ON LOCAL PROPERTIES OF NON-ARCHIMEDEAN ANALYTIC SPACES

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Introduction

Let k be a field complete with respect to a non-trivial non-Archimedean valuation. In [Ber1], V. Berkovich introduced and, in [Ber2], extended a new notion of a k-analytic space. The spaces from [Ber1] (called good in [Ber2]) are characterized among those from [Ber2] by the property that each point has an affinoid neighborhood (see Remark 2.7 for a translation of this property to the language of rigid geometry). The main result of this paper gives a criterion for a point of a k-analytic space to have an affinoid neighborhood. As applications of the main result, we establish the following two facts which were proven earlier by W. Lütkebohmert in [L] under the assumption that the valuation on k is discrete: (1) given a proper morphism $f : \mathcal{X} \longrightarrow \mathcal{Y}$ between formal schemes locally finitely presented over the ring of integers k°, the induced morphism $f_\eta : \mathcal{X}_\eta \longrightarrow \mathcal{Y}_\eta$ between their generic fibers is proper; (2) given a separated closed morphism $X \longrightarrow Y$ to an affinoid space Y, for every affinoid domain $U \subset X$ there extists a bigger affinoid domain $V \subset X$ such that $U \subset \mathrm{Int}(V/Y)$ and U is a Weierstrass subdomain of V.

In §1, we introduce for an abstract field k a category bir_k, whose objects are triples (X, K, f), where X is a connected quasi-compact and quasi-separated topological space, K is a field over k, and f is a local homeomorphism from X to the Riemann-Zariski space $\mathbf{P}_K$ of K, i.e., the set of all valuations on K trivial on k. (Morphisms in bir_k are defined in the evident way.) An object of bir_k as above is called affine if X is an open subset of $\mathbf{P}_K$ of the form $\{\nu \in \mathbf{P}_K \big| f_1, \dots, f_n \in \mathcal{O}_\nu\}$. A morphism $(X, K, f) \longrightarrow (Y, L, g)$ in bir_k is called proper if the canonical map $X \longrightarrow Y \times_{\mathbf{P}_L} \mathbf{P}_K$ is bijective.

Beginning with §2, the ground field k is complete with respect to a non-trivial non-Archimedean valuation, and all the k-analytic spaces considered are assumed to be strictly k-analytic, as defined in [Ber2]. (Because of this we, as a rule, suppress the word "strictly".) Let $\widetilde{k}$ be the residue field of k.

In §2 we construct a reduction functor $X_x \mapsto \widetilde{X}_x$ from the category of germs of k-analytic spaces at a point to the category $bir_{\widetilde{k}}$. This functor has also a natural interpretation in terms of R. Huber's adic spaces [H] (see

Remark 2.6). The main result of §2 states that the reduction functor induces a bijection between analytic subdomains of a k-germ X_x (i.e., equivalence classes of germs of analytic subdomains of X at the point x) and open quasi-compact subsets of its reduction $\widetilde{X}_x$.

In §3, we prove our main result which states that a k-germ X_x is good (i.e., the point x has an affinoid neighborhood in X) if and only if its reduction $\widetilde{X}_x$ is an affine object of $bir_{\widetilde{k}}$. In §4, we prove that a morphism of k-germs $f : X_x \longrightarrow Y_y$ is closed if and only if the corresponding morphism in $bir_{\widetilde{k}}$, $\widetilde{f} : \widetilde{X}_x \longrightarrow \widetilde{Y}_y$, is proper. As a corollary (resp. in §5), we prove the generalization of the first (resp. second) of W. Lütkebohmert's results mentioned at the beginning of the introduction. Notice that the proof in [L] is based completely on M. Raynaud's approach to rigid geometry and uses non-trivial results from algebraic geometry, whereas our proof uses more or less standard analytic tools.

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§1. The category bir_k

In this section we consider the category bir_k (mentioned in the introduction) and related to it categories Var_k and Bir_k.

The categories Var_k and Bir_k are defined as follows. Objects of Var_k are triples $(\mathcal{X}, K, \eta)$, where $\mathcal{X}$ is an integral scheme of finite type over k, K is a field over k, and η is a k-morphism $\mathrm{Spec}(K) \longrightarrow \mathcal{X}$ whose image is the generic point of $\mathcal{X}$ (i.e., η corresponds to an embedding of the field $R(\mathcal{X})$ of rational functions of $\mathcal{X}$ in K). A morphism $\phi : (\mathcal{X}, K, \eta) \longrightarrow (\mathcal{Y}, L, \varepsilon)$ is a pair of morphisms $f : \mathcal{X} \longrightarrow \mathcal{Y}$ and $i : \mathrm{Spec}(K) \longrightarrow \mathrm{Spec}(L)$ over k such that $f \circ \eta = \varepsilon \circ i$. A morphism (f, i) is said to be *separated* (resp. *proper*) if f is separated (resp. proper). It is said to be *birational* if f is proper and i is an isomorphism. The family of birational morphisms admits calculus of right fractions (see [GaZi]). The corresponding category of fractions is denoted by Bir_k.

Furthermore, for a field K over k, let $\mathbf{P}_K$ denote the set of all valuations on K trivial on k and, for $\nu \in \mathbf{P}_K$ let $\mathcal{O}_\nu$ be the valuation ring of ν. One endows $\mathbf{P}_K$ with the weakest topology with respect to which all sets of the form $\{\nu \in \mathbf{P}_K \big| f \in \mathcal{O}_\nu\}$ are open. Given subsets $X \subset \mathbf{P}_K$ and $A \subset K$, we set $X\{A\} = \{\nu \in X \big| f \in \mathcal{O}_\nu \text{ for all } f \in A\}$. Subsets of $\mathbf{P}_K$ of the form $\mathbf{P}_K\{f_1, \ldots, f_n\}$ are said to be *affine*.

1.1. Lemma. *Given k-subalgebras $A \subset B \subset K$ and an element $f \in K$, one has*

(i) $\mathbf{P}_K\{A[f]\} = \mathbf{P}_K\{A\}\{f\}$;

(ii) $\mathbf{P}_K\{A\} = \mathbf{P}_K\{B\}$ *if and only if B is integral over A.*

Proof. (i) is trivial, and (ii) follows from the following well known fact: the integral closure of a subalgebra A in K coincides with the intersection of all valuation subrings of K that contain A (see [Bou], ch. VI, §1, Th. 3). ∎

For an object $\overline{\mathcal{X}} = (\mathcal{X}, K, \eta)$ of Var_k, let $Val(\overline{\mathcal{X}})$ denote the set of pairs (ν, ϕ), where $\nu \in \mathbf{P}_K$ and ϕ is a morphism $\mathrm{Spec}(\mathcal{O}_\nu) \longrightarrow \mathcal{X}$ compatible with η. We endow $Val(\overline{\mathcal{X}})$ with the weakest topology with respect to which the canonical maps $\alpha : Val(\overline{\mathcal{X}}) \longrightarrow \mathcal{X}$ and $\beta : Val(\overline{\mathcal{X}}) \longrightarrow \mathbf{P}_K$ are continuous, where α takes (ν, ϕ) to the image of the closed point of $\mathrm{Spec}(\mathcal{O}_\nu)$ under ϕ, and β takes (ν, ϕ) to ν. (Notice that the map α is surjective.) By the valuative criterion of separatedness, $\mathcal{X}$ is separated if and only if the map β is injective. It follows that, for such $\overline{\mathcal{X}}$, β is an open embedding and, therefore, for an arbitrary $\overline{\mathcal{X}}$, β is a local homeomorphism. Furthermore, let $I(\overline{\mathcal{X}})$ denote the category whose objects are birational morphisms of the form $(\varphi, Id) : \overline{\mathcal{X}}' = (\mathcal{X}', K, \eta') \longrightarrow \overline{\mathcal{X}}$ and whose morphisms are the evident ones. Notice that $I(\overline{\mathcal{X}})$ is a filtered category.

1.2. Lemma. *Given a finite subset $S \subset K$, there exists an object $\overline{\mathcal{X}}' = (\mathcal{X}', K, \eta') \longrightarrow \overline{\mathcal{X}}$ of $I(\overline{\mathcal{X}})$ such that S is contained in the image of $R(\mathcal{X}')$ in K and each element $f \in S$ possesses the property that, for every point $x' \in \mathcal{X}'$, either f or f^{-1} is contained in the image of $\mathcal{O}_{\mathcal{X}',x'}$.*

Proof. We may assume that $S = \{f\}$. Consider the morphism $\mu : \mathrm{Spec}(K) \longrightarrow \mathbf{P}^1_k$ which is a composition of the morphism $\mathrm{Spec}(K) \longrightarrow \mathbf{A}^1$, defined by the homomorphism $k[T] \longrightarrow K$ that takes T to f, and the canonical embedding $\mathbf{A}^1 \longrightarrow \mathbf{P}^1$. It defines a morphism

$$\eta' : \mathrm{Spec}(K) \stackrel{(\eta,\mu)}{\longrightarrow} \mathcal{X} \times_k \mathbf{P}^1_k$$

Let $\mathcal{X}'$ be the closure of the image of η' (considered as a reduced scheme). The canonical projection $\mathcal{X}' \longrightarrow \mathcal{X}$ defines an object $\overline{\mathcal{X}}'$ of $I(\overline{\mathcal{X}})$ such that f coincides with the image of an element $g \in R(\mathcal{X}')$ in K, and there is a morphism $\mathcal{X}' \longrightarrow \mathbf{P}^1_k$ that takes the coordinate function of $\mathbf{P}^1_k$ to g. It follows that for any point $x' \in \mathcal{X}'$ either g or g^{-1} is contained in $\mathcal{O}_{\mathcal{X}',x'}$. ∎

1.3. Corollary. *There is a canonical homeomorphism $Val(\overline{\mathcal{X}}) \stackrel{\sim}{\rightarrow} \varprojlim_{I(\overline{\mathcal{X}})} \mathcal{X}'$.*

Proof. By the valuative criterion of properness, given a point $(\nu, \phi) \in Val(\overline{\mathcal{X}})$, the morphism $\phi : \mathrm{Spec}(\mathcal{O}_\nu) \longrightarrow \mathcal{X}$ can be lifted in a unique way to a morphism $\phi' : \mathrm{Spec}(\mathcal{O}_\nu) \longrightarrow \mathcal{X}'$ for each object $\overline{\mathcal{X}}' \longrightarrow \overline{\mathcal{X}}$ of $I(\overline{\mathcal{X}})$. In this way one gets continuous maps from $Val(\overline{\mathcal{X}})$ to all of $\mathcal{X}'$ and, therefore, to their projective limit. Suppose we are given a point of the projective limit, i.e., a compatible system of points $x' \in \mathcal{X}'$ for all $\overline{\mathcal{X}}' = (\mathcal{X}', K, \eta') \in Ob(I(\overline{\mathcal{X}}))$. The morphisms η' define embeddings $\mathcal{O}_{\mathcal{X}',x'} \hookrightarrow K$. From Lemma 1.2 it follows that the union $\mathcal{O}$ of all $\mathcal{O}_{\mathcal{X}',x'}$'s in K is a valuation ring in K. It defines a point $(\nu, \phi) \in Val(\overline{\mathcal{X}})$, i.e., we constructed a map from the projective limit to $Val(\overline{\mathcal{X}})$ which is inverse to the map we started with, and is continuous. ∎

From Corollary 1.3 it follows that the space $Val(\overline{\mathcal{X}})$ is quasi-compact and, in particular, the spaces $\mathbf{P}_K$ and their affine subsets are quasi-compact. Since the latter form a basis of topology on $\mathbf{P}_K$, and since the map $\beta : Val(\overline{\mathcal{X}}) \longrightarrow \mathbf{P}_K$ induces an open embedding of each open subset $Val(\overline{\mathcal{X}}') \subset Val(\overline{\mathcal{X}})$ in $\mathbf{P}_K$, where $\mathcal{X}'$ is an open affine subscheme of $\mathcal{X}$ and $\overline{\mathcal{X}}' = (\mathcal{X}', K, \eta)$, it follows that the space $Val(\overline{\mathcal{X}})$ is quasi-separated, i.e., the intersection of any two open quasi-compact subsets is quasi-compact.

Let now bir_k be the category whose objects are triples $\overline{X} = (X, K, \phi)$, where X is a connected quasi-compact and quasi-separated topological space, K is a field over k, and ϕ is a local homeomorphism $X \longrightarrow \mathbf{P}_K$. A morphism $\overline{X} = (X, K, \phi) \longrightarrow \overline{Y} = (Y, L, \psi)$ is a pair (h, i), where h is a continuous map $X \longrightarrow Y$ and i is a k-morphism $\mathrm{Spec}(K) \longrightarrow \mathrm{Spec}(L)$ such that $\psi \circ h = i^{\#} \circ \phi$, where $i^{\#} : \mathbf{P}_K \longrightarrow \mathbf{P}_L$ is the induced map. An object $\overline{X} = (X, K, \phi)$ of bir_k is said to be *affine* if ϕ induces a homeomorphism of X with an affine subset of $\mathbf{P}_K$. If $\overline{X} = (X, K, \phi) \in Ob(bir_k)$, then for any open quasi-compact subset $X' \subset X$ the triple $(X', K, \phi\big|_{X'})$ is an object of bir_k. If the latter object is affine, X' is said to be an *affine* subset of X.

Notice that the correspondence $\overline{\mathcal{X}} \mapsto (Val(\overline{\mathcal{X}}), K, \beta)$ gives rise to a functor $\mathcal{EF} : Var_k \longrightarrow bir_k$. Notice also that from the valuative criterion of separatedness (resp. properness) it follows that a morphism $\overline{\mathcal{X}} = (\mathcal{X}, K, \eta) \longrightarrow \overline{\mathcal{Y}} = (\mathcal{Y}, L, \varepsilon)$ in Var_k is separated (resp. proper) if and only if the map $Val(\overline{\mathcal{X}}) \longrightarrow Val(\overline{\mathcal{Y}}) \times_{\mathbf{P}_L} \mathbf{P}_K$ is injective (resp. bijective). In this case the above map is an open immersion (resp. a homeomorphism). In particular, a birational morphism in Var_k gives rise to an isomorphism in bir_k, i.e., the functor $\mathcal{EF}$ is a composition of the canonical functor $\mathcal{F} : Var_k \longrightarrow Bir_k$ with a functor $\mathcal{E} : Bir_k \longrightarrow bir_k$.

1.4. Proposition. *The functor $\mathcal{E}$ is an equivalence of categories.*

Proof. Let $\overline{\mathcal{X}} = (\mathcal{X}, K, \eta)$ be an object of Var_k, and let $\overline{X} = (X, K, \beta)$ be its image in bir_k. Given elements $f_1, \dots, f_n \in K$, let $X\{f_1, \dots, f_n\}$ denote the preimage of $\mathbf{P}_K\{f_1, \dots, f_n\}$ in X. If $f_1, \dots, f_n$ are contained in the image of $R(\mathcal{X})$, let $\mathcal{X}\{f_1, \dots, f_n\}$ denote the open subset of $\mathcal{X}$ that consists of the points x such that $f_1, \dots f_n$ are contained in the image of $\mathcal{O}_{\mathcal{X},x}$, and let $\overline{\mathcal{X}}\{f_1, \dots, f_n\}$ denote the corresponding object of Var_k. In this case, the canonical map $Val(\overline{\mathcal{X}}\{f_1, \dots, f_n\}) \longrightarrow X = Val(\overline{\mathcal{X}})$ identifies the first set with an open subset of $X\{f_1, \dots, f_n\}$. We remark that if the elements $f_1, \dots, f_n$ possess the property of Lemma 1.2 (i.e., for every i and every point $x \in \mathcal{X}$, either f_i or f_i^{-1} is contained in the image of $\mathcal{O}_{\mathcal{X},x}$), then $Val(\overline{\mathcal{X}}\{f_1, \dots, f_n\}) \widetilde{\rightarrow} X\{f_1, \dots, f_n\}$.

Let $\overline{\mathcal{X}} = (\mathcal{X}, K, \eta)$ and $\overline{\mathcal{Y}} = (\mathcal{Y}, L, \varepsilon)$ be objects in Var_k, and let $\overline{X} = (X, K, \phi)$ and $\overline{Y} = (Y, L, \psi)$ be their images in bir_k. To show that the functor $\mathcal{E}$ is faithful, it suffices to prove, that the functor $\mathcal{EF}$ is, i.e. that the canonical map $Hom_{Var_k}(\overline{\mathcal{X}}, \overline{\mathcal{Y}}) \longrightarrow Hom_{bir_k}(\overline{X}, \overline{Y})$ is injective. The latter follows from surjectivity of the maps $X \longrightarrow \mathcal{X}$ and $Y \longrightarrow \mathcal{Y}$ and the fact that any dominant morphism between integral schemes is uniquely determined by the embedding of their fields of rational functions and the map of their underlying spaces. Furthermore, let $\overline{X} \longrightarrow \overline{Y}$ be a morphism in bir_k. Take a covering of $\mathcal{Y}$ by open affine subschemes $\mathcal{Y}_i = \mathrm{Spec}(A_i)$, $1 \le i \le m$. (We identify A_i with its image in L.) Set $Y_i = Val(\overline{\mathcal{Y}}_i)$, then $\{Y_i\}_{1\le i\le m}$ is an open covering of Y. Let $\{X_j\}_{1\le j\le n}$ be an open affine covering of X, whose image in Y refines $\{Y_i\}$, i.e. the image of any X_j belongs to some $Y_{i(j)}$. Then $X_j \widetilde{\rightarrow} \mathbf{P}_K\{f_{j,1}, \dots, f_{j,l_j}\}$ for some elements $f_{j,p} \in K$, for any $1 \le j \le n$, let B_j be the subalgebra of K generated by k and $f_{j,1}, \dots, f_{j,l_j}$. By enlarging the set of elements $\{f_{j,p}\}$ we can achieve, that all algebras B_j are integrally closed. Apply Lemma 1.2 to the set $\{f_{j,p}\}_{1\le j\le n, 1\le p\le l_j} \subset K$, we get an object $\overline{\mathcal{X}}' = (\mathcal{X}', K, \eta') \longrightarrow \overline{\mathcal{X}}$ of $I(\overline{\mathcal{X}})$. By the construction, $\mathcal{X}'$ contains open affine subschemes $\mathcal{X}'_j$, such that $Val(\mathcal{X}'_j) \widetilde{\rightarrow} X_j$ ($\{\mathcal{X}'_j\}_{1\le j\le n}$ is a covering of $\mathcal{X}'$). The maps $f_{j,1}, \dots, f_{j,l_j} : \mathcal{X}'_j \to \mathbf{A}^1_k$ induce the map $\mathcal{X}'_j \to \mathrm{Spec}(B_j)$. By Lemma 1.1, for any j the integral closure of the algebra B_j (by our assumption it coincides with B_j) contains the image of $A_{i(j)}$ in K. So for each j there is a well defined and unique morphism from $\mathcal{X}'_j$ to $\mathcal{Y}_{i(j)}$ compatible with the embedding of L to K. It induces a morphism from $\mathcal{X}'_j$ to $\mathcal{Y}$, and the system of morphisms $\mathcal{X}'_j \longrightarrow \mathcal{Y}$ is compatible on joint intersections. The latter gives rise to a morphism of schemes $\mathcal{X}' \longrightarrow \mathcal{Y}$ and to a morphism $\overline{\mathcal{X}}' \longrightarrow \overline{\mathcal{Y}}$ in Var_k whose image in bir_k is the morphism $\overline{X} \longrightarrow \overline{Y}$ we started from. It follows that the functor $\mathcal{E}$ is fully faithful.

Let now $\overline{X} = (X, K, \phi)$ be an object of bir_k. Take a finite affine covering $\{X_i\}_{i\in I}$ of X and, for each pair $i, j \in I$, an affine covering $\{X_{i,j,l}\}$ of $X_i \cap X_j$. We apply Lemma 1.2 to $\overline{\mathcal{Z}} = (\mathrm{Spec}(k), K, \eta)$, where η is the canonical morphism $\mathrm{Spec}(K) \longrightarrow \mathrm{Spec}(k)$, and to the finite set of all of the elements of K that define the affine sets X_i and $X_{i,j,l}$, and we get an object $\overline{\mathcal{Y}} = (\mathcal{Y}, K, \varepsilon) \longrightarrow \overline{\mathcal{Z}}$ of $I(\overline{\mathcal{Z}})$ such that if $\mathcal{Y}_i$ and $\mathcal{Y}_{i,j,l}$ are the open subschemes of $\mathcal{Y}$ defined by the same elements that define the affine sets X_i and $X_{i,j,l}$, respectively, then $Val(\overline{\mathcal{Y}}_i) \widetilde{\rightarrow} X_i$ and $Val(\overline{\mathcal{Y}}_{i,j,l}) \widetilde{\rightarrow} X_{i,j,l}$. Finally, we glue the schemes $\mathcal{Y}_i$ along the subschemes $\cup_l \mathcal{Y}_{i,j,l}$. We get a scheme $\mathcal{X}$ and the corresponding object $\overline{\mathcal{X}}$ of Var_k whose image in bir_k is isomorphic to $\overline{X}$. Thus, the functor $\mathcal{E}$ is essentially surjective. ∎

Let K be a field over k, and let X be an open subset of $\mathbf{P}_K$. A *Laurent covering* of X is a covering of the form $\{X\{f_1^{\varepsilon_1}, \dots, f_n^{\varepsilon_n}\}\}_{(\varepsilon_1,\dots,\varepsilon_n)\in\{\pm 1\}^n}$, where $f_1, \dots, f_n$ are non-zero elements of K. If the set X is affine, then all of the above subsets are also affine.

1.5. Lemma. *Any finite covering of X by open sets of the form $X\{f_1, \dots, f_n\}$ has a Laurent refinement.*

Proof (see [BGR], §8.2.2). Suppose first that our covering is *rational*, i.e., it is of the form $U_i = \{X\{\frac{f_1}{f_i}, \dots, \frac{f_n}{f_i}\}\}_{1\le i\le n}$, where $f_1, \dots, f_n$ are non-zero elements of K. We claim that the Laurent covering of X defined by the elements $g_{ij} = \frac{f_i}{f_j}$ with $1 \le i < j \le n$ refines the above covering. Indeed, let

V be a subset from the Laurent covering. It is defined by a choice of $\varepsilon_{ij} = \pm 1$ for $1 \le i < j \le n$. If $1 \le i, j \le n$ and $i \ne j$, we write $i \prec j$ if either $i < j$ and $\varepsilon_{ij} = 1$, or $i > j$ and $\varepsilon_{ji} = -1$. This defines an ordering on the set $\{1, \dots, n\}$ (the ordering is non-strict, i.e. one may have $i_1 \prec i_2 \prec \dots \prec i_m \prec i_1$ and thus $i_1 \sim i_2 \sim \dots \sim i_m$). Let i be the maximal element with respect to this ordering. Then $\frac{f_j}{f_i} \in \mathcal{O}_\nu$ for all $\nu \in V$, i.e., $V \subset X\{\frac{f_1}{f_i}, \dots, \frac{f_n}{f_i}\}$.

Thus, it suffices to show that any finite covering $X = \cup_{i=1}^m U_i$ by sets of the form, given in the lemma, has a rational refinement. Removing, if necessary, zeros and adding ones, we may assume that $U_i = X\{f_{i1}, \dots, f_{in}\}$, where f_{ij} are non-zero elements of K and $f_{in} = 1$. Let J be the set of all sequences $\mathbf{j} = (j_1, \dots, j_m)$ such that $1 \le j_i \le n$ for all $1 \le i \le m$ and $\max\limits_{1 \le i \le m} \{j_i\} = n$. We claim that the rational covering of X defined by the elements $g_{\mathbf{j}} = f_{1j_1} \dots f_{mj_m}$ refines the covering we started from. Indeed, given $\mathbf{j} \in J$, let i be such that $j_i = n$. To verify the claim, it suffices to show that the set $V_{\mathbf{j}} = X\{\frac{g_{\mathbf{j}'}}{g_{\mathbf{j}}}\}_{\mathbf{j}' \in J}$ is contained in U_i. We have to check that, given $\nu \in V_{\mathbf{j}}$, one has $f_{ik} \in \mathcal{O}_\nu$ for all $1 \le k \le n$. The point ν is contained in some U_l (we may assume that $l \ne i$) and, in particular, $f_{lj_l} \in \mathcal{O}_\nu$. On the other hand, if $\mathbf{j}'$ is the element of J with $j'_l = n$ and $j'_k = j_k$ for $k \ne l$, then $\frac{g_{\mathbf{j}'}}{g_{\mathbf{j}}} = \frac{1}{f_{lj_l}} \in \mathcal{O}_\nu$, and we get $f_{lj_l} \in \mathcal{O}_\nu^*$. If we are now given $1 \le k \le n$, let $\mathbf{j}'$ be the element of J with $j'_i = k$, $j'_l = n$ and $j'_p = j_p$ for $p \ne i, l$. One has $\frac{g_{\mathbf{j}'}}{g_{\mathbf{j}}} = \frac{f_{ik}}{f_{lj_l}} \in \mathcal{O}_\nu$ and, therefore, $f_{ik} \in \mathcal{O}_\nu$. ∎

1.6. Remark. One can define a notion of affine morphism in bir_k by the condition that preimage of any affine subspace is affine. However as the following example shows, this property is not local with respect to the base. Consider two fields $K = k(x)$ and $L = k(x, y)$. Set $X_1 = \mathbf{P}_K\{x\}$, $X_2 = \mathbf{P}_K\{x^{-1}\}$, $X = X_1 \cup X_2$ and $Y_1 = \mathbf{P}_L\{x, xy\}$, $Y_2 = \mathbf{P}_L\{x^{-1}, x^{-1}y\}$, $Y = Y_1 \cup Y_2$. We have a natural morphism $f : Y \longrightarrow X$ such that $f^{-1}(X_1) = Y_1$ and $f^{-1}(X_2) = Y_2$. Clearly X, X_1, X_2, Y_1 and Y_2 are affine, but it is easily seen, that Y is not. For example, Y has no non-constant functions, but $Y \ne \mathbf{P}_L$.

§2. The reduction functor

Beginning with this section the ground field k is a non-Archimedean field with a non-trivial valuation. All of the k-analytic spaces, morphisms between them and their analytic domains considered are assumed to be strictly k-analytic (i.e., we work with the category of strictly k-analytic spaces *st-k-$\mathcal{A}n$* in the sense of [Ber2]).

Recall ([Ber2], §3.4) that the category *Germs* of germs of a k-analytic space at a point is the localization of the category of punctual k-analytic spaces with respect to the system of morphisms $\varphi : (X, x) \longrightarrow (Y, y)$ that induce an isomorphism of X with an open neighborhood of y in Y. The germ corresponding to a punctual k-analytic space (X, x) will be denoted by X_x.

A germ X_x is said to be *good* if the point x has an affinoid neighborhood in X. A morphism of germs $\varphi : X_x \longrightarrow Y_y$ is said to be *separated* (resp. *closed*) if it is induced by a separated (resp. closed) morphism $X' \longrightarrow Y$, where X' is an open neighborhood of x in X.

Let X_x be a germ. One can define as follows an equivalence relation on the set of analytic domains in X that contain the point x: X' is equivalent to X'' if the intersection $X' \cap X''$ is a neighborhood of x in both X' and X''. An equivalence class is said to be a *subdomain of* X_x. It will be denoted by X'_x, where X' is an analytic subdomain of X from the corresponding equivalence class. One defines in the evident way on the set of subdomains of X_x the inclusion relation and the operations of union and intersection. Notice that any isomorphism of germs $X_x \widetilde{\to} Y_y$ gives rise to a bijection from the set of subdomains of X_x to that of Y_y, and this bijection commutes with the inclusion relation and the union and intersection. Furthermore, assume that X_x is a good germ. Then for a family of elements $f = (f_1, \dots, f_n) \subset \mathcal{O}_{X,x}$ we define a subdomain $X_x\{f\}$ as the equivalence class of the subdomain $X'\{f\} = \{x \in X' \big| |f_i(x)| \le 1, 1 \le i \le n\}$ of X, where X' is an analytic neighborhoods of x such that all f_i come from analytic functions on X'. If Y_x is a subdomain of X_x, we set $Y_x\{f\} = Y_x \cap X_x\{f\}$. Our next purpose is to define a functor Red: $Germs \longrightarrow bir_{\widetilde{k}}$. For this we use results of M. Raynaud ([Ray]) from his approach to rigid analytic geometry, which were elaborated in [BL].

A formal $\mathrm{Spf}(k^\circ)$-scheme is *locally finitely presented* if it has a locally finite open covering by formal schemes of the form $\mathrm{Spf}(k^\circ\{T_1, \dots, T_n\}/J)$. If in addition it has no π-torsion (for $\pi \in k^{\circ\circ} \setminus \{0\}$), it is said to be admissible. (The condition of a locally finite covering is added to the usual definition from [BL] in order to define the generic fibre as an analytic space in the sense of [Ber2].) Let *Adm* be the category of quasi-compact admissible formal schemes over k°. Such a formal scheme $\mathcal{X}$ has the special (or closed) fiber $\mathcal{X}_s$, which is a scheme of finite type over $\widetilde{k}$, and the generic fiber $\mathcal{X}_\eta$, which is a compact (strictly) k-analytic space, see [Ber3]. (Recall that by [Ber2], §1.6, the category of compact strictly k-analytic spaces is equivalent to the category of quasi-compact quasi-separated rigid analytic spaces over k, which are considered in [Ray] and [BL] as the generic fibers of quasi-compact admissible formal schemes.) There is a surjective reduction map $\mathcal{X}_\eta \longrightarrow \mathcal{X}_s : x \mapsto \widetilde{x}$. By [BL], Theorem 4.1, the correspondence $\mathcal{X} \mapsto \mathcal{X}_\eta$ gives rise to an equivalence between the localization of the category *Adm* by admissible formal blow-ups and the category of compact k-analytic spaces.

Let Adm^p denote the category of punctual quasi-compact admissible formal schemes over k°, i.e., the category of pairs $(\mathcal{X}, x)$, where $\mathcal{X} \in Ob(Adm)$ and $x \in \mathcal{X}_\eta$. From [BL], 4.1, it follows that the correspondence $(\mathcal{X}, x) \mapsto (\mathcal{X}_\eta, x)$ gives rise to an equivalence of categories $Adm^p_S \widetilde{\to} Germs$, where Adm^p_S is the localization of Adm^p with respect to the system S of morphisms $\varphi : (\mathcal{Y}, y) \longrightarrow (\mathcal{X}, x)$ such that φ_η induces an isomorphism of $\mathcal{Y}_\eta$ with a compact analytic domain in $\mathcal{X}_\eta$ which is a neighborhood of the point x in $\mathcal{X}_\eta$.

We now define a functor r : $Adm^p \longrightarrow Var_{\widetilde{k}}$ as follows. Given $(\mathcal{X}, x) \in Ob(Adm^p)$, $\mathrm{r}(\mathcal{X}, x)$ is the triple $(V_{\widetilde{x}}, \widetilde{\mathcal{H}(x)}, \varepsilon)$, where $V_{\widetilde{x}}$ is the closure of the point $\widetilde{x}$ in $\mathcal{X}_s$, and ε is the morphism $\mathrm{Spec}(\widetilde{\mathcal{H}(x)}) \longrightarrow V_{\widetilde{x}}$ that corresponds to the canonical embedding of fields $\widetilde{k}(\widetilde{x}) \longrightarrow \widetilde{\mathcal{H}(x)}$ (see [Ber1], §2.4). Let red: $Adm^p \longrightarrow Bir_{\widetilde{k}}$ be the composition of r with the functor $Var_{\widetilde{k}} \longrightarrow Bir_{\widetilde{k}}$.

2.1. Lemma. *The functor* red *takes morphisms from S to isomorphisms.*

Proof. Let $\varphi : (\mathcal{Y}, y) \longrightarrow (\mathcal{X}, x)$ be a morphism from S. Since $\mathcal{Y}_\eta$ is a neighborhood of x in $\mathcal{X}_\eta$, we can find a compact analytic subdomain Z of $\mathcal{X}_\eta$ such that $\mathcal{X}_\eta = \mathcal{Y}_\eta \cup Z$ and $x \notin Z$. By [BL], 4.4, there are admissible formal blow-ups $\mathcal{X}' \longrightarrow \mathcal{X}$, $\mathcal{Y}' \longrightarrow \mathcal{Y}$ and an open immersion $\varphi' : \mathcal{Y}' \hookrightarrow \mathcal{X}'$ such that the diagram

$$\begin{array}{ccc} \mathcal{Y}' & \longrightarrow & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{X} \end{array}$$

is commutative and $\mathcal{X}' = \mathcal{Y}' \cup \mathcal{Z}'$, where $\mathcal{Z}'$ is an open subscheme of $\mathcal{X}'$ and $\mathcal{Z}'_\eta \widetilde{\to} Z$. Notice that r takes admissible formal blow-ups to birational morphisms, hence we should prove only, that red takes φ' to an isomorphism. Since the reduction $\widetilde{x}$ of x in $\mathcal{X}'_s$ is not contained in the open subset $\mathcal{Z}'_s$, it follows that the closure $V'_{\widetilde{x}}$ of $\widetilde{x}$ in $\mathcal{X}'_s$ is contained in $\mathcal{Y}'_s$, and the statement of the lemma follows. ∎

By Lemma 2.1, the functor red goes through a functor $Adm^p_S \longrightarrow Bir_{\widetilde{k}}$. Its composition with a fixed functor $Germs \longrightarrow Adm^p_S$, inverse to the equivalence $Adm^p_S \longrightarrow Germs$, and the functor $Bir_{\widetilde{k}} \longrightarrow bir_{\widetilde{k}}$ gives rise to the required functor Red: $Germs \longrightarrow bir_{\widetilde{k}}$. The image of a germ X_x under Red is a triple $(\widetilde{X}_x, \widetilde{\mathcal{H}(x)}, \varepsilon)$. Our first aim is to describe $\widetilde{X}_x$ in the case when X is a k-affinoid space. Recall that for a k-affinoid algebra $\mathcal{A}$ one sets $\widetilde{\mathcal{A}} = \mathcal{A}^\circ / \mathcal{A}^{\circ\circ}$, where $\mathcal{A}^\circ = \{f \in \mathcal{A} \big| |f|_{\sup} \le 1\}$ and $\mathcal{A}^{\circ\circ} = \{f \in \mathcal{A} \big| |f|_{\sup} < 1\}$. Each point $x \in \mathcal{M}(\mathcal{A})$ gives rise to a bounded character $\chi_x : \mathcal{A} \longrightarrow \mathcal{H}(x)$ which induces a character $\widetilde{\chi}_x : \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{H}(x)}$.

2.2. Lemma. *If $X = \mathcal{M}(\mathcal{A})$ and $x \in X$, then $\widetilde{X}_x \widetilde{\to} \mathbf{P}_{\widetilde{\mathcal{H}(x)}}\{\widetilde{\chi}_x(\widetilde{\mathcal{A}})\}$.*

Proof. By [BL], one can find an admissible affine formal model $\mathcal{X} = \mathrm{Spf}(A)$ of X. The canonical homomorphism $A \longrightarrow \mathcal{A}^\circ$ gives rise to a character $\psi : A \longrightarrow \mathcal{H}(x)$ which, in its turn, gives rise to a character $\widetilde{\psi} : \widetilde{A} = A/k^{\circ\circ}A \longrightarrow \widetilde{\mathcal{H}(x)}$. By the construction, $\widetilde{X}_x \widetilde{\to} \mathbf{P}_{\widetilde{\mathcal{H}(x)}}\{\widetilde{\psi}(\widetilde{A})\}$ and, therefore, it suffices to show that $\widetilde{\chi}_x(\widetilde{\mathcal{A}})$ is integral over $\widetilde{\psi}(\widetilde{A})$. We claim that in fact the canonical homomorphism $\widetilde{A} \longrightarrow \widetilde{\mathcal{A}}$ is finite. Indeed, take an epimorphism $k^\circ\{T\} = k^\circ\{T_1, \dots, T_n\} \longrightarrow A$. It induces epimorphisms $\widetilde{k}[T] \longrightarrow \widetilde{A}$ and

$k\{T\} \longrightarrow \mathcal{A}$. By [BGR], 6.3.5/1, the homomorphism $\widetilde{k}[T] \longrightarrow \widetilde{\mathcal{A}}$ is finite, and the claim follows. ∎

2.3. Proposition. *Let X_x be a germ.*

(i) Given an analytic subdomain $Y \subset X$ that contains x, the canonical map $\widetilde{Y}_x \longrightarrow \widetilde{X}_x$ identifies $\widetilde{Y}_x$ with an open quasi-compact subset of $\widetilde{X}_x$, and this subset depends only on the equivalence class of Y.

(ii) Given subdomains Y_x and Z_x of X_x, one has $\widetilde{Y_x \cap Z_x} = \widetilde{Y}_x \cap \widetilde{Z}_x$ and $\widetilde{Y_x \cup Z_x} = \widetilde{Y}_x \cup \widetilde{Z}_x$.

(iii) If X_x is good, then for any family of elements $f = (f_1, \dots, f_n) \subset \mathcal{O}_{X,x}$ with $|f_i(x)| \le 1$ one has $\widetilde{X_x\{f\}} = \widetilde{X}_x\{\widetilde{f}\}$, where $\widetilde{f}_i$ is the image of f_i in $\widetilde{\mathcal{H}(x)}$.

Proof. We may assume that $X = \mathcal{X}_\eta$ for an admissible formal scheme $\mathcal{X}$. To verify (i), we apply again [BL], 4.4, and get an admissible formal blow-up $\mathcal{X}' \longrightarrow \mathcal{X}$ and an open formal subscheme $\mathcal{Y}' \subset \mathcal{X}'$ with $\mathcal{Y}'_\eta \widetilde{\rightarrow} Y$. Then (i) follows from the construction of the reduction functor and Lemma 2.1. The statement (ii) is verified in the same way.

(iii) We may assume that $X = \mathrm{Spf}(\mathcal{A})$ and $n = 1$. By Lemma 2.2, it suffices to check that $\widetilde{\chi}_x(\widetilde{\mathcal{A}\{f\}})$ is finite over $\widetilde{\chi}_x(\widetilde{\mathcal{A}})[\widetilde{f}]$. As in the proof of Lemma 2.2, we apply [BGR], 6.3.5/1, to the surjective bounded homomorphism $\mathcal{A}\{T\} \longrightarrow \mathcal{A}\{f\}$ that takes T to f. It follows that the induced homomorphism $\widetilde{\mathcal{A}}[T] \longrightarrow \widetilde{\mathcal{A}\{f\}}$ is finite. This implies the required fact. ∎

2.4. Theorem. *Given a germ X_x, the reduction functor establishes a one-to-one correspondence between subdomains of X_x and open quasi-compact subsets of $\widetilde{X}_x$.*

Proof. In Steps 1-4 we assume that the germ X_x is good, and prove the theorem in this case. The general case is deduced from the particular one in Step 5.

Step 1. *Any subdomain Y_x of X_x has a finite covering by subdomains of the form $X_x\{f_1, \dots, f_n\}$ with $f_i \in \mathcal{O}_{X,x}$ and $|f_i(x)| = 1$.* We may assume that $X = \mathcal{M}(\mathcal{A})$ is k-affinoid, and Y is a compact analytic domain in X. The Gerritzen-Grauert theorem ([BGR], 7.3.5/3) implies that Y is a finite union of rational domains and, therefore, we may assume that Y is a rational domain, i.e., $Y = X\{\frac{g_1}{h}, \dots, \frac{g_n}{h}\}$, where $g_1, \dots, g_n, h$ are elements of $\mathcal{A}$ without common zeros in X. Since $x \in Y$, it follows that $h(x) \neq 0$. We can therefore shrink X and assume that $h \in \mathcal{A}^*$, i.e., $Y = X\{f_1, \dots, f_n\}$ for $f_i = \frac{g_i}{h} \in \mathcal{A}$. If $|f_i(x)| < 1$ for some i, then we can shrink X so that $|f_i(x')| < 1$ for all $x' \in X$, i.e., we may remove all f_i's with $|f_i(x)| < 1$.

Step 2. *Let $f = (f_1, \dots, f_l)$ and $g = (g_1, \dots, g_m)$ be two families of elements of $\mathcal{O}_{X,x}$ with $|f_i(x)| = |g_j(x)| = 1$ and suppose, that $\widetilde{X}_x\{\widetilde{f}\} \subset \widetilde{X}_x\{\widetilde{g}\}$, then $X_x\{f\} \subset X_x\{g\}$.* We can assume, that $X = \mathcal{M}(\mathcal{A})$ is k-affinoid and $f_i, g_j \in \mathcal{A}$. Let $\chi_x : \mathcal{A} \longrightarrow \mathcal{H}(x)$ be the character of x and set $B = \widetilde{\chi_x}(\widetilde{\mathcal{A}}) \subset \widetilde{\mathcal{H}(x)}$, then Lemma 2.2 and Proposition 2.3 (iii) imply, that $\widetilde{X}_x \widetilde{\rightarrow} \mathbf{P}_{\widetilde{\mathcal{H}(x)}}\{B\}$,

$\widetilde{X_x\{f\}} \widetilde{\to} \mathbf{P}_{\widetilde{\mathcal{H}(x)}}\{B[\widetilde{f}]\}$ and $\widetilde{X_x\{g\}} \widetilde{\to} \mathbf{P}_{\widetilde{\mathcal{H}(x)}}\{B[\widetilde{g}]\}$. Since $\widetilde{X}_x\{\widetilde{f}\} = \widetilde{X}_x\{\widetilde{f}, \widetilde{g}\}$, Lemma 1.1 implies, that $B[\widetilde{f}, \widetilde{g}]$ is integral over $B[\widetilde{f}]$. Each element $\widetilde{g}_j$ satisfies an equation of the form $\widetilde{g}_j^n + \sum_{k=0}^{n-1} \widetilde{a}_k \widetilde{g}_j^k = 0$, where $\widetilde{a}_k \in B[\widetilde{f}]$. The coefficients $\widetilde{a}_k$ may be lifted to elements $a_k \in (\mathcal{A}\{f\})^\circ$ and obviously $|(g_j^n + \sum_{k=0}^{n-1} a_k g_j^k)(x)| < 1$. The last inequality holds also in a neighborhood V of x in $X\{f\}$ and $|a_k| \leq 1$ in $X\{f\}$, therefore $|g_j| \leq 1$ in V, i.e. $X_x\{f\} \subset X_x\{g\}$.

Step 3. *Let Y_x be a subdomain of X_x such that $\widetilde{Y}_x \widetilde{\to} \widetilde{X}_x$, then $Y_x = X_x$.* By Step 1, Y_x has a finite covering by subdomains of the form $X_x\{f_1, \ldots, f_n\}$ (where $|f_p(x)| = 1$), say $Y_x = \cup_{i=1}^m V_i$. Our assumption implies, that $\{\widetilde{V}_i\}_{1 \leq i \leq m}$ is a covering of $\widetilde{X}_x$. By Lemma 1.5, this covering has a Laurent refinement $\{\widetilde{U}_j\}_{j \in \{\pm 1\}^l} = \{\widetilde{X}_x\{\widetilde{g}_1^{j_1}, \ldots, \widetilde{g}_l^{j_l}\}\}_j$ (i.e. for any $j \in \{\pm 1\}^l$, $\widetilde{U}_j \subset \widetilde{V}_{i(j)}$). Let $g_q \in \mathcal{O}_{X,x}$ be some liftings of $\widetilde{g}_q$, then $\{U_j\}_{j \in \{\pm 1\}^l} = \{X_x\{g_1^{j_1}, \ldots, g_l^{j_l}\}\}_j$ is a covering of X_x whose reduction coincides with $\{\widetilde{U}_j\}_j$. By the previous step, for any $j \in \{\pm 1\}^l$ we have $U_j \subset V_{i(j)}$, hence $\{V_i\}_i$ is also a covering of X_x, i.e. $X_x = Y_x$.

Step 4. *The theorem holds if the germ X_x is good.* It suffices to prove the following two statements: (1) any open quasi-compact subset $\widetilde{Y}_x$ of $\widetilde{X}_x$ is a reduction of some subdomain of X_x, and (2) if the reductions of two subdomains Y_x and Z_x of X_x coincide, then the subdomains are equal. To prove the first statement, find a representation $\widetilde{Y}_x = \cup_i \widetilde{X}_x\{\widetilde{f}_{i,1}, \ldots, \widetilde{f}_{i,n_i}\}$ and let $f_{i,j} \in \mathcal{O}_{X,x}$ be some liftings, then $\cup_i X_x\{f_{i,1}, \ldots, f_{i,n_i}\}$ is a lifting of $\widetilde{Y}_x$. Suppose now, that $\widetilde{Y}_x = \widetilde{Z}_x$. Find representations $Y_x = \cup_i Y_x^i = \cup_{i=1}^p X_x\{f_{i,1}, \ldots, f_{i,m_i}\}$ and $Z_x = \cup_j Z_x^j = \cup_{j=1}^q X_x\{g_{j,1}, \ldots, g_{j,n_j}\}$, then $\widetilde{Y}_x = \cup_i \widetilde{Y}_x^i = \cup_j \widetilde{Z}_x^j$. Therefore for any fixed $i \in \{1, \ldots, p\}$, the sets $\widetilde{Y_x^i \cap Z_x^j}$ $(1 \leq j \leq q)$ form a covering of $\widetilde{Y}_x^i$. By the previous step, the sets $Y_x^i \cap Z_x^j$ cover Y_x^i (obviously, the germs Y_x^i are good). It follows, that $Y_x \subset Z_x$, and by the symmetry the converse inclusion is also satisfied.

Step 5. *The general case.* Let $\{X_x^i\}_{i \in I}$ be a good covering of X_x. Again, it suffices to check the conditions (1) and (2) from the previous step. Let $\widetilde{Y}_x$ be an open quasi-compact subset of $\widetilde{X}_x$. By the previous step we can lift all sets $\widetilde{Y}_x \cap \widetilde{X}_x^i$ to subdomains Y_x^i, then the union of all Y_x^i's is the required lifting of $\widetilde{Y}_x$. Suppose now, that for subdomains Y_x and Z_x of X_x we have $\widetilde{Y}_x = \widetilde{Z}_x$. Then for any $i \in I$ we have $\widetilde{Y_x \cap X_x^i} = \widetilde{Y}_x \cap \widetilde{X}_x^i = \widetilde{Z}_x \cap \widetilde{X}_x^i = \widetilde{Z_x \cap X_x^i}$. By the case of a good germ, the liftings $Y_x \cap X_x^i$ and $Z_x \cap X_x^i$ coincide, and since X_x^i cover X_x, we obtain, that $Y_x = Z_x$. ∎

2.5. Proposition. *A morphism of germs $f : X_x \longrightarrow Y_y$ is separated if and only if its image under* Red *is separated.*

Proof. The direct implication follows from [BL], 4.7, so we shall prove the converse one. We can assume, that X and Y are compact, let $\overline{f} : (\mathcal{X}, x)$

$\longrightarrow (\mathcal{Y}, y)$ be a morphism of Adm^p inducing f. Let $V_{\widetilde{x}}$ (resp. $V_{\widetilde{y}}$) be the closure of $\widetilde{x}$ (resp. $\widetilde{y}$) in $\mathcal{X}_s$ (resp. $\mathcal{Y}_s$), by our assumption the induced morphism $\overline{f}_s : V_{\widetilde{x}} \longrightarrow V_{\widetilde{y}}$ is separated. Since $V_{\widetilde{x}}$ is a closed subscheme of $\mathcal{X}_s$, its preimage X_0 in X is open. Let $Z \subset X_0$ be a compact neighborhood of x, by [BL], 4.4., we can find an admissible formal blow up $\phi : \mathcal{X}' \longrightarrow \mathcal{X}$ together with an open subscheme $\mathcal{Z}' \subset \mathcal{X}'$ such that $\mathcal{Z}'_\eta \widetilde{\rightarrow} Z$. Let U be the preimage of $V_{\widetilde{x}}$ under ϕ_s, the morphism $U \longrightarrow V_{\widetilde{x}}$ is proper, hence the composition morphism $U \longrightarrow V_{\widetilde{y}}$ is separated. But $\mathcal{Z}'$ is obviously an open subscheme of U, hence the morphism $\mathcal{Z}' \longrightarrow \mathcal{Y}$ is separated. Now [BL], 4.7, implies that the morphism $Z \longrightarrow Y$ is separated and the lemma follows. ■

2.6. Remark. The reduction functor Red may be defined also as follows. One can attach to any analytic space X an adic space X^{ad} of R. Huber (see [Hu], 1.1.11), where instead of valuations of height one arbitrary valuations are considered. Now to any point $x \in X$ we attach the set of all its specializations (see [Hu], 1.1.9) with the induced topology, it coincides with the closure $\overline{x}$ of x. Notice, that to any specialization y of x corresponds some valuation $\nu(y)$ on the field $\widetilde{\mathcal{H}(x)}$ and the triple $(\overline{x}, \widetilde{\mathcal{H}(x)}, \nu)$ defines an object of $bir_{\widetilde{k}}$.

In [BGR], 9.6.2., given a morphism of affinoid spaces $X \longrightarrow Y$, one defines a notion of relatively compact affinoid subdomains of X (notation $U \Subset_Y X$). Given a separated analytic space Y and its affinoid subdomains $X' \subset X$, we say that X' is relatively compact in X over Y and denote $X' \Subset_Y X$, if for any affinoid domain $Z \subset Y$ the affinoid domains $X_Z = X \cap Z$ and $X'_Z = X' \cap Z$ satisfy $X'_Z \Subset_Z X_Z$.

2.7. Remark. In [Ber2], §1.6, a fully faithful functor $X \mapsto X_0$ from the category of Hausdorff analytic spaces to the category of (quasi-separated) rigid analytic spaces was constructed. One can easily see that a separated analytic space X is good if and only if the corresponding rigid space X_0 possesses the following property: there exist admissible affinoid coverings $\{U_0^i\}_{i\in I}$ and $\{V_0^i\}_{i\in I}$ such that $U_0^i \Subset_{X_0} V_0^i$ for all $i \in I$.

§3. A characterization of good germs

3.1. Theorem. *A germ X_x is good if and only if its reduction $\widetilde{X}_x = (\widetilde{X}_x, \widetilde{\mathcal{H}(x)}, \varepsilon)$ is affine.*

Proof. The direct implication follows from Lemma 2.2. To establish the converse implication, we shall reduce it, first of all, to a problem of showing that, under certain conditions, the analytic space obtained by gluing two affinoid spaces along an affinoid subdomain is affinoid. Assume that $\widetilde{X}_x$ is an affine subset of $\mathbf{P}_{\widetilde{\mathcal{H}(x)}}$. By Proposition 2.5, we may assume that X is a compact separated analytic space. Take a finite affinoid covering of X. It gives rise to a finite affine covering of $\widetilde{X}_x$. By Lemma 1.5, the latter has a Laurent refinement $\{V_\alpha = \widetilde{X}_x\{\lambda_1^{\alpha_1}, \dots, \lambda_n^{\alpha_n}\}\}_{\alpha\in\{\pm 1\}^n}$, where $\lambda_1, \dots, \lambda_n$ are non-zero elements of $\widetilde{\mathcal{H}(x)}$. By Theorem 2.4, each V_α is the reduction $\widetilde{Y}_x^{(\alpha)}$ of some affinoid subdomain $Y^{(\alpha)} \subset X$, and we can shrink X and assume that $X = \cup_\alpha Y^{(\alpha)}$. Induction on n reduces the theorem to verification of the following fact.

Given a separated analytic space X and a point $x \in X$, assume that $\widetilde{X}_x$ is affine and that X is a union of two affinoid subdomains Y and Z such that $x \in Y \cap Z$, $\widetilde{Y}_x = \widetilde{X}_x\{\lambda\}$ and $\widetilde{Z}_x = \widetilde{X}_x\{\lambda^{-1}\}$ for a non-zero element $\lambda \in \widetilde{\mathcal{H}(x)}$. Then the point x has an affinoid neighborhood in X.

In the construction which follows we replace X by a subdomain of the form $Y' \cup Z'$, where Y' and Z' are affinoid neighborhoods of the point x in Y and Z, respectively. (Such a subdomain is a neighborhood of x in X.) Let $Y = \mathcal{M}(\mathcal{B})$, $Z = \mathcal{M}(\mathcal{C})$ and $Y \cap Z = \mathcal{M}(\mathcal{A})$.

Step 1. *One can shrink X so that the following is true. There exist surjective homomorphisms*

$$k\{T_1, \dots, T_n, S_1, r^{-1}S_2\} \longrightarrow \mathcal{B}: \ T_i \mapsto f_i, \ S_1 \mapsto f, \ S_2 \mapsto f^{-1}$$

$$k\{T_1, \dots, T_n, r^{-1}S_1, S_2\} \longrightarrow \mathcal{C}: \ T_i \mapsto g_i, \ S_1 \mapsto g, \ S_2 \mapsto g^{-1}$$

such that $r > 1$, $r \in \sqrt{|k|}$, $Y \cap Z = Y\{f^{-1}\} = Z\{g\}$ and all of the numbers $||f_i - g_i||$ and $||f - g||$ are strictly less than 1, where $\| \ \|$ denotes the quotient norm on $\mathcal{A}$ induced from the canonical norm of $k\{T_1, \dots, T_n, S_1, S_2\}$ with respect to the surjective homomorphism $k\{T_1, \dots, T_n, S_1, S_2\} \longrightarrow \mathcal{A} : T_i \mapsto f_i$, $S_1 \mapsto f$, $S_2 \mapsto f^{-1}$. (The latter homomorphism is surjective because $Y \cap Z = Y\{f^{-1}\}$.)

Shrinking X, we can find invertible elements $f \in \mathcal{B}$ and $g \in \mathcal{C}$ with $|f|_Y \le 1$, $|g^{-1}|_Z \le 1$ and $\lambda = \widetilde{f(x)} = \widetilde{g(x)}$ (here $|f|_Y = \max_{y \in Y} |f(y)|$). Since $|(f-g)(x)| < 1$, we can shrink X so that $|f - g|_{Y \cap Z} < 1$ and, in particular, $Y \cap Z \subset Y\{f^{-1}\} \cap Z\{g\}$.

Since the reductions of the germs of $Y \cap Z$, $Y\{f^{-1}\}$ and $Z\{g\}$ at the point x coincide, there are affinoid neighborhoods $Y' = \mathcal{M}(\mathcal{B}')$ and $Z' = \mathcal{M}(\mathcal{C}')$ of x in Y and Z, respectively, such that $Y' \cap Z = Y'\{f^{-1}\}$ and $Y \cap Z' = Z'\{g\}$. Furthermore, since $Y' \cap Z'$ is an affinoid neighborhood of x in $Y \cap Z$, we can find a Laurent neighborhood W of x in $Y \cap Z$ which is contained in $Y' \cap Z'$ and is of the form $(Y \cap Z)\{u_i, v_j^{-1}\}$ with $u_i, v_j \in \mathcal{A}$, $|u_i(x)| < 1$ and $|v_j(x)| > 1$. Since W is also a Laurent neighborhood of x in $Y' \cap Z = Y'\{f^{-1}\}$ and $Y \cap Z' = Z'\{g\}$, and the latter are Weierstrass domains in Y' and Z', there are elements $u_i', v_j' \in \mathcal{B}'$ and $u_i'', v_j'' \in \mathcal{C}'$ sufficiently closed to u_i, v_j over $Y' \cap Z$ and $Y \cap Z'$ so that $Y'' = Y'\{u_i', v_j'\}$ and $Z'' = Z'\{u_i'', v_j''\}$ are neighborhoods of x in Y' and Z', respectively, and $W = Y'' \cap (Y' \cap Z) = Z'' \cap (Y \cap Z')$. It follows that $W = Y''\{f^{-1}\} = Z''\{g\} = Y'' \cap Z''$. Thus, we can replace Y by Y'' and Z by Z'' and assume that $Y \cap Z = Y\{f^{-1}\} = Z\{g\}$.

Since $\widetilde{X}_x$ is affine, it coincides with $\mathbf{P}_{\widetilde{\mathcal{H}(x)}}\{\alpha_1, \dots, \alpha_m\}$ for non-zero elements $\alpha_i \in \widetilde{\mathcal{H}(x)}$. We can shrink X so that $\alpha_i = \widetilde{f_i(x)}$ for elements

$f_i \in \mathcal{B}^\circ$. Let $\mathcal{D} = k\{T_1, \dots, T_m, S_1, r^{-1}S_2\}$, where $r \in \sqrt{|k|}$ is a number with $r > \max |f^{-1}|_Y$. Notice that $\widetilde{\mathcal{D}} = \widetilde{k}[T_1, \dots, T_m, S_1, U]$, where U is the image of the element $\frac{S_2^m}{a}$, m is the minimal positive integer with $r^m \in |k|$, and $a \in k$ is such that $r^m = |a|$. Let $\phi : \mathcal{D} \longrightarrow \mathcal{B}$ be the continuous homomorphism that takes T_i to f_i, S_1 to f and S_2 to f^{-1}, and let φ be its composition with the character $\chi_x : \mathcal{B} \longrightarrow \mathcal{H}(x)$. Since $|\varphi(\frac{S_2^m}{a})(x)| = \frac{|f(x)|^m}{|a|} = \frac{1}{r^m} < 1$, one has $\widetilde{\varphi}(U) = 0$ and, therefore, $\widetilde{\varphi}(\widetilde{\mathcal{D}}) = \widetilde{k}[\alpha_1, \dots, \alpha_m, \lambda]$. On the other hand, one has $\widetilde{Y}_x = \mathbf{P}_{\widetilde{\mathcal{H}(x)}}\{\alpha_1, \dots, \alpha_m, \lambda\}$. From Lemmas 2.2 and 1.1 it follows that the algebra $\widetilde{\chi}_x(\widetilde{\mathcal{B}})$ is finite over $\widetilde{\varphi}(\widetilde{\mathcal{D}})$. By [Ber1], 2.5.2(d), the character χ_x is inner with respect to $\mathcal{D}$. From *loc. cit.* it follows that ϕ can be extended to a continuous epimorphism

$$k\{T_1, \dots, T_m, S_1, r^{-1}S_2, U_1, \dots, U_p\} \longrightarrow \mathcal{B} : U_i \mapsto u_i$$

with $|u_i(x)| < 1$ for all $1 \le i \le p$.

We provide $\mathcal{A}$ with the quotient norm $\| \ \|$ induced from the canonical norm on $k\{T_1, \dots, T_m, S_1, S_2, U_1, \dots, U_q\}$. Since $Y \cap Z$ is a Weierstrass domain in Z, we can find elements $g_1, \dots g_m, g' \in \mathcal{C}$ with $||g_i - f_i|| < 1$ and $||g' - f|| < 1$ (then $\alpha_i = \widetilde{g_i(x)}$ and $\lambda = \widetilde{g'(x)}$). Since $|g' - g|_{Y \cap Z} < 1$, we can replace Z by a Weierstrass domain which is a neighborhood of $Y \cap Z$ so that $|g' - g|_Z < 1$. We therefore can replace g by g' so that the equality $Z\{g\} = Y \cap Z$ remains to be true. In the same way as above one constructs for $r > \max |g|_Z$, $r \in \sqrt{|k|}$, a continuous epimorphism

$$k\{T_1, \dots, T_m, r^{-1}S_1, S_2, V_1, \dots, V_q\} \longrightarrow \mathcal{C} : T_i \mapsto g_i, S_1 \mapsto g, S_2 \mapsto g^{-1}, V_j \mapsto v_j$$

with $|v_j(x)| < 1$ for all $1 \le j \le q$.

Since $Y \cap Z$ is a Weierstrass domain in both Y and Z, we can find elements $v'_1, \dots, v'_q \in \mathcal{B}$ and $u'_1, \dots, u'_p \in \mathcal{C}$ with $\|u_i - u'_i\| < 1$ and $\|v_j - v'_j\| < 1$. The affinoid domains $Y' = Y\{v'_1, \dots, v'_q\}$ and $Z' = Z\{u'_1, \dots, u'_p\}$ are neighborhoods of the point x in Y and Z, respectively, and $Y'\{f^{-1}\} = Y \cap Z = Z'\{g\}$. Thus, we can replace Y by Y' and Z by Z', and find the required epimorphisms with

$$(f_1, \dots, f_n) = (f_1, \dots, f_m, u_1, \dots, u_p, v'_1, \dots, v'_q)$$
$$(g_1, \dots, g_n) = (g_1, \dots, g_m, u'_1, \dots, u'_p, v_1, \dots, v_q)$$

(Notice that the new epimorphism induces a norm on $\mathcal{A}$, which is majorated by the old one, and so all inequalities involving the norm of $\mathcal{A}$ are also true for the new norm.)

Step 2. *An analytic space X satisfying conditions of Step 1 is affinoid.*

Since this step is of its own interest and will be used in §5, we formulate it as a lemma in a slightly more general form. Let $\mathcal{K}$ be an affinoid algebra and $\theta : k\{T'_1, \dots, T'_m\} \longrightarrow \mathcal{K}$ an admissible epimorphism, we provide the algebra $\mathcal{K}$ with the quotient norm (say $|| \ ||_{\mathcal{K}}$) and the algebras $\mathcal{K}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ with the following norm, $|| \sum a_i T^i || = \max(r^i \|a_i\|_{\mathcal{K}})$.

3.2. Lemma. *Let X be an analytic space over $\mathcal{M}(\mathcal{K})$ which is a union of two affinoid domains $Y = \mathcal{M}(\mathcal{B})$ and $Z = \mathcal{M}(\mathcal{C})$ such that the intersection $Y \cap Z = \mathcal{M}(\mathcal{A})$ is affinoid. Assume that there exist positive numbers $r_1, \dots, r_n, p, q \in \sqrt{|k|}$ with $p \le 1 \le q$ and continuous $\mathcal{K}$-epimorphisms*

$$\mathcal{K}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, q^{-1}S, S^{-1}\} \xrightarrow{\psi} \mathcal{C} : \ T_i \mapsto g_i, \ S \mapsto g$$

$$\mathcal{K}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, S, pS^{-1}\} \xrightarrow{\phi} \mathcal{B} : \ T_i \mapsto f_i, \ S \mapsto f$$

(resp. $\mathcal{K}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, S\} \xrightarrow{\phi} \mathcal{B} : \ T_i \mapsto f_i, \ S \mapsto f$) such that $Y \cap Z = Y\{f^{-1}\} = Z\{g\}$ and $||f_i - g_i|| < r_i$, $||f - g|| < 1$, where the norm $\| \ \|$ on $\mathcal{A}$ is the quotient norm induced by the epimorphism $\mathcal{K}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, S, S^{-1}\} \longrightarrow \mathcal{A} : T_i \mapsto f_i, S \mapsto f$. Then

(i) the space X is affinoid, let $X = \mathcal{M}(\mathcal{D})$;

(ii) for any positive δ, there exists a continuous $\mathcal{K}$-epimorphism

$$\mathcal{K}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, q^{-1}S, pS^{-1}\} \longrightarrow \mathcal{D} : T_i \mapsto h_i, S \mapsto h$$

(resp. $\mathcal{K}\{r_1^{-1}, \dots, r_n^{-1}T_n, q^{-1}S\} \longrightarrow \mathcal{D} : T_i \mapsto h_i, S \mapsto h$) such that $||f_i - h_i||_{\mathcal{B}} < ||f_i - g_i|| + \delta$, $||g_i - h_i||_{\mathcal{C}} < ||f_i - g_i|| + \delta$, $||f - h||_{\mathcal{B}} < ||f - g|| + \delta$ and $||g - h||_{\mathcal{C}} < ||f - g|| + \delta$, where the norms on $\mathcal{B}$ and $\mathcal{C}$ are the quotient norms induced by ϕ and ψ, respectively.

In our case $\mathcal{K} = k$. Note that the above norm on $\mathcal{A}$ coincides with the quotient norm induced from $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, S_1, S_2\}$ with respect to the epimorphism $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, S_1, S_2\} \longrightarrow \mathcal{A} : T_i \mapsto f_i$, $S_1 \mapsto f$, $S_2 \mapsto f^{-1}$, and so Lemma 3.2 can be applied to the statement of Step 2.

Proof. We suppose that the source of ϕ is $\mathcal{K}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, S, pS^{-1}\}$, the second case is treated in the same way.

Case 1: *$\mathcal{K} = k$ and the homomorphisms ϕ and ψ are isomorphisms.* (In this case the proof is very close to [BGR], 9.7.1.) Notice that the canonical homomorphisms $\mathcal{B} \longrightarrow \mathcal{A}$ and $\mathcal{C} \longrightarrow \mathcal{A}$ are injective. Each element of $\mathcal{B}$ (resp. $\mathcal{C}$; resp. $\mathcal{A}$) has a unique representation in the form $\sum_{\nu \ge 0} \sum_{j=-\infty}^{\infty} \lambda_{\nu,j} T^\nu S^j$, where $\nu = (\nu_1, \dots \nu_n)$, and the spectral norm of such an element is equal to $\max\{\max_{\nu \ge 0, j \ge 0} \alpha^j r^\nu |\lambda_{\nu,j}|, \max_{\nu \ge 0, j < 0} \beta^j r^\nu |\lambda_{\nu,j}|\}$, where $\alpha = 1$ and $\beta = p^{-1}$ (resp. $\alpha = q$ and $\beta = 1$; resp. $\alpha = \beta = 1$). Let $\mathcal{B}_+$ and $\mathcal{C}_-$ be the subspaces of $\mathcal{B}$ and $\mathcal{C}$ consisting of elements of the form $\sum_{\nu \ge 0, j > 0} \lambda_{\nu,j} f_1^{\nu_1} \dots f_n^{\nu_n} f^j$ and $\sum_{\nu \ge 0, j \le 0} \lambda_{\nu,j} g_1^{\nu_1} \dots g_n^{\nu_n} g^j$, respectively. *We claim that each element $a \in \mathcal{A}$ can be represented in the form $a = b + c$ with $b \in \mathcal{B}_+$, $c \in \mathcal{C}_-$ and $||b||_{\mathcal{B}}, ||c||_{\mathcal{C}} \le ||a||$.* Indeed, let $0 < \varepsilon < 1$ be a number with $||f_i - g_i|| < \varepsilon ||f_i||$ and $||f - g|| < \varepsilon ||f||$. Since the spectral norm on $\mathcal{A}$ is multiplicative, it follows that $||f_1^{\nu_1} \dots f_n^{\nu_n} f^j - g_1^{\nu_1} \dots g_n^{\nu_n} g^j|| < \varepsilon ||f_1^{\nu_1} \dots f_n^{\nu_n} f^j||$ for all $\nu \ge 0$ and j. Let now $a = \sum_{\nu \ge 0} \sum_{j=-\infty}^{\infty} \lambda_{\nu,j} f_1^{\nu_1} \dots f_n^{\nu_n} f^j$ be an element of $\mathcal{A}$. Setting $b' = \sum_{\nu \ge 0, j > 0} \lambda_{\nu,j} f_1^{\nu_1} \dots f_n^{\nu_n} f^j \in \mathcal{B}_+$ and $c' = \sum_{\nu \ge 0, j \le 0} \lambda_{\nu,j} g_1^{\nu_1} \dots g_n^{\nu_n} g^j \in \mathcal{C}_-$, we get $||a - b' - c'|| < \varepsilon ||a||$. Applying the same procedure to the element $a' = a - b' - c'$ and iterating it, we get the claim.

By the above claim, one has $f_i - g_i = b_i + c_i$ and $f - g = b + c$, where $b_i, b \in \mathcal{B}_+$, $c_i, c \in \mathcal{C}_-$, $||b_i||_\mathcal{B}, ||c_i||_\mathcal{C} \le ||f_i - g_i||$ and $||b||_\mathcal{B}, ||c||_\mathcal{C} \le ||f - g||$. Then the elements $h_i = f_i - b_i$ and $h = f - b$ are contained in $\mathcal{D} := \mathcal{B} \cap \mathcal{C}$, and satisfy the condition (ii) of the lemma. It follows (see [BGR], 9.7.1/1 and 9.7.1/2) that the continuous homomorphisms $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, S, pS^{-1}\} \longrightarrow \mathcal{B}$ and $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, q^{-1}S, S\} \longrightarrow \mathcal{C}$ that take T_i to h_i and S to h are isomorphisms, and therefore $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, q^{-1}S, pS^{-1}\} \widetilde{\rightarrow} \mathcal{D}$. This gives rise to an isomorphism of analytic spaces $X \widetilde{\rightarrow} \mathcal{M}(\mathcal{D})$.

Case 2: $\mathcal{K} = k$. We set $\mathcal{A}' = k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, S, S^{-1}\}$ and consider the surjective homomorphism $\mathcal{A}' \longrightarrow \mathcal{A} : T_i \mapsto f_i, S \mapsto f$. We can find preimages G_i and G in $\mathcal{A}'$ of the elements g_i and g such that $||T_i - G_i||_{\mathcal{A}'} < \min(||f_i - g_i|| + \delta, r_i)$ and $||S - G||_{\mathcal{A}'} < \min(||f - g|| + \delta, 1)$. Notice that the homomorphism $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, S, S^{-1}\} \longrightarrow \mathcal{A}' : T_i \mapsto G_i$, $S \mapsto G$ is an isomorphism. Let $\mathcal{B}' = k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, S, pS^{-1}\}$ and $\mathcal{C}' = k\{r_1^{-1}G_1, \dots, r_n^{-1}G_n, q^{-1}G, G^{-1}\}$ be subalgebras of $\mathcal{A}'$, set $Y' = \mathcal{M}(\mathcal{B}')$ and $Z' = \mathcal{M}(\mathcal{C}')$. By the construction there are canonical isomorphism $Y'\{S^{-1}\} \widetilde{\rightarrow} Z'\{G\} \widetilde{\rightarrow} \mathcal{M}(\mathcal{A}')$ and, by the previous case, the space X' obtained by gluing Y' and Z' along $\mathcal{M}(\mathcal{A}')$ is affinoid. The homomorphisms $\mathcal{B}' \longrightarrow \mathcal{A}$ and $\mathcal{C}' \longrightarrow \mathcal{A}$ factors through $\mathcal{B}$ and $\mathcal{C}$, respectively, and give rise to closed immersions $Y \longrightarrow Y'$ and $Z \longrightarrow Z'$. The latter give rise to the same closed immersion $Y \cap Z \longrightarrow Y' \cap Z'$ and, therefore, to a closed immersion $X \longrightarrow X'$. It follows that the space X is also affinoid. Let $X = \mathcal{M}(\mathcal{D})$ and $X' = \mathcal{M}(\mathcal{D}')$. By the first step, $\mathcal{D}' \widetilde{\rightarrow} k\{r_1^{-1}H_1, \dots, r_n^{-1}H_n, q^{-1}H, pH^{-1}\}$ and $||H_i - G_i||_{\mathcal{C}'} < ||f_i - g_i|| + \delta$, $||H_i - T_i||_{\mathcal{B}'} < ||f_i - g_i|| + \delta$, $||H - G||_{\mathcal{C}'} < ||f - g|| + \delta$ and $||H - S||_{\mathcal{B}'} < ||f - g|| + \delta$. So the epimorphism $\mathcal{D}' \longrightarrow \mathcal{D}$ and the images of H_i and H in $\mathcal{D}$ satisfy the claim of the lemma.

Case 3: $\mathcal{K}$ *is arbitrary*. Note, that θ and ϕ induce an epimorphism

$$\phi' : k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, T'_1, \dots, T'_m, S, pS^{-1}\} \longrightarrow \mathcal{B} : T_i \mapsto f_i, T'_j \mapsto \theta(T'_j), S \mapsto f$$

and the norm induced by ϕ' coincides with the norm induced by ϕ. Define ψ' analogously. Note, that ϕ' and ψ' satisfy the conditions of the lemma. By the previous step, X is affinoid and we can find an epimorphism

$$k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, T'_1, \dots, T'_m, q^{-1}S, pS^{-1}\} \longrightarrow \mathcal{D} : T_i \mapsto h_i, T'_j \mapsto h'_j, S \mapsto h$$

(where h_i satisfy the required inequalities). Moreover, since $\phi'(T'_j) = \psi'(T'_j) = \theta(T'_j)$, we can choose the above epimorphism such that $h'_j = \theta(T'_j)$ (see the construction from step 1). But then we obtain a $\mathcal{K}$-epimorphism

$$\mathcal{K}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, q^{-1}S, pS^{-1}\} \longrightarrow \mathcal{D} : T_i \mapsto h_i, S \mapsto h$$

which satisfies all claims of the lemma. ∎

§4. A characterization of closed morphisms

Recall that the notion of a closed morphism was introduced in [Ber1], §2.5 and §3.1, for good analytic spaces, and it was extended in [Ber2], §1.5, for arbitrary analytic spaces as follows. A morphism $Y \longrightarrow X$ is closed if, for any morphism $X' \longrightarrow X$ from a good space X' defined over a non-Archimedean field $K \supset k$, the space $Y' = Y \times_X X'$ is a good K-analytic space and the induced morphism $Y' \longrightarrow X'$ is closed. The notion of a closed morphism we work with here is the above one restricted to the category of strictly analytic spaces. (It is a priori broader than that from [Ber2], 1.5.3, but it is very likely that both notions are equivalent.) Recall that a morphism of germs $Y_y \longrightarrow X_x$ is said to be *closed* if it is induced by a closed morphism $Y' \longrightarrow X$, where Y' is an open neighborhood of y in Y. In what follows, the reduction of a germ X_x will be denoted by $\widetilde{X}_x$ (instead of $(\widetilde{X}_x, \widetilde{\mathcal{H}(x)}, \varepsilon)$).

4.1. Theorem. *A morphism of germs $\varphi_y : Y_y \longrightarrow X_x$ is closed if and only if the induced morphism between their reductions $\widetilde{\varphi}_y : \widetilde{Y}_y \longrightarrow \widetilde{X}_x$ is proper.*

Given an extension of non-Archimedean fields K/k, let $\mathcal{E}_{K/k}$ denote the natural functor $bir_{\widetilde{k}} \longrightarrow bir_{\widetilde{K}}$.

4.2. Lemma. *Let X be a k-analytic space, K a non-Archimedean field over k, $y \in Y = X\widehat{\otimes}K$ a point and x its image in X. Then $\widetilde{Y}_y \widetilde{\rightarrow} \mathcal{E}_{K/k}(\widetilde{X}_x)$.*

Proof. It suffices to check the case of an affinoid space X. Let $\mathrm{Spf}(A)$ be an affine admissible formal model of X and $\widetilde{A} = A/k^{\circ\circ}A$, then $\widetilde{X}_x$ may be obtained from $\mathrm{Spec}(\widetilde{A})$ by the construction of the reduction functor red from §2. Notice also, that $\mathrm{Spf}(A\widehat{\otimes}K^\circ)$ is a formal model of $X\widehat{\otimes}K$ and that the natural homomorphism $\phi : \widetilde{A} \otimes \widetilde{K} \longrightarrow \widetilde{A\widehat{\otimes}K^\circ}$ induces a morphism $\widetilde{Y}_y \longrightarrow \mathcal{E}_{K/k}(\widetilde{X}_x)$. Finally, notice that ϕ is finite (it is even surjective) and, therefore, the latter morphism is an isomorphism. ∎

4.3. Lemma. *Given a cartesian diagram of morphisms of k-analytic spaces X and Y and K-analytic spaces X' and Y' (where $K \supset k$)*

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

and a point $y' \in Y'$ whose images in X', Y and X are x', y and x, respectively, assume that the morphism $\widetilde{Y}_y \longrightarrow \widetilde{X}_x$ is proper. Then the morphism $\widetilde{Y}'_{y'} \longrightarrow \widetilde{X}'_{x'}$ is also proper.

Proof. The diagram appearing in the statement may be factored as follows

$$\begin{array}{ccccc} Y' & \longrightarrow & X'\widehat{\otimes}K & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & X\widehat{\otimes}K & \longrightarrow & X \end{array}$$

(left and right squares are cartesian). Thus we should prove the lemma in the two particular cases: (1) all spaces X, Y, X' and Y' are defined over the same field, (2) the diagram is the natural diagram

$$\begin{array}{ccc} X'\widehat{\otimes}K & \longrightarrow & X' \\ \downarrow & & \downarrow \\ X\widehat{\otimes}K & \longrightarrow & X \end{array}$$

The second case follows from the previous lemma, so it is enough to consider only the case when all spaces are k-analytic. We can assume, that the spaces X, Y and X' are compact and that the morphisms $Y \longrightarrow X$ and $X' \longrightarrow X$ are the generic fibers of morphisms of formal schemes $\mathcal{Y} \longrightarrow \mathcal{X}$ and $\mathcal{X}' \longrightarrow \mathcal{X}$, respectively. Set $\mathcal{Y}' = \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$, notice, that $\mathcal{Y}'_\eta \widetilde{\rightarrow} Y'$ and $\mathcal{Y}'_s \widetilde{\rightarrow} \mathcal{Y}_s \times_{\mathcal{X}_s} \mathcal{X}'_s$. Let $\widetilde{x}$ be the image of x under the map $\mathcal{X}_\eta \longrightarrow \mathcal{X}_s$ and let $V_{\widetilde{x}}$ be its Zariski closure, define $V_{\widetilde{y}}$, $V_{\widetilde{x}'}$ and $V_{\widetilde{y}'}$ analogously. The morphism $\mathcal{Y} \longrightarrow \mathcal{X}$ induces a morphism $f : V_{\widetilde{y}} \longrightarrow V_{\widetilde{x}}$ and its image in $bir_{\widetilde{k}}$ is the morphism $\widetilde{Y}_y \longrightarrow \widetilde{X}_x$. Thus f is proper and, therefore, its base change $f' : U = V_{\widetilde{y}} \times_{V_{\widetilde{x}}} V_{\widetilde{x}'} \longrightarrow V_{\widetilde{x}'}$ is also proper. Note, that U is a closed subscheme of $\mathcal{Y}'_s$ and, therefore, $V_{\widetilde{y}'}$ is a closed subscheme of U. Hence the natural morphism $V_{\widetilde{y}'} \longrightarrow V_{\widetilde{x}'}$ is proper. But the latter morphism induces the morphism $\widetilde{Y}'_{y'} \longrightarrow \widetilde{X}'_{x'}$ and the lemma follows.

■

Proof of Theorem 4.1. Assume first that the two germs are good. In this case the morphism of germs is induced by a morphism of affinoid spaces $\varphi : Y = \mathcal{M}(\mathcal{B}) \longrightarrow X = \mathcal{M}(\mathcal{A})$. From Lemma 2.2 it follows that the morphism $\widetilde{\varphi}_y$ is proper if and only if the morphism of affine schemes $\mathrm{Spec}(\widetilde{\chi}_y(\widetilde{\mathcal{B}})) \longrightarrow \mathrm{Spec}(\widetilde{\chi}_x(\widetilde{\mathcal{A}}))$ is proper, where $\chi_x : \mathcal{A} \longrightarrow \mathcal{H}(x)$ and $\chi_y : \mathcal{B} \longrightarrow \mathcal{H}(y)$ are the characters corresponding to the points x and y, respectively. It follows that $\widetilde{\varphi}_y$ is proper if and only if $\widetilde{\chi}_y(\widetilde{\mathcal{B}})$ is integral over $\widetilde{\chi}_x(\widetilde{\mathcal{A}})$ and, by [Ber1], 2.5.2(d), the latter is equivalent to the fact that φ is closed at the point y.

Consider now the general case. The direct implication is easily reduced to the case of good germs. Assume that the morphism $\widetilde{\varphi}_y$ is proper. We may assume that φ_y is induced by a morphism of compact spaces $\varphi : Y \longrightarrow X$. Let $(\mathcal{Y}, y) \longrightarrow (\mathcal{X}, x)$ be a morphism of punctual admissible formal schemes that gives rise to φ, and let $U_{\widetilde{x}}$ and $V_{\widetilde{y}}$ be the closures of $\widetilde{x}$ and $\widetilde{y}$ in $\mathcal{X}_s$ and $\mathcal{Y}_s$, respectively. By our assumption the morphism $V_{\widetilde{y}} \longrightarrow U_{\widetilde{x}}$ is proper. Since $V_{\widetilde{y}}$ is closed in $\mathcal{Y}_s$, its preimage Z under the reduction map $Y = \mathcal{Y}_\eta \longrightarrow \mathcal{Y}_s$ is an open subset of Y. Notice that for each point $y' \in Z$ the morphism $\widetilde{\varphi}_{\widetilde{y}'} : \widetilde{Y}_{y'} \longrightarrow \widetilde{X}_{x'}$ is proper. *We claim that the induced morphism* $\psi : Z \longrightarrow X$ *is closed.*

Let $X' \longrightarrow X$ be a morphism from a good K-analytic space X', where K is a non-Archimedean field over k. We have to show that the K-analytic space $Z' = Z \times_X X'$ is also good and that the induced morphism $\psi' : Z' \longrightarrow X'$ is closed. Let y' be a point in Z', and let x', y and x be its images in X', Z and X, respectively. (The points y and x here are not necessarily the original y and x.) The morphism $\widetilde{Z}_y = \widetilde{Y}_y \longrightarrow \widetilde{X}_x$ is proper and therefore, by

Lemma 4.3, the morphism $\widetilde{Z}'_{y'} \longrightarrow \widetilde{X}'_{x'}$ is also proper. Since $\widetilde{X}'_{x'}$ is affine, it follows that $\widetilde{Z}'_{y'}$ is affine and, by Theorem 3.1, the germ $Z'_{y'}$ is good. Thus, the required claim follows from the case of good germs. ∎

The notion of a proper morphism we work with here is that from [Ber2], 1.5.3, restricted to the category of strictly analytic spaces, i.e., a morphism of analytic spaces is proper if it is proper as a map of topological spaces and is closed in the above sense. A morphism $\varphi : \mathcal{Y} \longrightarrow \mathcal{X}$ of admissible formal schemes over k° is said to be *proper* if the induced morphism of schemes $\varphi_s : \mathcal{Y}_s \longrightarrow \mathcal{X}_s$ is proper, and it is said to be *locally proper* if the induced morphism from the Zariski closure of any point of $\mathcal{Y}_s$ to $\mathcal{X}_s$ is proper. (Recall that the definition of an admissible formal scheme from §2 is slightly more restrictive than that from [BL].)

4.4. Corollary. *A morphism $\varphi : \mathcal{Y} \longrightarrow \mathcal{X}$ of admissible formal schemes over k° is proper (resp. locally proper) if and only if the induced morphism of analytic spaces $\varphi_\eta : \mathcal{Y}_\eta \longrightarrow \mathcal{X}_\eta$ is proper (resp. closed).*

Proof. Theorem 4.1 implies immediately, that φ_η is closed if and only if φ is locally proper. We should prove only, that $\varphi_s : \mathcal{Y}_s \longrightarrow \mathcal{X}_s$ is of finite type if and only if $\varphi_\eta : |Y_\eta| \longrightarrow |X_\eta|$ is a proper map of topological spaces. Since the question is local on the base, we can assume, that $\mathcal{X}$ is affine. Now, if φ_s is of finite type, then $\mathcal{Y}$ is a finite union of open affine subschemes and $\mathcal{Y}_\eta$ is a finite union of affinoid subspaces. In particular Y is compact and φ_η is a proper map of topological spaces. Conversely, suppose φ_η is a proper map. Let $y \in \mathcal{Y}_s$ be a generic point and V_y its Zariski closure, then the preimage of V_y in $\mathcal{Y}_\eta$ is an open subspace. Since by our assumption $\mathcal{Y}_\eta$ is compact, it has a finite covering by such subspaces. Thus $\mathcal{Y}_s$ consists of a finite number of components and so φ_s is of finite type. ∎

Recall the definition (due to Kiehl) of proper morphisms in rigid geometry (see [BGR], 9.6.2/2). A separated morphism $\varphi : Y_0 \longrightarrow X_0$ of rigid spaces is proper if there exists an affinoid covering $\{X_0^i\}_{i \in I}$ of X_0 and finite affinoid coverings $\{Y_0^{i,j}\}_{j \in J_i}$ and $\{Z_0^{i,j}\}_{j \in J_i}$ of the spaces $\varphi^{-1}(X_0^i)$ such that $Z_0^{i,j} \Subset_{X_0^i} Y_0^{i,j}$.

4.5. Corollary. *A morphism of Hausdorff analytic spaces $\varphi : Y \longrightarrow X$ is proper if and only if the corresponding morphism of rigid analytic spaces $\varphi_0 : Y_0 \longrightarrow X_0$ is proper in the sense of Kiehl.*

Proof. In the case of good spaces both notions are easily seen to be equivalent. By the definition, a morphism φ_0 is proper if for some affinoid covering $\{X_0^i\}$ of X_0 the induced morphisms $\varphi^{-1}(X_0^i) \longrightarrow X_0^i$ are proper. Theorem 4.1 implies, that the same criterion holds for the notion of analytic properness (though apriori the definition is much more restrictive), hence the corollary is valid for arbitrary spaces.
∎

If one defines the relative interior $\mathrm{Int}(Y/X)$ of a morphism $\varphi : Y \longrightarrow X$ as in [Ber2], 1.5.4, but with the above notion of a closed morphism, one gets the following corollary.

4.6. Corollary. *Given two morphisms* $Z \xrightarrow{\psi} Y \xrightarrow{\varphi} X$, *assume that* φ *is locally separated. Then* $\mathrm{Int}(Z/X) = \mathrm{Int}(Z/Y) \cap \psi^{-1}(\mathrm{Int}(Y/X))$.

Proof. The corollary follows from theorem 4.1 and the following simple fact. Let $f : \mathcal{Z} \longrightarrow \mathcal{Y}$ and $g : \mathcal{Y} \longrightarrow \mathcal{X}$ be dominant morphisms of irreducible schemes, and suppose, that g is separated, then $g \circ f$ is proper if and only if both f and g are proper. ∎

Recall that the above fact was shown in [Ber2], 1.5.5(ii), under the additional assumption that the morphism φ is good. Notice that Theorem 4.1 implies that the properties of a morphism to be closed or proper are local with respect to the G-topology. But the latter is not true for the property of a morphism to be a good one (see [Ber2], 1.5.3) as can be shown using Remark 1.6.

§5. Extension of affinoid domains

5.1. Theorem. *Let* $X \longrightarrow R$ *be a separated closed morphism to an affinoid space* R. *Then for every affinoid domain* $U \subset X$ *there exists a bigger affinoid domain* $V \subset X$ *such that* $U \subset \mathrm{Int}(V/R)$ *and* U *is a Weierstrass subdomain of* V.

Proof. Let $R = \mathcal{M}(\mathcal{K})$, we fix an epimorphism $k\{T'_1, \ldots, T'_m\} \longrightarrow \mathcal{K}$ and provide $\mathcal{K}$ with the quotient norm (thus we obtain also natural norms on the algebras $\mathcal{K}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$).

Let us say that V is a *relative* W*-extension of* U *in* X if it satisfies the properties stated in the theorem. Our first step is to use the reasoning from [BGR], §8.2.2, to reduce the theorem to the following statement.

If for an affinoid domain $U \subset X$ *there exists an analytic function* h *on* U *such that the affinoid subdomains* $U\{h\}$ *and* $U\{h^{-1}\}$ *have relative* W*-extensions in* X, *then* U *itself has a relative* W*-extension in* X.

Indeed, assume that the above statement is true. Then to apply [BGR], Lemmas 8.2.2/2-4, we have to verify the following two facts: (a) if U has a relative W-extension in X, then the same is true for every rational subdomain of U; and (b) any affinoid domain $U \subset X$ has a finite covering by affinoid subdomains which have relative W-extensions in X.

We start with the proof of (a). Let U' be a relative W-extension of U, say $U = U'\{f_1, \ldots, f_n\}$, and let $V = U\{\frac{P_1}{Q}, \ldots, \frac{P_m}{Q}\}$ be a rational subdomain of U (P_i and Q have no common zero on U). By the definition of inner homomorphism (see [Ber1], 2.5.1), for $\varepsilon > 0$ the domain $U_\varepsilon = U'\{(1 + \varepsilon)^{-1}f_1, \ldots, (1+\varepsilon)^{-1}f_n\}$ is a relative W-extension of U. Since $\mathcal{O}(U')$ is dense in $\mathcal{O}(U)$, we can assume, that P_i and Q are defined on U'. Note, also, that for some $\varepsilon > 0$, the functions P_i and Q have no common zero on the domain

U_ε. Now, for any $\delta > 0$, the domain $U_\varepsilon\{(1+\delta)^{-1}\frac{P_1}{Q}, \dots, (1+\delta)^{-1}\frac{P_m}{Q}\}$ is a relative W-extension of V.

Next we prove (b). It suffices to show, that for any point $x \in U$, some neighborhood V in U has a relative W-extension in X. The reduction $\widetilde{U}_x$ is affine, let $\widetilde{U}_x \widetilde{\to} \mathbf{P}_{\widetilde{\mathcal{H}(x)}}\{\widetilde{f}_1, \dots, \widetilde{f}_n\}$. Lift $\widetilde{f}_i$ to elements f_i of $\mathcal{O}_{X,x}$ (X is good). By theorem 2.4, $U_x = X_x\{f_1, \dots, f_n\}$, i.e. for sufficiently small neighborhood Y of x in X the functions f_i are defined on Y, and $Y\{f_1, \dots, f_n\}$ is a neighborhood of x in U. Since the map $X \longrightarrow R$ is closed at x and R is good, the space X is good at x. Thus the space Y above can be chosen affinoid, say $Y = \mathcal{M}(\mathcal{A})$. Note that $x \in \mathrm{Int}(Y/R)$, hence the homomorphism $\chi_x : \mathcal{A} \longrightarrow \mathcal{H}(x)$ is inner with respect to $\mathcal{K}$. It means, that there exists an epimorphism $\mathcal{K}\{T_1, \dots, T_l\} \longrightarrow \mathcal{A} : T_i \mapsto g_i$ such that $|\chi_x(g_i)| < 1$. Notice that for $\varepsilon > 0$, Y is a relative W-extension of $Y_\varepsilon = Y\{(1-\varepsilon)^{-1}g_1, \dots, (1-\varepsilon)^{-1}g_l\}$ (the homomorphism $\mathcal{A} \longrightarrow \mathcal{A}\{(1-\varepsilon)^{-1}g\}$ is inner with respect to $\mathcal{K}$) and for some $\varepsilon > 0$, Y_ε is a neighborhood of x in X. For such ε, Y is a relative W-extension of $Y_\varepsilon\{f_1, \dots, f_n\}$ and the last domain is a required neighborhood of x in U.

We return to the statement mentioned at the beginning of the proof. Let $U = \mathcal{M}(\mathcal{D})$. We set $Y_0 = U\{h\}$ and $Z_0 = U\{h^{-1}\}$. Then $Y_0 = \mathcal{M}(\mathcal{B}_0)$, $Z_0 = \mathcal{M}(\mathcal{C}_0)$ and $Y_0 \cap Z_0 = \mathcal{M}(\mathcal{A}_0)$, where $\mathcal{B}_0 = \mathcal{D}\{h\}$, $\mathcal{C}_0 = \mathcal{D}\{h^{-1}\}$ and $\mathcal{A}_0 = \mathcal{D}\{h, h^{-1}\}$. Furthermore, let $Y = \mathcal{M}(\mathcal{B})$ and $Z = \mathcal{M}(\mathcal{C})$ be relative W-extensions of Y_0 and Z_0, respectively. We will show that there exist affinoid domains $Y_0 \subset Y' \subset Y$ and $Z_0 \subset Z' \subset Z$ such that $Y' \cup Z'$ is a relative W-extension of U.

Step 1. Let us fix a continuous $\mathcal{K}$-epimorphism $\mathcal{K}\{T_1, \dots, T_n\} \longrightarrow \mathcal{D} : T_i \mapsto h_i$. It induces continuous $\mathcal{K}$-epimorphisms $\mathcal{K}\{T_1, \dots, T_n, S\} \longrightarrow \mathcal{B}_0$, $\mathcal{K}\{T_1, \dots, T_n, S, S^{-1}\} \longrightarrow \mathcal{A}_0$ and $\mathcal{K}\{T_1, \dots, T_n, r^{-1}S, S^{-1}\} \longrightarrow \mathcal{C}_0 : S \mapsto h$, where $r \in \sqrt{|k|}$ and $r \geq |h|_{Z_0} := \max_{x \in Z_0}\{|h(x)|\}$. We provide $\mathcal{A}_0$ with the quotient norm with respect to the above epimorphism. Since $Y_0 \subset Y$ and $Z_0 \subset Z$ are Weierstrass domains, the images of $\mathcal{B}$ in $\mathcal{B}_0$ and of $\mathcal{C}$ in $\mathcal{C}_0$ are dense. By [BGR], 7.3.4/3, we can find elements $f_i, f \in \mathcal{B}$ and $g_i, g \in \mathcal{C}$ whose images in $\mathcal{B}_0$ and $\mathcal{C}_0$ are sufficiently close to h_i, h, respectively, so that the following is true: (a) the continuous $\mathcal{K}$-homomorphisms $\mathcal{K}\{T_1, \dots, T_n, S\} \longrightarrow \mathcal{B}_0 : T_i \mapsto f_i, S \mapsto f$ and $\mathcal{K}\{T_1, \dots, T_n, r^{-1}S, S^{-1}\} \longrightarrow \mathcal{C}_0 : T_i \mapsto g_i, S \mapsto g$ are surjective, (b) the norm on $\mathcal{A}_0$ coincides with the quotient norm induced by the $\mathcal{K}$-epimorphism $\mathcal{K}\{T_1, \dots, T_n, S, S^{-1}\} \longrightarrow \mathcal{A}_0 : T_i \mapsto f_i, S \mapsto f$, and (c) all of the numbers $||f_i - g_i||_{\mathcal{A}_0}$ and $||f - g||_{\mathcal{A}_0}$ are strictly less than 1.

Step 2. *One can replace Y and Z by smaller neighborhoods of Y_0 and Z_0 so that $Y_0 = Y\{f_1, \dots, f_n, f\}$ and $Z_0 = Z\{g_1, \dots, g_n, r^{-1}g, g^{-1}\}$.* Indeed, one evidently has $Y_0 \subset Y\{f_1, \dots, f_n, f\}$ and $Z_0 \subset Z\{g_1, \dots, g_n, r^{-1}g, g^{-1}\}$. The canonical homomorphisms $\mathcal{B}\{f_1, \dots, f_n, f\} \longrightarrow \mathcal{B}_0$ and $\mathcal{C}\{f_1, \dots, f_n, r^{-1}g, g^{-1}\} \longrightarrow \mathcal{C}_0$ are surjective, hence the complements of Y_0 in $Y\{f_1, \dots, f_n, f\}$ and of Z_0 in $Z\{g_1, \dots, g_n, r^{-1}g, g^{-1}\}$ are affinoid domains and, therefore, we can shrink Y and Z so that the desired fact is true.

Step 3. For a pair of non-negative numbers $\varepsilon = (\varepsilon_1, \varepsilon_r)$ with $1+\varepsilon_1, r+\varepsilon_r \in \sqrt{|k|}$, we set $Y_\varepsilon = Y\{(1+\varepsilon_1)^{-1}f_1, \dots, (1+\varepsilon_1)^{-1}f_n, f\}$ and $Z_\varepsilon = Z\{(1+\varepsilon_1)^{-1}g_1, \dots, (1+\varepsilon_1)^{-1}g_n, (r+\varepsilon_r)^{-1}g, g^{-1}\}$. One has $Y_\varepsilon = \mathcal{M}(\mathcal{B}_\varepsilon)$ and $Z_\varepsilon = \mathcal{M}(\mathcal{C}_\varepsilon)$, where $\mathcal{B}_\varepsilon = \mathcal{B}\{(1+\varepsilon_1)^{-1}f_1, \dots, (1+\varepsilon_1)^{-1}f_n, f\}$ and $\mathcal{C}_\varepsilon = \mathcal{C}\{(1+\varepsilon_1)^{-1}f_1, \dots, (1+\varepsilon_1)^{-1}f_n, (r+\varepsilon_r)^{-1}g, g^{-1}\}$. *We claim that, for every sufficiently small positive* ε, *one has* $Y_\varepsilon \cap Z_\varepsilon = Y_\varepsilon\{f^{-1}\} = Z_\varepsilon\{g\}$. Indeed, since $||f_i - g_i|| < 1$ and $||f-g|| < 1$, it follows that $|(f_i-g_i)(x)| < 1$ and $|(f-g)(x)| < 1$ for all points x from an open neighborhood V of $Y_0 \cap Z_0$ in $Y \cap Z$. But $Y_\varepsilon\{f^{-1}\} \subset V$ and $Z_\varepsilon\{g\} \subset V$ for all sufficiently small ε, and it immediately implies that $Y_\varepsilon\{f^{-1}\} = Z_\varepsilon\{g\}$ for such ε. Since the underlying topological space of X is Hausdorff, $Y_\varepsilon \cap Z_\varepsilon$ is a disjoint union of the latter set and a compact set outside V. Decreasing again ε, we can achieve the inclusion $Y_\varepsilon \cap Z_\varepsilon \subset V$ which gives the required fact.

Step 4. *For every sufficiently small* ε *the continuous* $\mathcal{K}$*-homomorphisms*

$$\mathcal{K}\{(1+\varepsilon_1)^{-1}T_1, \dots, (1+\varepsilon_1)^{-1}T_n, S\} \longrightarrow \mathcal{B}_\varepsilon : T_i \mapsto f_i, S \mapsto f$$

$$\mathcal{K}\{(1+\varepsilon_1)^{-1}T_1, \dots, (1+\varepsilon_1)^{-1}T_n, (r+\varepsilon_r)^{-1}S, S^{-1}\} \longrightarrow \mathcal{C}_\varepsilon : T_i \mapsto g_i, S \mapsto g$$

are surjective and all of the numbers $||f_i - g_i||_{\mathcal{A}_\varepsilon}$ *and* $||f-g||_{\mathcal{A}_\varepsilon}$ *are strictly less than* 1, *where the norm on* $\mathcal{A}_\varepsilon = \mathcal{B}_\varepsilon\{f^{-1}\}$ *is induced by the first of the above epimorphisms.* The statement follows from the next lemma where the following situation is considered. Let $X = \mathcal{M}(\mathcal{A})$ be an affinoid space over $R = \mathcal{M}(\mathcal{K})$, $f_1, \dots, f_n$ elements of $\mathcal{A}$, $r_1, \dots, r_n \in \sqrt{|k|}$ positive numbers. For a set of non-negative numbers $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ with $r_i + \varepsilon_i \in \sqrt{|k|}$, let $X_\varepsilon = X\{(r_1+\varepsilon_1)^{-1}f_1, \dots, (r_n+\varepsilon_n)^{-1}f_n\}$. One has $X_\varepsilon = \mathcal{M}(\mathcal{A}_\varepsilon)$, where $\mathcal{A}_\varepsilon = \mathcal{A}\{(r_1+\varepsilon_1)^{-1}f_1, \dots, (r_n+\varepsilon_n)^{-1}f_n\}$.

5.2. Lemma. *Assume that* $X_0 \subset \mathrm{Int}(X/R)$ *and that the continuous* $\mathcal{K}$*-homomorphism* $\mathcal{K}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \longrightarrow \mathcal{A}_0 : T_i \mapsto f_i$ *is surjective. Let* $g \in \mathcal{A}$ *be an element with* $||g||_{\mathcal{A}_0} < 1$, *then there exists* $\delta > 0$ *such that for every* ε *with* $\varepsilon_i \le \delta$ *the following is true:*

(i) the continuous $\mathcal{K}$*-homomorphism* $\mathcal{K}\{(r_1+\varepsilon_1)^{-1}f_1, \dots, (r_n+\varepsilon_n)^{-1}f_n\} \longrightarrow \mathcal{A}_\varepsilon : T_i \mapsto f_i$ *is surjective;*

(ii) $||g||_{\mathcal{A}_\varepsilon} < 1$, *where the norm on* $\mathcal{A}_\varepsilon$ *is induced by the epimorphism from (i).*

Proof. Consider the $\mathcal{K}$-homomorphism $\phi^\# : \mathcal{K}[T_1, \dots, T_n] \longrightarrow \mathcal{A}$ taking T_i to f_i, it induces a morphism $\phi : X \longrightarrow \mathbf{A}^n_R$. For non-negative numbers $\varepsilon_1, \dots, \varepsilon_n$, let V_ε denotes the polydisc (over R) in $\mathbf{A}^n_R$ given by inequalities $|T_i| \le r_i + \varepsilon_i$ and let $\phi_\varepsilon : X_\varepsilon = \phi^{-1}(V_\varepsilon) \longrightarrow V_\varepsilon$ be the restriction of ϕ. By assumption, ϕ_0 is a closed immersion and we should prove, that ϕ_ε remains a closed immersion for sufficiently small positive ε. By [Ber1], 2.5.9, the homomorphism $\mathcal{A} \longrightarrow \mathcal{A}_0$ is inner with respect to $\mathcal{K}$, hence there exists a $\mathcal{K}$-epimorphism $\mathcal{K}\{T_1, \dots, T_p\} \longrightarrow \mathcal{A} : T_l \mapsto a_l$, such that $\rho_{\mathcal{A}_0}(a_l) < 1$ (see

[Ber1], 2.5.1). Let $a'_1,\dots,a'_{p'}$ be generators of $\mathcal{K}$ (i.e. the homomorphism $k\{T_1,\dots,T_{p'}\}\longrightarrow\mathcal{K}:T_j\mapsto a'_j$ is surjective), note that a_l,a'_j are generators of $\mathcal{A}$ over k. Finally, we choose elements $s_1,\dots,s_p\in\mathcal{K}[T_1,\dots,T_n]$ such that $\rho_{\mathcal{A}_0}(\phi^{\#}(s_l)-a_l)<1$ and set $s'_j=a'_j\in\mathcal{K}\subset\mathcal{K}[T_1,\dots,T_n]$ (obviously $\rho_{\mathcal{A}_0}(\phi^{\#}(s'_j)-a'_j)<1$). As shown at the proof of [BGR], 7.3.4/10, for sufficiently small ε, ϕ_ε factors through a closed immersion $\psi_\varepsilon:X_\varepsilon\longrightarrow V_\varepsilon\{s_l,s'_j\}$. By our choice, $V_\varepsilon\{s_l,s'_j\}=V_\varepsilon\{s_l\}$. Since $\rho_{\mathcal{A}_0}(a_l)<1$, for some positive ε,α, we have $\rho_{\mathcal{A}_\varepsilon}(a_l)<1-\alpha$. Then the image of X_ε under ψ_ε belongs to $V_\varepsilon\{(1-\alpha)^{-1}s_1,\dots,(1-\alpha)^{-1}s_p\}$. Therefore for small ε the map $\phi_\varepsilon:X_\varepsilon\longrightarrow V_\varepsilon$ is also a closed immersion.

For a non-negative ε, set $\mathcal{B}_\varepsilon=\mathcal{K}\{(r_1+\varepsilon_1)^{-1}T_1,\dots,(r_n+\varepsilon_n)^{-1}T_n\}$. Let $\phi^{\#}_\varepsilon:\mathcal{B}_\varepsilon\longrightarrow\mathcal{A}_\varepsilon$ be the continuous $\mathcal{K}$-homomorphism taking T_i to f_i, I_ε its kernel ideal and $||\ ||_{\mathcal{B}_\varepsilon}$ the natural norm of $\mathcal{B}_\varepsilon$. Choose a positive ε for which $\phi^{\#}_\varepsilon$ is an epimorphism, then $\phi_\varepsilon:\mathcal{M}(\mathcal{A}_\varepsilon)\longrightarrow\mathcal{M}(\mathcal{B}_\varepsilon)$ is a closed immersion and $\phi_0:\mathcal{M}(\mathcal{A}_0)\longrightarrow\mathcal{M}(\mathcal{B}_0)$ is its restriction. Therefore $I_0=I_\varepsilon\cdot\mathcal{B}_0$, in particular I_ε is dense in I_0. Let $G\in\mathcal{B}_\varepsilon$ be a preimage of g, then G (as an element of $\mathcal{B}_0$) is a preimage of g under $\phi^{\#}_0$. Since $||g||_{\mathcal{A}_0}<1$, there exists an element $a\in I_0$ such that $||G+a||_{\mathcal{B}_0}<1$. Since I_ε is dense in I_0, we can choose a in I_ε. Then $G+a$ is a preimage of g_ε under $\phi^{\#}_\varepsilon$. Clearly for some positive $\delta\le\varepsilon$ we have $||G+a||_{\mathcal{B}_\delta}<1$ and then $||g_\delta||_{\mathcal{A}_\delta}<1$. ∎

Step 5. *If ε is small enough so that the properties stated in Steps 3 and 4 are true, then the analytic domain $U_\varepsilon=Y_\varepsilon\cup Z_\varepsilon$ is affinoid. Furthermore, if $U_\varepsilon=\mathcal{M}(\mathcal{D}_\varepsilon)$, then there is a continuous $\mathcal{K}$-epimorphism*

$$\mathcal{K}\{(1+\varepsilon_1)^{-1}T_1,\dots,(1+\varepsilon_1)^{-1}T_n,(r+\varepsilon_r)^{-1}S\}\longrightarrow\mathcal{D}_\varepsilon:T_i\mapsto h_i,S\mapsto h'$$

such that all of the numbers $|f_i-h_i|_{Y_\varepsilon}$, $|f-h'|_{Y_\varepsilon}$, $|g_i-h_i|_{Z_\varepsilon}$ and $|g-h'|_{Z_\varepsilon}$ are strictly less than 1. The statement follows from lemma 3.2.

Step 6. *The affinoid domain U_ε is a relative W-extension of U.* Since $|(f_i-h_i)(x)|<1$ (resp. $|(g_i-h_i)(x)|<1$ and $|(g-h')(x)|<1$) for all points $x\in Y_\varepsilon$ (resp. Z_ε), it follows that $Y_0=Y_\varepsilon\{h_1,\dots,h_n\}$ (resp. $Z_0=Z_\varepsilon\{h_1,\dots,h_n,r^{-1}h'\}$) and, therefore, $U_\varepsilon\{h_1,\dots,h_n,r^{-1}h'\}=Y_0\cup Z_0=U$. In particular, U is a Weierstrass domain in U_ε. Finally, since the continuous $\mathcal{K}$-homomorphism $\mathcal{K}\{(1+\varepsilon_1)^{-1}T_1,\dots,(1+\varepsilon_1)^{-1}T_n,(r+\varepsilon_r)^{-1}S\}\longrightarrow\mathcal{D}_\varepsilon:T_i\mapsto h_i,S\mapsto h'$ is surjective, $|h_i|_U\le 1$ and $|h'|_U\le r$, it follows that $U\subset\mathrm{Int}(U_\varepsilon/R)$. ∎

5.3. Conjecture. *Given a good separated analytic space X, for every affinoid domain $U\subset X$ there exists a bigger affinoid domain $V\subset X$ such that $U\Subset_X V$ and U is a Weierstrass subdomain of V.*

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