{G}}_{n-1}\times\boldsymbol{R}^{n-1},$$

where, for $\xi=(\xi_1,\cdots,\xi_{n-1})\in\boldsymbol{R}^{n-1}$, we mean by $\tilde{\xi}$ the skew symmetric matrix of the form:

$$\tilde{\xi}=\begin{pmatrix} & & -\xi_1\\ & 0 & \vdots\\ & & -\xi_{n-1}\\ \xi_1 & \cdots\ \xi_{n-1} & 0\end{pmatrix}.$$

Since $\varphi(\lambda,\hat{A})\in\bar{\mathscr{G}}_n$, we have $\Phi(\lambda,\hat{A},\xi)\in\bar{\mathscr{G}}_n$. Let $(\lambda,\hat{A})\in\boldsymbol{R}_*\times\bar{\mathscr{G}}_{n-1}$. We denote by $\Phi_*=\Phi_{*(\lambda,\hat{A},0)}$ the differential of the map Φ at $(\lambda,\hat{A},0)$, where 0 means the zero vector $(0,\cdots,0)$ in $\boldsymbol{R}^{n-1}$. In a natural way, the tangent space the variety $\boldsymbol{R}_*\times\bar{\mathscr{G}}_{n-1}\times\boldsymbol{R}^{n-1}$ at $(\lambda,\hat{A},0)$ may be identified with the direct sum $\boldsymbol{R}\oplus(\bar{\mathfrak{g}}_{n-1})_{\hat{A}}\oplus\boldsymbol{R}^{n-1}$. Hence Φ_* induces a linear map of $\boldsymbol{R}\oplus(\bar{\mathfrak{g}}_{n-1})_{\hat{A}}\oplus\boldsymbol{R}^{n-1}$ into $\boldsymbol{R}^{n^3}$. We also denote it by Φ_*.

Let $\mu\in\boldsymbol{R}$, $\hat{B}=(\hat{B}_{ij}{}^{k})_{1\leq i,j,k\leq n-1}\in(\bar{\mathfrak{g}}_{n-1})_{\hat{A}}$ and $\xi=(\xi_1,\cdots,\xi_{n-1})\in\boldsymbol{R}^{n-1}$. We note that the vectors $\Phi_*(\mu)$, $\Phi_*(B)$ and $\Phi_*(\xi)$ in $\boldsymbol{R}^{n^3}$ are necessarily contained in $(\bar{\mathfrak{g}}_n)_{\varphi(\lambda,\hat{A})}$. By using the coefficients they are explicitly represented as follows:

$$\begin{aligned}
&\Phi_*(\mu)_{ij}{}^{k}=(\lambda/\sqrt{1+\lambda^2})\mu\hat{A}_{ij}{}^{k},\ \Phi_*(\mu)_{ij}{}^{n}=\mu\delta_{ij},\\
&\Phi_*(\mu)_{in}{}^{n}=0,\ \Phi_*(\mu)_{nn}{}^{n}=(1/\lambda^2)(1+\lambda^2)\mu;\\
&\Phi_*(\hat{B})_{ij}{}^{k}=\sqrt{1+\lambda^2}\hat{B}_{ij}{}^{k},\\
(**)\qquad&\Phi_*(\hat{B})_{ij}{}^{n}=\Phi_*(\hat{B})_{in}{}^{n}=\Phi_*(\hat{B})_{nn}{}^{n}=0;\\
&\Phi_*(\xi)_{ij}{}^{k}=\lambda(\xi_i\delta_{jk}+\xi_j\delta_{ki}+\xi_k\delta_{ij}),\\
&\Phi_*(\xi)_{ij}{}^{n}=-\sqrt{1+\lambda^2}\,(\textstyle\sum_{p=1}^{n}\xi_p\hat{A}_{ij}{}^{p}),\\
&\Phi_*(\xi)_{in}{}^{n}=-(1/\lambda)(1+\lambda^2)\xi_i,\ \Phi_*(\xi)_{nn}{}^{n}=0\qquad\text{for}\quad 1\leqq i,j,k\leqq n-1.
\end{aligned}$$

Lemma 3.3. *For each* $B\in(\bar{\mathfrak{g}}_n)_{\varphi(\lambda,\hat{A})}$, *there exists a unique* $(\mu,\hat{B},\xi)\in\boldsymbol{R}\oplus(\bar{\mathfrak{g}}_{n-1})_{\hat{A}}\oplus\boldsymbol{R}^{n-1}$ *such that*

$$B=\Phi_*(\mu)+\Phi_*(\hat{B})+\Phi_*(\xi).$$

Proof. We first suppose that $\Phi_*(\mu)+\Phi_*(\hat{B})+\Phi_*(\xi)=0$ for some $(\mu,\hat{B},\xi)\in\boldsymbol{R}\oplus(\bar{\mathfrak{g}}_{n-1})_{\hat{A}}\oplus\boldsymbol{R}^{n+1}$. Then by $(**)$ we can easily obtain $\mu=\hat{B}=\xi=0$. This implies the uniqueness. Next we show the decomposition. Let $B=(B_{ij}{}^{k})_{1\leq i,j,k\leq n}\in(\bar{\mathfrak{g}}_n)_{\varphi(\lambda,\hat{A})}$. Take $\mu\in\boldsymbol{R}$ and $\xi=(\xi_1,\cdots,\xi_{n-1})\in\boldsymbol{R}^{n-1}$ so that $B_{nn}{}^{n}=(1/\lambda^2)(1+\lambda^2)\mu$, $B_{in}{}^{n}=-(1/\lambda)(1+\lambda^2)\xi_i$ for $1\leqq i\leqq n$. We set $\bar{B}=B-(\Phi_*(\mu)+\Phi_*(\xi))$. Then $\bar{B}\in(\bar{\mathfrak{g}}_n)_{\varphi(\lambda,\hat{A})}$, $\bar{B}_{nn}{}^{n}=0$ and $\bar{B}_{in}{}^{n}$

$=0$ for $1\leqq i\leqq n$. Substituting $B=\bar{B}$ into (3.6), we have $\bar{B}_{ij}{}^{n}=0$ for $1\leqq i,j\leqq n-1$ and

$$\sum_{p=1}^{n}(\bar{B}_{ik}{}^{p}\hat{A}_{jl}{}^{p}+\hat{A}_{ik}{}^{p}\bar{B}_{jl}{}^{p}-\bar{B}_{il}{}^{p}\hat{A}_{jk}{}^{p}-\hat{A}_{il}{}^{p}\bar{B}_{jk}{}^{p})=0 \qquad \text{for} \quad 1\leqq i,j,k,l\leqq n-1.$$

Thus if we set $\hat{B}_{ij}{}^{k}=(1/\sqrt{1+\lambda^2})\bar{B}_{ij}{}^{k}$ for $1\leqq i,j,k\leqq n-1$, then $\hat{B}=(\hat{B}_{ij}{}^{k})_{1\leqq i,j,k\leqq n-1}\in(\mathfrak{g}_{n-1})_{\hat{A}}$ and $\bar{B}=\Phi_*(\hat{B})$. Hence we have $B=\Phi_*(\mu)+\Phi_*(\hat{B})+\Phi_*(\xi)$. Q.E.D.

Proof of Proposition 3.2. We proceed by induction on n. It is easy to see that the proposition holds for $n=1$. We now assume that the proposition holds $n-1$ with $n\geqq 2$. Then there exists an open set $\hat{U}$ in $\boldsymbol{R}^{(n-1)^3}$ such that:

(1′) $\bar{\mathscr{G}}_{n-1}\cap\hat{U}$ is a $\frac{1}{2}n(n-1)$-dimensional submanifold of $\boldsymbol{R}^{(n-1)^3}$.

(2′) For each $\hat{A}\in\bar{\mathscr{G}}_{n-1}\cap\hat{U}$, there exists a vector $x=(x_1,\cdots,x_{n-1})\in\boldsymbol{R}^{n-1}$ having the following property:

If $\hat{B}=(\hat{B}_{ij}{}^{k})_{1\leqq i,j,k\leqq n-1}\in(\mathfrak{g}_{n-1})_{\hat{A}}$ and $B\neq 0$, then

$$\sum_{p=1}^{n-1}x_k B_{ij}{}^{k}\neq 0 \qquad \text{for some } (i,j)\ (1\leqq i,j\leqq n-1).$$

(3′) $\dim(\mathfrak{g}_{n-1})_{\hat{A}}=\frac{1}{2}n(n-1)$ for any $\hat{A}\in\bar{\mathscr{G}}_{n-1}\cap\hat{U}$.

Let us take any $\lambda_0\in\boldsymbol{R}_*$, $\hat{A}_0\in\bar{\mathscr{G}}_{n-1}\cap\hat{U}$ and a sufficiently small open set $\tilde{U}$ in $\boldsymbol{R}_*\times(\bar{\mathscr{G}}_{n-1}\cap\hat{U})\times\boldsymbol{R}^{n-1}$ containing $(\lambda_0,\hat{A}_0,0)$. In view of Lemma 3.3, we know that the differential Φ_* of at $(\lambda_0,\hat{A}_0,0)$ is injective. Hence we may assume that the restriction $\Phi_{|\tilde{U}}\colon\tilde{U}\to\boldsymbol{R}^{n^3}$ of the map Φ to $\tilde{U}$ is an imbedding. Because of the assumption (1′), the image $\Phi(\tilde{U})$ of $\tilde{U}$ forms *a* $\frac{1}{2}n(n-1)$-dimensional submanifold of $\boldsymbol{R}^{n^3}$. Moreover we know $(\mathfrak{g}_n)_{\varphi(\lambda_0,\hat{A}_0)}=T_{\varphi(\lambda_0,\hat{A}_0)}\Phi(\tilde{U})$. We now notice the following

Lemma 3.4. *Let $f_1,\cdots,f_r$ be differentiable functions on a manifold M and let $\mathscr{V}$ be the subvariety of M defined by the system of equations $f_1=\cdots=f_r=0$. Assume that there is a submanifold S of M and a point $p\in S$ such that*

(i) $S\subset\mathscr{V}$;

(ii) $T_p(S)=\{x\in T_p(M)\,|\,df_1(x)=\cdots=df_r(x)=0\}$.

Then there exists a neighborhood U of p in M such that $\mathscr{V}\cap U=S\cap U$.

The proof of the lemma is left to the readers.

By virtue of Lemma 3.4, we know that there exists an open neighborhood U of $\varphi(\lambda_0,\hat{A}_0)$ in $\boldsymbol{R}^{n^3}$ such that $\bar{\mathscr{G}}_n\cap U=\Phi(\tilde{U})\cap U$. This implies (1) of the proposition.

Next we show (2). Let $A\in\bar{\mathscr{G}}_n\cap U$. Since $A\in\Phi(\hat{U})$, we may write $A=\varphi(\lambda,\hat{A})^{\exp\xi}$ by using a suitable $(\lambda,\hat{A},\xi)\in\tilde{U}$. Hence, in order to show (2), it suffices to deal with the case $A=\varphi(\lambda,\hat{A})$. Let $\hat{x}=(\hat{x}_1,\cdots,\hat{x}_{n-1})\in\boldsymbol{R}^{n-1}$ be the vector stated in (2′). We may assume that $|\hat{x}_i|\ll 1$ for $1\leqq i\leqq n-1$. We define $x=(x_1,\cdots,x_n)\in\boldsymbol{R}^n$ by $x_i=\hat{x}_i$ for $1\leqq i\leqq n-1$ and $x_n=1$. Let us show the vector x defined above has the property stated in (2). Let $B=(B_{pq}{}^{r})_{1\leqq p,q,r\leqq n}\in(\mathfrak{g}_n)_{\varphi(\lambda,\hat{A})}$ and suppose $\sum_{r=1}^{n}x_rB_{pq}{}^{r}=0$ for any (p,q) $(1\leqq p,q\leqq n)$. By using the decomposition $B=\Phi_*(\mu)+\Phi_*(\hat{B})+\Phi_*(\xi)$ in

Lemma 3.3 and the formulas (**) for $\Phi_*(\mu)$, $\Phi_*(\hat{B})$, $\Phi_*(\xi)$, we have

$$\begin{aligned}(3.10)\qquad &\sum_{k=1}^{n-1}\{\lambda(\hat{x}_j\delta_{ik}+\hat{x}_i\delta_{jk}+\hat{x}_k\delta_{ij})-\sqrt{1+\lambda^2}\hat{A}_{ij}{}^k\}\xi_k\\ &+\left(\delta_{ij}+\frac{\lambda}{\sqrt{1+\lambda^2}}\sum_{k=1}^{n-1}\hat{x}_k\hat{A}_{ij}{}^k\right)\mu+\sqrt{1+\lambda^2}\left(\sum_{k=1}^{n-1}\hat{x}_k\hat{B}_{ij}{}^k\right)=0\\ &\text{for}\quad 1\leqq i,j\leqq n-1;\end{aligned}$$

$$(3.11)\qquad -\frac{1}{\lambda}(1+\lambda^2)\sum_{j=1}^{n-1}\left(\delta_{ij}+\frac{\lambda}{\sqrt{1+\lambda^2}}\sum_{j=1}^{n-1}\hat{x}_k\hat{A}_{ij}{}^k\right)\xi_j+\hat{x}_i\mu=0 \quad \text{for}\quad 1\leqq i\leqq n-1;$$

$$(3.12)\qquad \mu-\lambda\sum_{k=1}^{n-1}\hat{x}_k\xi_k=0.$$

Since $|\hat{x}_i|\ll 1$, it follows from (3.11) and (3.12) that $\mu=\xi_1=\cdots=\xi_{n-1}=0$. Hence by (3.10) we have $\sum_{k=1}^{n}\hat{x}_k\hat{B}_{ij}{}^k=0$ for any $1\leqq i,j\leqq n-1$. Therefore by the assumption (2′), we obtain $\hat{B}=0$. This completes the proof of (2).

Finally we show (3) of the proposition. As in the proof of (2), we may assume $A=\varphi(\lambda,\hat{A})$. Let $C=({}^sC_{pq}{}^r)_{1\leqq p,q,r,s\leqq n}\in(\bar{\mathfrak{g}}_n^{(1)})_{\varphi(\lambda,\hat{A})}$. For each s ($1\leqq s\leqq n$), we denote by ${}^sC$ the element of $\boldsymbol{R}^{n^3}$ given by ${}^sC=({}^sC_{pq}{}^r)_{1\leqq p,q,r\leqq n}$. Clearly we have ${}^sC\in(\bar{\mathfrak{g}}_n)_{\varphi(\lambda,\hat{A})}$ for $1\leqq s\leqq n$. By Lemma 3.3, there exist ${}^s\mu\in\boldsymbol{R}$, ${}^s\hat{B}\in(\bar{\mathfrak{g}}_{n-1})_{\hat{A}}$ and ${}^s\xi\in\boldsymbol{R}^{n-1}$ such that

$${}^sC=\Phi_*({}^s\mu)+\Phi_*({}^s\hat{B})+\Phi_*({}^s\xi).$$

Since ${}^sC_{pq}{}^r={}^pC_{sq}{}^r$, we have

$$(3.13)\qquad {}^i\mu=-\lambda{}^n\xi_i \qquad \text{for}\quad 1\leqq i\leqq n-1;$$

$$(3.14)\qquad {}^j\xi_i=\frac{\lambda}{\sqrt{1+\lambda^2}}\left(\sum_{l=1}^{n-1}{}^n\xi_l\hat{A}_{ij}{}^l-\frac{1}{\sqrt{1+\lambda^2}}{}^n\mu\delta_{ij}\right) \qquad \text{for}\quad 1\leqq i,j\leqq n-1;$$

$$\begin{aligned}(3.15)\qquad {}^n\hat{B}_{ij}{}^k=&-\frac{\lambda}{\sqrt{1+\lambda^2}}(2{}^n\xi_i\delta_{jk}+{}^n\xi_j\delta_{ik}+{}^n\xi_k\delta_{ij})-\frac{\lambda}{\sqrt{1+\lambda^2}}\left(\sum_{p,q=1}^{n-1}{}^n\xi_q\hat{A}_{iq}{}^p\hat{A}_{jk}{}^p\right)\\ &\text{for}\quad 1\leqq i,j,k\leqq n-1;\end{aligned}$$

$$\begin{aligned}(3.16)\qquad {}^l\hat{B}_{ij}{}^k-{}^i\hat{B}_{lj}{}^k=&\frac{\lambda}{\sqrt{1+\lambda^2}}({}^i\xi_j\delta_{lk}+{}^i\xi_k\delta_{lj}-{}^l\xi_j\delta_{ik}-{}^l\xi_k\delta_{ij})\\ &+\frac{\lambda}{1+\lambda^2}({}^i\mu\hat{A}_{lj}{}^k-{}^l\mu\hat{A}_{ij}{}^k) \qquad \text{for}\quad 1\leqq i,j,k,l\leqq n-1.\end{aligned}$$

Here we set

$$\begin{aligned}(3.17)\qquad {}^l_0\hat{B}_{ij}{}^k=&-\frac{\lambda}{\sqrt{1+\lambda^2}}(2{}^l\xi_i\delta_{jk}+{}^l\xi_j\delta_{ik}+{}^l\xi_k\delta_{ij})\\ &-\frac{\lambda}{\sqrt{1+\lambda^2}}\left(\sum_{p,q=1}^{n-1}{}^l\xi_p\hat{A}_{ip}{}^q\hat{A}_{jk}{}^q\right) \qquad \text{for}\quad 1\leqq i,j,k,l\leqq n-1.\end{aligned}$$

Then by (3.13) and (3.14) we have ${}^{l}_{0}\hat{B}=({}^{l}_{0}\hat{B}_{ij}{}^{k})_{1\leq i,j,k\leq n-1}\in(\bar{\mathfrak{g}}_{n-1})_{\hat{A}}$ for $1\leqq l\leqq n-1$ and ${}^{l}\hat{B}_{ij}{}^{k}-{}^{l}_{0}\hat{B}_{ij}{}^{k}={}^{i}\hat{B}_{lj}{}^{k}-{}^{i}_{0}\hat{B}_{lj}{}^{k}$ for $1\leqq i,j,k,l\leqq n-1$. Hence if we set

$$(3.18)\qquad {}^{l}\hat{C}_{ij}{}^{k}={}^{l}\hat{B}_{ij}{}^{k}-{}^{l}_{0}\hat{B}_{ij}{}^{k}\qquad \text{for}\quad 1\leqq i,j,k,l\leqq n-1,$$

then $\hat{C}=({}^{l}\hat{C}_{ij}{}^{k})_{1\leq i,j,k,l\leq n-1}\in(\bar{\mathfrak{g}}^{(1)}_{n-1})_{\hat{A}}$. By these arguments we know that any $C\in(\bar{\mathfrak{g}}^{(1)}_{n})_{\varphi(\lambda,\hat{A})}$ can be completely determined by ${}^{n}\eta\in\boldsymbol{R}$, $\hat{C}\in(\bar{\mathfrak{g}}^{(1)}_{n-1})_{\hat{A}}$ and ${}^{n}\xi=({}^{n}\xi_{1},\cdots,{}^{n}\xi_{n-1})\in\boldsymbol{R}^{n-1}$. Conversely it is clear that these variables are independent. Hence we have

$$\dim(\bar{\mathfrak{g}}^{(1)}_{n})_{\varphi(\lambda,\hat{A})}=1+\tfrac{1}{2}n(n-1)+n-1=\tfrac{1}{2}n(n+1).$$

This completes the proof the of the proposition. Q.E.D.

§ 4. The proof of Theorem 3.1

As in the previous section, we shall fix an orthnormal basis $\{e_a\}_{0\leq a\leq n}$ (resp. $\{v_i\}_{1\leq i\leq n}$) of $\boldsymbol{T}$ (resp. $\boldsymbol{N}$). As usual every $\sigma\in\mathfrak{o}(\boldsymbol{T})$ (resp. $\rho\in\mathfrak{o}(\boldsymbol{N})$) can be represented by a skew symmetric matrix $(\sigma_a{}^b)_{0\leq a,b\leq n}$ (resp. $(\rho_i{}^j)_{1\leq i,j\leq n}$) with respect to $\{e_a\}$ (resp. $\{v_i\}$).

Let $A\in\bar{\mathscr{G}}_n$. By α_A, we mean the element of $\mathring{\mathscr{G}}$ corresponding to A. Let us define a map $\psi\colon\mathfrak{o}(N)\times\bar{\mathscr{G}}_n\times\boldsymbol{R}^n\to S^2\boldsymbol{T}^*\otimes\boldsymbol{N}$ by

$$\psi(\rho,A,\tau)=\exp\rho\alpha_A{}^{\exp\tilde{\tau}}\qquad \text{for } (\rho,A,\tau)\in\mathfrak{o}(N)\times\bar{\mathscr{G}}_n\times\boldsymbol{R}^n,$$

where for each $\tau=(\tau_1,\cdots,\tau_n)$ we denote by $\tilde{\tau}$ the element of $\mathfrak{o}(\boldsymbol{T})$ such that

$$\tilde{\tau}=\begin{pmatrix} 0, & \tau_1, \cdots, \tau_n \\ -\tau_1 & \\ \vdots & 0 \\ -\tau_n & \end{pmatrix}.$$

Note that since $\alpha_A\in\mathring{\mathscr{G}}$, we have $\psi(\rho,A,\tau)\in\mathscr{G}$. Let us denote by $\psi_*=\psi_{*(0,A,0)}$ the differential of ψ at $(0,A,0)\in\mathfrak{o}(N)\times\bar{\mathscr{G}}_n\times\boldsymbol{R}^n$, where 0 means the zero matrix of degree n or the vector $(0,\cdots,0)\in\boldsymbol{R}^n$. Naturally the tangent space to the variety $\mathfrak{o}(N)\times\bar{\mathscr{G}}_n\times\boldsymbol{R}^n$ at $(0,A,0)$ may be identified with the direct sum $\mathfrak{o}(N)\oplus(\bar{\mathfrak{g}}_n)_A\oplus\boldsymbol{R}^n$. Hence ψ_* induces a linear map of $\mathfrak{o}(N)\oplus(\bar{\mathfrak{g}}_n)_A\oplus\boldsymbol{R}^n$ into $S^2\boldsymbol{T}^*\otimes\boldsymbol{N}$. We also denote it by ψ_*.

Let $\rho=(\rho_i{}^j)_{1\leq i,j\leq n}\in\mathfrak{o}(N)$, $B=(B_{ij}{}^k)_{1\leq i,j,k\leq n}\in(\bar{\mathfrak{g}}_n)_A$, and $\tau=(\tau_1,\cdots,\tau_n)\in\boldsymbol{R}^n$. It is noted that the vectors $\psi_*(\rho)$, $\psi_*(B)$ and $\psi_*(\tau)$ in $S^2\boldsymbol{T}^*\otimes\boldsymbol{N}$ are necessarily contained in $\mathfrak{g}_{\alpha_A}$. In terms of coefficients they are represented as follows:

$$\begin{aligned} &\psi_*(\rho)_{00}{}^k=0,\quad \psi_*(\rho)_{0i}{}^k=\delta_{ik},\quad \psi_*(\rho)_{ij}{}^k=\sum_{l=1}^{n}\rho_l{}^kA_{ij}{}^l;\\ (***)\qquad &\psi_*(B)_{00}{}^k=\psi_*(B)_{0i}{}^k=0,\quad \psi_*(B)_{ij}{}^k=B_{ij}{}^k;\\ &\psi_*(\tau)_{00}{}^k=-2\tau_k,\quad \psi_*(\tau)_{0i}{}^k=-\sum_{l=1}^{n}\tau_lA_{li}{}^k,\quad \psi_*(\tau)_{ij}{}^k=\tau_i\delta_{jk}+\tau_j\delta_{ik}\\ &\hspace{22em}\text{for}\quad 1\leqq i,j,k\leqq n.\end{aligned}$$

Lemma 4.1. *For each* $\beta \in \mathfrak{g}_{\alpha_A}$, *there exists a unique* $(\rho, B, \tau) \in \mathfrak{o}(N) \oplus (\bar{\mathfrak{g}}_n)_A \oplus \boldsymbol{R}^n$ *such that*

$$\beta = \psi_*(\rho) + \psi_*(B) + \psi_*(\tau).$$

Proof. We first assume that $\psi_*(\rho) + \psi_*(B) + \psi_*(\tau) = 0$ for some $(\rho, B, \tau) \in \mathfrak{o}(N) \oplus (\bar{\mathfrak{g}}_n)_A \oplus \boldsymbol{R}^n$. Then by (***), we immediately have $\rho = B = \tau = 0$. This proves the uniqueness. We next show the decomposition. Let $\beta \in \mathfrak{g}_{\alpha_A}$ and let $\tilde{B} = (\tilde{B}_{ab}{}^k)_{\substack{0 \leq a, b \leq n \\ 1 \leq k \leq n}} \in \boldsymbol{R}^{n(n+1)^2}$ be the coefficients of β. Take $\tau = (\tau_1, \cdots, \tau_n) \in \boldsymbol{R}^n$ so that $\tau_k = -\frac{1}{2}\tilde{B}_{00}{}^k$ for $1 \leq k \leq n$. We set $\bar{\beta} = \beta - \psi_*(\tau)$. Since $\bar{\beta} \in \mathfrak{g}_{\alpha_A}$, we have $\Omega_{*\alpha_A}(\bar{\beta}) = 0$, i.e.,

$$\begin{aligned} \langle \bar{\beta}(x, z), \alpha_A(y, w) \rangle + \langle \alpha_A(x, z), \bar{\beta}(y, w) \rangle - \langle \bar{\beta}(x, w), \alpha_A(y, z) \rangle \\ - \langle \alpha_A(x, w), \bar{\beta}(y, z) \rangle = 0 \qquad \text{for} \quad x, y, z, w \in \boldsymbol{T}. \end{aligned} \tag{4.1}$$

Let $(\bar{B}_{ab}{}^k)_{\substack{0 \leq a, b \leq n \\ 1 \leq k \leq n}}$ be the coefficients of $\bar{\beta}$. By the choice of τ, we have $\bar{B}_{00}{}^k = 0$ for $1 \leq i \leq n$. Thus if we put $x = e_0$, $y = e_i$, $z = e_0$, $w = e_j$ $(1 \leq i, j \leq n)$ into (4.1), then we obtain $\bar{B}_{0i}{}^j + \bar{B}_{0j}{}^i = 0$. Let us set $\rho = (\bar{B}_{0i}{}^j)_{1 \leq i, j \leq n}$ and set $\mathring{\beta} = \bar{\beta} - \psi_*(\rho) = \beta - (\psi_*(\tau) + \psi_*(\rho))$. Since $\rho \in \mathfrak{o}(N)$, we have $\mathring{\beta} \in \mathfrak{g}_{\alpha_A}$. Hence

$$\begin{aligned} \langle \mathring{\beta}(x, z), \alpha_A(y, w) \rangle + \langle \alpha_A(x, z), \mathring{\beta}(y, w) \rangle - \langle \mathring{\beta}(x, w), \alpha_A(y, z) \rangle \\ - \langle \alpha_A(x, w), \mathring{\beta}(y, z) = 0 \qquad \text{for} \quad x, y, z, w \in \boldsymbol{T}. \end{aligned} \tag{4.2}$$

Let $\mathring{B} = (\mathring{B}_{a,b}{}^k)_{\substack{0 \leq a, b \leq n \\ 1 \leq k \leq n}}$ be the coefficients of $\mathring{\beta}$. Then we have $\mathring{B}_{00}{}^k = \mathring{B}_{0i}{}^k = 0$ for $1 \leq i$, $k \leq n$. By this relation, the equation (4.2) may be reduced to the system of equations:

$$\mathring{B}_{ij}{}^k = \mathring{B}_{ik}{}^j; \qquad \sum_{p=1}^{n} (\mathring{B}_{ik}{}^p A_{jl}{}^p + A_{ik}{}^p \mathring{B}_{jl}{}^p - \mathring{B}_{il}{}^p A_{jk}{}^p - A_i{}^{lp} \mathring{B}_{jk}{}^p) = 0 \qquad \text{for} \quad 1 \leq i, j, k, l \leq n.$$

Thus if we set $B = (\mathring{B}_{ij}{}^k)_{1 \leq i, j, k \leq n}$, we have $B \in (\bar{\mathfrak{g}}_n)_A$ and $\mathring{\beta} = \psi_*(B)$. Hence we have $\beta = \psi_*(\rho) + \psi_*(B) + \psi_*(\tau)$. Q.E.D.

These being prepared, we start the proof of Theorem 3.1. Let U be the open set in $\boldsymbol{R}^{n^3}$stated in Proposition 3.2. Let $A_0 \in \bar{\mathscr{G}}_n \cap U$. By Lemma 4.1. we know that the differential ψ_* of ψ at $(0, A_0, 0)$ is injective. Taking a sufficiently small open neighborhood $\tilde{O}$ of $(0, A_0, 0)$ in $\mathfrak{o}(N) \times (\bar{\mathscr{G}}_n \cap U) \times \boldsymbol{R}^n$, we may assume that the restriction of the map ψ to $\tilde{O}$ is an imbedding. Therefore the image $\psi(\tilde{O})$ of $\tilde{O}$ forms an $n(n-1)$-dimensional submanifold of $S^2\boldsymbol{T}^* \otimes N$. Note that $\bar{\mathscr{G}}_n \cap U$ is a $\frac{1}{2}n(n-1)$ dimensional submanifold of $\boldsymbol{R}^{n^3}$. Moreover we know $\mathfrak{g}_{\alpha_{A_0}} = T_{\alpha_{A_0}}\psi(\tilde{O})$. Hence by Lemma 3.4, there exists an open neighborhood O of α_{A_0} in $S^2\boldsymbol{T}^* \otimes N$ such that $\mathscr{G} \cap O = \psi(\tilde{O}) \cap O$. This shows (1) of the theorem.

Next we show (2) of the theorem. Let $\alpha \in \mathscr{G} \cap O$. Since $\alpha \in \psi(\tilde{O})$, we may write $\alpha = \psi(\rho, A, \tau)$ by using suitable $(\rho, A, \tau) \in \mathfrak{o}(N) \times (\bar{\mathscr{G}}_n \cap U) \times \boldsymbol{R}^n$. Hence in order to show (2), it suffices to deal with the case where $\alpha = \psi(0, A, 0) = \alpha_A$. Since $A \in \bar{\mathscr{G}}_n \cap U$ there exists a vector $x = (x_1, \cdots, x_n) \in \boldsymbol{R}^n$ satisfying (2) of Proposition 3.2. Here we may assume that $|x_i| \ll 1$ for $1 \leq i \leq n$. We set $e = e_0 + \sum_{i=1}^n x_i e_i$. Then we can see that the vector e satisfies the condition in (2). In fact, let us suppose $e \rfloor \beta=0$ for some $\beta \in \mathfrak{g}_{\alpha_A}$. Then by using the decomposition $\beta=\psi_*(\rho)+\psi_*(B)+\psi_*(\tau)$ in Lemma 4.1. and the formulas (***), we have

$$-2\tau_k+\sum_{i=1}^n x_i\Big(\rho_i{}^k-\sum_{j=1}^n \tau_j A_{ji}{}^k\Big)=0 \qquad \text{for} \quad 1\leq k\leq n; \tag{4.3}$$

$$\begin{aligned}&\rho_i{}^k-\sum_{j=1}^n \tau_j A_{ji}{}^k+\sum_{j=1}^n x_j\Big(\tau_j\delta_{ik}+\tau_i\delta_{jk}+\sum_{l=1}^n \rho_l{}^k A_{ji}{}^l\Big)\\ &\quad+\sum_{j=1}^n x_j B_{ji}{}^k=0 \qquad \text{for} \quad 1\leq i,\, k\leq n.\end{aligned} \tag{4.4}$$

The equation (4.3) and the skew symmetric part of (4.4) with respect to the pair (i, k) form a system of homogeneous linear equations with variables τ_i, $\rho_j{}^k$ $(1\leq i, j, k\leq n)$. Since $|x_i|\ll 1$ it follows that $\tau_i=\rho_j{}^k=0$ for $1\leq i, j, k\leq n$. Hence we have $\sum_{j=1}^n x_j B_{ji}{}^k=0$ for any $1\leq i, k\leq n$. Therefore we obtain $B_{ji}{}^k=0$ for any $1\leq i, j, k\leq n$. This shows $\beta=0$.

Finally we show (3) of the theorem. As in the proof of (2), we may assume that $\alpha=\alpha_A$. Let $\gamma \in \mathfrak{g}_{\alpha_A}^{(1)}$. Since $e_s \rfloor \gamma \in \mathfrak{g}_{\alpha_A}$ for $0\leq s\leq n$, there are ${}^s\rho=({}^s\rho_i{}^j)_{1\leq i,j\leq n} \in \mathfrak{o}(N)$, ${}^sB=({}^sB_{ij}{}^k)_{1\leq i,j,k\leq n} \in (\bar{\mathfrak{g}}_n)_A$ and ${}^s\tau=({}^s\tau_1, \cdots, {}^s\tau_n) \in \boldsymbol{R}^n$ such that $e_s \rfloor \gamma=\Psi_*({}^s\rho)+\Psi_*({}^sB)+\Psi_*({}^s\tau)$. Then, by the relation $e_t \rfloor e_s \rfloor \gamma=e_s \rfloor e_t \rfloor \gamma$ for $1\leq s, t\leq n$, we have

$${}^i\tau_k=\tfrac{1}{2}{}^0\rho_k{}^i+\tfrac{1}{2}\sum_{j=1}^n {}^0\tau_j A_{jk}{}^i \qquad \text{for} \quad 1\leq i, k\leq n; \tag{4.5}$$

$$\begin{aligned}&{}^i\rho_j{}^k-\sum_{l=1}^n {}^i\tau_l A_{lj}{}^k\\ &\quad={}^0\tau_i\delta_{jk}+{}^0\tau_j\delta_{ik}+\sum_{l=1}^n {}^0\rho_l{}^k A_{ij}{}^l+{}^0B_{ij}{}^k \qquad \text{for} \quad 1\leq i, j, k\leq n;\end{aligned} \tag{4.6}$$

$$\begin{aligned}{}^lB_{ij}{}^k-{}^iB_{lj}{}^k&=\tfrac{1}{2}({}^0\rho_l{}^i\delta_{jk}+{}^0\rho_l{}^j\delta_{ik}+{}^0\rho_l{}^k\delta_{ji})-\tfrac{1}{2}({}^0\rho_i{}^l\delta_{jk}+{}^0\rho_i{}^j\delta_{lk}+{}^0\rho_i{}^k\delta_{jl})\\ &\quad+\tfrac{1}{2}\sum_{p,q=1}^n {}^0\rho_p{}^q(A_{lk}{}^pA_{ij}{}^q-A_{ik}{}^pA_{lj}{}^q) \qquad \text{for} \quad 1\leq i, j, k, l\leq n.\end{aligned} \tag{4.7}$$

From (4.6) we obtain

$${}^i\rho_j{}^k=\tfrac{1}{2}({}^0\tau_j\delta_{ik}-{}^0\tau_k\delta_{ij})+\tfrac{1}{2}\sum_{l=1}^n ({}^0\rho_l{}^kA_{ij}{}^l-{}^0\rho_l{}^jA_{ik}{}^l); \tag{4.8}$$

$$\begin{aligned}{}^0B_{ij}{}^k&=-\tfrac{1}{2}({}^0\tau_j\delta_{ik}+{}^0\tau_k\delta_{ij}+2{}^0\tau_i\delta_{jk})-\tfrac{1}{2}\sum_{p,q=1}^n {}^0\tau_qA_{qi}{}^pA_{jk}{}^p\\ &\quad-\tfrac{1}{2}\sum_{p=1}^n ({}^0\rho_p{}^kA_{ij}{}^p+{}^0\rho_p{}^jA_{ik}{}^p+{}^0\rho_p{}^iA_{pj}{}^k) \qquad \text{for} \quad 1\leq i, j, k\leq n.\end{aligned} \tag{4.9}$$

We set

$$\begin{aligned}{}_0^lB_{ij}{}^k&=-\tfrac{1}{2}({}^l\tau_j\delta_{ik}+{}^l\tau_k\delta_{ij}+2{}^l\tau_i\delta_{jk})-\tfrac{1}{2}\sum_{p,q=1}^n {}^l\tau_qA_{qi}{}^pA_{jk}{}^p\\ &\quad-\tfrac{1}{2}\sum_{p=1}^n ({}^l\rho_p{}^kA_{ij}{}^p+{}^l\rho_q{}^jA_{ik}{}^p+{}^l\rho_p{}^iA_{pj}{}^k) \qquad \text{for} \quad 1\leq i, j, k, l\leq n.\end{aligned} \tag{4.10}$$

Then we have ${}^l_0B=({}^l_0B_{ij}{}^k)_{1\leqq i,j,k\leqq n}\in(\bar{\mathfrak{g}}_n)_A$ for $1\leqq l\leqq n$. Moreover by (4.5), (4.7) and (4.8) we obtain ${}^lB_{ij}{}^k-{}^l_0B_{ij}{}^k={}^iB_{lj}{}^k-{}^i_0B_{lj}{}^k$ for $1\leqq i,j,k,l\leqq n$. Let us set

$$(4.11)\qquad {}^lC_{ij}{}^k={}^lB_{ij}{}^k-{}^l_0B_{ij}{}^k \qquad \text{for } 1\leqq i,j,k,l\leqq n.$$

Then we have $C=({}^lC_{ij}{}^k)_{1\leqq i,j,n,l\leqq n}\in(\bar{\mathfrak{g}}_n^{(1)})_A$. Therefore any $\gamma\in\mathfrak{g}_{\alpha A}^{(1)}$ can be completely determined by ${}^0\rho==({}^0\rho_i{}^j)_{1\leqq i,j\leqq n}\in\mathfrak{o}(N)$, $C=({}^lC_{ij}{}^k)_{1\leqq i,j,k,l\leqq n}\in(\bar{\mathfrak{g}}_n^{(1)})_A$ and ${}^0\tau=({}^0\tau_1,\cdots,{}^0\tau_n)\in \boldsymbol{R}^n$. Conversely it is clear that these variables are independent. Hence we have $\dim\mathfrak{g}_{\alpha A}^{(1)}=\frac{1}{2}n(n-1)+\frac{1}{2}n(n+1)+n=n(n+1)$. Thus we have completed the proof of Theorem 3.1.

§ 5. Isometric immersions of the spaces of negative constant curvature

Let (M,g) be the space of constant curvature of dimension n with sectional curvature $k<0$. Then at each $p\in M$, we have

$$-g(R(x,y)z,w)=k\{g(x,z)g(y,w)-g(x,w)g(y,z)\} \qquad \text{for } x,y,z,w\in T_p.$$

Moreover we have $\nabla R\equiv 0$ on M.

We now show the following

Theorem 5.1. *If $m=2n-1$, then there exists an open fibered submanifold $\pi_1^2\colon P_{\#}^{(1)}\to P$ of the vector bundle $\pi_1^2\colon P^{(1)}\to P$ such that the intersection $Q_{\#}=Q\cap P_{\#}^{(1)}$ forms an involutive differential equation.*(**)

Proof. Let $\alpha\in P$ and let V be a sufficiently small neighborhood of α in P. From the local triviality of the vector bundle $\pi\colon N\to P$, we may assume that $\pi^{-1}(V)$ is isomorphic to the vector bundle $V\times N$, where $N=N_\alpha$. Furthermore we may assume that the isomorphism gives an isometric isomorphism between each fiber of $\pi^{-1}(V)$ and N. (Note that since each fiber of the vector bundle $\pi\colon N\to P$ is a subspace of $\boldsymbol{R}^{2n-1}$, it is endowed with an inner product.) Similarly the restriction $T_{|\pi_{-1}^1(v)}$ of the tangent bundle $T=T(M)$ to $\pi_{-1}^1(V)$ may be assumed to be isomorphic to the bundle $\pi_{-1}^1(V)\times T$, where we set $T=T_p$. We may assume that the isomorphism gives an isometric isomorphism between each fiber of $T_{|\pi_{-1}^1(v)}$ and T with respect to the given Riemannian metric g. Under these observations the restriction $P_{|v}^{(1)}$ of the vector bundle $\pi_1^2\colon P^{(1)}\to P$ to V may be considered to be isomorphic to the bundle $V\times S^2T^*\otimes N$. By this isomorphism the set $Q\cap(\pi_1^2)^{-1}(V)$ is mapped onto $V\times\mathscr{G}(G_k)$ (see the equation (1.8)). Hence there exists an open set O in $S^2T^*\otimes N$ having the property stated in Theorem 3.1. By O_α we denote the open set in $P^{(1)}$ that corresponds to the open set $V\times O$ in $V\times S^2T^*\otimes N$. For any $\alpha\in P$, we take such an open set O_α in $P^{(1)}$ and set $P_{\#}^{(1)}=\bigcup_{\alpha\in P}O_\alpha$. Then it is clear that $\pi_1^2\colon P_{\#}^{(1)}\to P$ forms an open fibered submanifold of the vector bundle $\pi_1^2\colon P^{(1)}\to P$. We set $Q_{\#}=Q\cap P_{\#}^{(1)}$ and

(**) For the definition of involutive differential equations, see [8] or [4].

$Q_\sharp^{(1)} = Q^{(1)} \cap (\pi_2^3)^{-1}(P_\sharp^{(1)})$. Then we can easily verify that $Q_\sharp$ forms a fibered submanifold of the vector bundle $\pi_1^2: P^{(1)} \to P$. Hence $Q_\sharp$ is a differential equation. In order to show the involutiveness of $Q_\sharp$, we must show the following:

(a) The map $\pi_2^3: Q_\sharp^{(1)} \to Q_\sharp$ is surjective.

(b) The union $\mathfrak{q}^{(1)} = \bigcup_{\beta \in Q_\sharp} \mathfrak{q}_\beta^{(1)}$ is a vector bundle over $Q_\sharp$.

(c) For each $\beta \in Q_\sharp$, the symbol $\mathfrak{q}_\beta$ of $Q_\sharp$ at β is involutive.

Proof of (*a*). Let $\beta = (p; \omega_0, \omega_1, \omega_2) \in Q_\sharp$. Let us define $\bar{\omega}_3 \in \otimes^3 T_p^* \otimes \boldsymbol{R}^{2n-1}$ by setting

$$\langle \bar{\omega}_3(w, z, x), \omega_1(y) \rangle = -\langle \omega_2(z, x), \omega_2(w, y) \rangle,$$
$$\langle \bar{\omega}_3(w, z, x), n \rangle = 0 \qquad \text{for } x, y, z, w \in T_p,\ n \in N_\alpha(\alpha = \pi_1^2(\beta)).$$

Then we have $\gamma = (p; \omega_0, \omega_1, \omega_2, \bar{\omega}_3) \in J^3(M, m)$ and $\pi_2^3(\gamma) = \beta$. Moreover since $\nabla R \equiv 0$ we have $\gamma \in Q_\sharp^{(1)}$ (see the equations (1.9) and (1.10)). Hence the map $\pi_2^3: Q_\sharp^{(1)} \to Q_\sharp$ is surjective.

Proof of (*b*) *and* (*c*). Let $\beta = (p; \omega_0, \omega_1, \omega_2) \in Q_\sharp$. We set $\alpha = \pi_1^2(\beta)$. We may assume $\beta \in O_\alpha$. Then by the definitions of the vector spaces $\mathfrak{q}_\beta$ and $\mathfrak{q}_\beta^{(1)}$, we have $\mathfrak{q}_\beta = \mathfrak{g}_{\omega_2}$ and $\mathfrak{q}_\beta^{(1)} = \mathfrak{g}_{\omega_2}^{(1)}$ (Note that we are assuming $\boldsymbol{T} = T_p$ and $\boldsymbol{N} = N_\alpha$.) Since $\dim \boldsymbol{N} = \dim \boldsymbol{T} - 1 = n - 1$, we have $\dim \mathfrak{q}_{\omega_2}^{(1)} = \dim \mathfrak{g}_{\omega_2}^{(1)} = n(n-1)$. This indicates that the union $\mathfrak{q}^{(1)} = \bigcup_{\beta \in Q_*} \mathfrak{q}_\beta^{(1)}$ is a vector bundle over Q_*. We next show (c). Since $\beta \in O_\alpha$, there exists a vector $e \in \boldsymbol{T} = T_p$ such that $e \rfloor \xi \neq 0$ for any $\xi \in \mathfrak{q}_\beta$; $\xi \neq 0$. Then we can easily see that any basis $\{e_1, \cdots, e_n\}$ of $\boldsymbol{T} = T_p$ such that $e_1 = e$ is regular for the symbol $\mathfrak{q}_\beta$. Hence $\mathfrak{q}_\beta$ is involutive. Thus we have completed the proof of Theorem 5.1. Q.E.D.

Note that any space of constant curvature is a real analytic Riemanian manifold. Then the varieties P, $P^{(1)}$, $Q^{(1)}$ and Q are also considered to be real analytic. We can easily see that Theorem 5.1 still holds if we consider everything in the real analytic category. Then $Q_\sharp$ forms a real analytic differential equation. From the existence theroem of local solutions of real analytic involutive differential equations (cf. [8], [4]) follows

Theorem 5.2. *Any space of negative constant curvature of dimension n can be locally isometrically immersed into the euclidean space* $\boldsymbol{R}^{2n-1}$.

Department of Mathematics,
Kyoto University

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Added in proof: After submitting this paper, the author knew the following classical work:

E. Cartan, Sur les variétés de courbure constante d'un espace euclidien ou non-euclidean, Oeuvres complètes, Partie III, vol. 1, Gauthier-Villars, Paris, 1955.