KYUNGPOOK Math. J. 49(2009), 313-319

On Local Properties of Factored Fourier Series

Hüseyin Bor
Department of Mathematics, Erciyes University, 38039 Kayseri, Turkey
e-mail : bor@erciyes.edu.tr

Abstract. In the present paper, a theorem dealing with local property of $\mid \bar{N}, p_n, \theta_n \mid_k$ summability of factored Fourier series which generalizes a result of Mazhar [8], has been proved. Some new results have also been obtained.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by t_n the n-th (C,1) mean of the sequence (na_n). A series $\sum a_n$ is said to be summable $\mid C, 1 \mid_k$, $k \geq 1$, if (see [6])

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid t_n \mid^k < \infty. \tag{1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0,\ i \geq 1). \tag{2}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{3}$$

defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [7]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k, k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid \Delta\sigma_{n-1} \mid^k < \infty, \tag{4}$$

where

$$\Delta\sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{5}$$

Received May 29, 2008; accepted September 3, 2008.
2000 Mathematics Subject Classification: 40G99, 42A24, 42B24.
Key words and phrases: absolute summability, infinite series, local property, Fourier series.

In the special case $p_n = 1$ for all values of n, $\mid \bar{N}, p_n \mid_k$ summability is the same as $\mid C, 1 \mid_k$ summability. Also, if we take $k = 1$ and $p_n = 1/(n+1)$, then summability $\mid \bar{N}, p_n \mid_k$ is equivalent to the summability $\mid R, \log n, 1 \mid$. Let (θ_n) be any sequence of positive constants. The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n, \theta_n \mid_k, k \geq 1$, if (see [12])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid \Delta\sigma_{n-1} \mid^k < \infty. \tag{6}$$

If we take $\theta_n = \dfrac{P_n}{p_n}$, then $\mid \bar{N}, p_n, \theta_n \mid_k$ summability reduces to $\mid \bar{N}, p_n \mid_k$ summability. Also, if we take $\theta_n = n$ and $p_n = 1$ for all values of n, then we get $\mid C, 1 \mid_k$ summability.

Furthermore, if we take $\theta_n = n$, then $\mid \bar{N}, p_n, \theta_n \mid_k$ summability reduces to $\mid R, p_n \mid_k$ (see [4]) summability. A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n \geq 0$ for every positive integer n, where $\Delta^2\lambda_n = \Delta(\Delta\lambda_n)$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0 \tag{7}$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t). \tag{8}$$

It is well known (see [13]) that the convergence of the Fourier series at $t = x$ is a local property of the generating function f(t) (i.e., it depends only on the behaviour of f in a arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of the generating function f(t).

2. Known result

Mohanty [11] has demonstrated that the summability $\mid R, \log n, 1 \mid$ of

$$\sum A_n(t)/\log(n+1), \tag{9}$$

at $t = x$, is a local property of the generating function of $\sum A_n(t)$. Later on Matsumoto [9] improved this result by replacing the series (9) by

$$\sum A_n(t)/\log\log(n+1)^{1+\epsilon}, \ \epsilon > 0. \tag{10}$$

Generalizing the above result Bhatt [1] proved the following theorem.

Theorem A. *If* (λ_n) *is a convex sequence such that* $\sum n^{-1}\lambda_n$ *is convergent, then the summability* $\mid R, \log n, 1 \mid$ *of the series* $\sum A_n(t)\lambda_n \log n$ *at a point can be ensured by a local property.*

Mishra [10] has also proved the following most general theorem dealing with local property.

Theorem B. *If* (p_n) *is a sequence such that*

$$P_n = O(np_n), \tag{11}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{12}$$

then the summability $\mid \bar{N}, p_n \mid$ *of the series*

$$\sum_{n=1}^{\infty} A_n(t)\lambda_n P_n / np_n \tag{13}$$

at a point can be ensured by local property, where (λ_n) *is as in Theorem A.*

On the other hand, Bor [3] has extended Theorem B and he proved that under the conditions of Theorem B the result also holds for the summability $\mid \bar{N}, p_n \mid_k$, $k \geq 1$. Later on, he [5] has further generalized his result in the following way :

Theorem C. *Let* $k \geq 1$, (p_n) *and* (λ_n) *be a sequences such that*

$$\Delta X_n = O(1/n), \quad X_n = \frac{P_n}{np_n}, \tag{14}$$

$$\sum_{n=1}^{\infty} X_n^{k-1} \frac{\mid \lambda_n \mid^k + \mid \lambda_{n+1} \mid^k}{n} < \infty, \tag{15}$$

$$\sum_{n=1}^{\infty} (X_n^k + 1) \mid \Delta\lambda_n \mid < \infty. \tag{16}$$

Then the summability $\mid \bar{N}, p_n \mid_k$ *of the series* $\sum A_n(t)\lambda_n X_n$ *at a point can be ensured by local property.*

Mazhar [8] has generated Theorem C in the following form.

Theorem D. *Let* $k \geq 1$, (p_n) *and* (λ_n) *be sequences such that*

$$\Delta(P_{n-1}X_n) = O\left(\frac{P_n}{n}\right), \quad X_n = \frac{P_n}{np_n}, \tag{17}$$

$$\sum_{n=1}^{\infty} X_n^{k-1} \frac{\mid \lambda_n \mid^k}{n} < \infty, \tag{18}$$

$$\sum_{n=1}^{\infty} X_{n+1} \mid \Delta\lambda_n \mid < \infty. \tag{19}$$

Then the summability $\mid \bar{N}, p_n \mid_k$ *of the series* $\sum A_n(t)\lambda_n X_n$ *at point can be ensured by a local property.*

3. The main result

The aim of this paper is to generalize Theorem D for $\mid \bar{N}, p_n, \theta_n \mid_k$ summability. That is, we shall prove the following theorem.

Theorem. *Let* $k \geq 1$, (p_n) *and* (λ_n) *be sequences such that conditions* (17), (19) *are satisfied. If* $\left(\frac{\theta_n p_n}{P_n}\right)$ *is a non-decreasing sequence and*

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \mid \lambda_n \mid^k X_n^k < \infty, \tag{20}$$

then the summability $\mid \bar{N}, p_n, \theta_n \mid_k$ *of the series* $\sum A_n(t)\lambda_n X_n$ *at a point can be ensured by a local property.*

It should be noted that if we take $\theta_n = \frac{P_n}{p_n}$, then we get Theorem D. In this case condition (20) reduces to condition (18). In fact

$$\begin{aligned}
\sum_{n=1}^{\infty} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \mid \lambda_n \mid^k X_n^k &= \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^k \frac{p_n}{P_n} \left(\frac{p_n}{P_n}\right)^k \mid \lambda_n \mid^k X_n^k \\
&= \sum_{n=1}^{\infty} X_n^k \frac{p_n}{P_n} \mid \lambda_n \mid^k \\
&= \sum_{n=1}^{\infty} X_n^k \frac{1}{nX_n} \mid \lambda_n \mid^k \\
&= \sum_{n=1}^{\infty} X_n^{k-1} \frac{\mid \lambda_n \mid^k}{n} < \infty.
\end{aligned}$$

Proof of theorem. Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only and hence to complete the proof of the theorem it is sufficent to prove that if (s_n) is bounded, then under the conditions of our theorem $\sum a_n \lambda_n X_n$ is summable $\mid \bar{N}, p_n, \theta_n \mid_k$, $k \geq 1$.
Let (T_n) denotes the $(\bar{N}, p_n)$ mean of the series $\sum a_n \lambda_n X_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r \lambda_r X_r = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v \lambda_v X_v.$$

Then, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v X_v, \quad n \geq 1, \quad (P_{-1} = 0).$$

Now using Abel's transformation, we have

$$\begin{aligned}
& T_n - T_{n-1} \\
= \; & \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v \lambda_v \Delta(P_{v-1} X_v) + \frac{p_n s_n \lambda_n X_n}{P_n} \\
= \; & \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v \lambda_v \Delta(P_{v-1} X_v) + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v P_v X_{v+1} \Delta \lambda_v + \frac{p_n s_n \lambda_n X_n}{P_n} \\
= \; & T_{n,1} + T_{n,2} + T_{n,3}, \quad \text{say.}
\end{aligned}$$

To complete the proof of the Theorem, by Minkowski's inequality for $k > 1$, it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid T_{n,r} \mid^k < \infty, \quad \text{for} \quad r = 1, 2, 3. \tag{21}$$

Now applying Hölder's inequality, we have that

$$\begin{aligned}
& \sum_{n=2}^{m+1} \theta_n^{k-1} \mid T_{n,1} \mid^k \\
\leq \; & \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k} \left| \sum_{v=1}^{n-1} s_v \lambda_v \Delta(P_{v-1} X_v) \right|^k \\
= \; & O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{\mid \lambda_v \mid P_v}{v} \right\}^k \\
= \; & O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \mid \lambda_v \mid X_v p_v \right\}^k \\
= \; & O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \mid \lambda_v \mid^k X_v^k \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
= \; & \sum_{v=1}^{m} p_v \mid \lambda_v \mid^k X_v^k \sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} \\
= \; & O(1) \sum_{v=1}^{m} p_v \mid \lambda_v \mid^k X_v^k \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}}
\end{aligned}$$

$$= O(1)\sum_{v=1}^{m} p_v \mid \lambda_v \mid^k X_v^k \theta_v^{k-1} \left(\frac{p_v}{P_v}\right)^{k-1} \frac{1}{P_v}$$
$$= O(1)\sum_{v=1}^{m} \theta_v^{k-1} \left(\frac{p_v}{P_v}\right)^{k} \mid \lambda_v \mid^k X_v^k = O(1) \quad \text{as} \quad m \to \infty,$$

by virtue of the hypotheses of the Theorem. Again

$$\sum_{n=2}^{m+1} \theta_n^{k-1} \mid T_{n,2} \mid^k$$
$$\leq \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k} \left\{\sum_{v=1}^{n-1} \mid s_v \mid P_v X_{v+1} \mid \Delta\lambda_v \mid\right\}^k \times \left\{\sum_{v=1}^{n-1} X_{v+1} \mid \Delta\lambda_v \mid\right\}^{k-1}$$
$$= O(1)\sum_{v=1}^{n-1} P_v^k X_{v+1} \mid \Delta\lambda_v \mid \sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^k \frac{p_n}{P_n P_{n-1}^k}$$
$$= O(1)\sum_{v=1}^{m} P_v^k X_{v+1} \mid \Delta\lambda_v \mid \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{P_v^{k-1}} \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}}$$
$$= O(1)\sum_{v=1}^{m} P_v^k X_{v+1} \mid \Delta\lambda_v \mid \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{P_v^{k-1}} \frac{1}{P_v}$$
$$= O(1)\left(\frac{\theta_1 p_1}{P_1}\right)^{k-1} \sum_{v=1}^{m} X_{v+1} \mid \Delta\lambda_v \mid$$
$$= O(1)\sum_{v=1}^{m} X_{v+1} \mid \Delta\lambda_v \mid = O(1) \quad \text{as} \quad m \to \infty,$$

in view of the hypotheses of the Theorem.
Finally, we have that

$$\sum_{n=1}^{m} \theta_n^{k-1} \mid T_{n,3} \mid^k = \sum_{n=1}^{m} \theta_n^{k-1} \left|\frac{p_n s_n \lambda_n X_n}{P_n}\right|^k$$
$$= O(1)\sum_{n=1}^{m} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \mid \lambda_n \mid^k X_n^k = O(1) \quad \text{as} \quad m \to \infty,$$

by virtue of the hypotheses of the Theorem. Therefore we get that

$$\sum_{n=1}^{m} \theta_n^{k-1} \mid T_{n,r} \mid^k = O(1) \quad as \quad m \to \infty, \quad \text{for} \quad r = 1, 2, 3.$$

which completes the proof of the Theorem. If we take $\theta_n = n$ and $p_n = 1$ for all values of n, then we get a new result for $\mid C, 1 \mid_k$ summability. Also, if we take

$\theta_n = n$, then we have another new result for $| R, p_n |_k$ summablity. □

References

[1] S. N. Bhatt, *An aspect of local property of* $| R, \log n, 1 |$ *summability of the factored Fourier series*, Proc. Nat. Inst. India, **26**(1960), 69-73.

[2] H. Bor, *On two summability methods*, Math. Proc. Cambridge Philos. Soc., **97**(1985), 147-149.

[3] H. Bor, *Local property of* $| \bar{N}, p_n |_k$ *summability of factored Fourier series*, Bull. Inst. Math. Acad. Sinica, **17**(1989), 165-170.

[4] H. Bor, *On the relative strength of two absolute summability methods*, Proc. Amer. Math. Soc., **113**(1991), 1009-1012.

[5] H. Bor, *On the local property of* $| \bar{N}, p_n |_k$ *summability of factored Fourier series*, J. Math. Anal. Appl., **163**(1992), 220-226.

[6] T. M. Flett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. London Math. Soc., **7**(1957), 113-141.

[7] G. H. Hardy, Divergent series, Oxford Univ. Press, Oxford, 1949.

[8] S.M. Mazhar, *A note on the localization of* $| \bar{N}, p_n |_k$ *summability of Fourier series*, Kumamoto J. Math., **12**(1999), 1-8.

[9] K. Matsumoto, *Local property of the summability* $| R, \lambda_n, 1 |$, Tohoku Math. J., **8(2)**(1956), 114-124.

[10] K. N. Mishra, *Multipliers for* $| \bar{N}, p_n |$ *summability of Fourier series*, Bull. Inst. Math. Acad. Sinica, **14**(1986), 431-438.

[11] R. Mohanty, *On the summability* $| R, \log w, 1 |$ *of Fourier series*, J. London Math. Soc., **25**(1950), 67-72.

[12] W. T. Sulaiman, *On some summability factors of infinite series*, Proc. Amer. Math. Soc., **115**(1992), 313-317.

[13] E.C. Titchmarsh, The Theory of Functions, Oxford Univ. Press, London, 1961.