Czechoslovak Mathematical Journal

Pavla Vrbová

On local spectral properties of operators in Banach spaces

Czechoslovak Mathematical Journal, Vol. 23 (1973), No. 3, 483–492

Persistent URL: `http://dml.cz/dmlcz/101189`

Czechoslovak Mathematical Journal, 23 (98) 1973, Praha

ON LOCAL SPECTRAL PROPERTIES OF OPERATORS IN BANACH SPACES

PAVLA VRBOVÁ, Praha

(Received July 12, 1972)

In his paper [5] P. R. HALMOS has introduced the notion of capacity in a Banach algebra, particularly in the algebra of operators on a Hilbert space. The notion turns out to be of use in investigations of properties of quasialgebraic elements of a Banach algebra (an element a in the algebra A is called quasialgebraic if there exists a sequence (p_n) of monic polynomials, with degree $d(p_n)$, such that $|p_n(a)|^{1/d(p_n)} \to 0$ as $n \to \infty$). In this note we shall prove some additional properties of quasialgebraic elements.

As regards operators, there arises the natural question how to define a locally quasialgebraic operator T acting on a Banach space X. F. H. VASILESCU proposed in [8] a definition of to be locally quasialgebraic which relates this notion to spectra of elements $x \in X$ with respect to T. In the present note we prove the existence of a comparatively large set of elements $x \in X$ with extremal spectra, namely $\gamma_T(x) \supset$ $\supset \operatorname{bd} \sigma(T)$ and $\sigma_T(x) = \sigma(T)$ (See [4]). This result is then used to obtain a proof of the fact that every locally quasialgebraic operator is quasialgebraic.

The paper is divided into two parts. In the first we shall describe some spectral properties of elements of X, the second part is devoted to the notion of capacity. To make the paper self-contained we shall give some proofs included in [5], [7], [8].

1. Let X be a Banach space, let $T \in B(X)$ (the algebra of all linear bounded operators from X to X) be given. Denote by $\Omega_T \subset \mathbf{C}$ (the complex plane) the set of all complex λ for which there exists a neighbourhood U_λ with the property that for every open set $\omega \subset U_\lambda$ the unique holomorphic solution of the equation $(\lambda - T) f(\lambda) \equiv 0$ on ω is the function $f \equiv 0$ in ω. The set Ω_T is open by its definition. The set Ω_T is called the set of analytic uniqueness of T and the set $S_T \doteq \mathbf{C} \setminus \Omega_T$ is called the analytic residuum of T [7]. If $S_T = \emptyset$ we say that T has the single-valued extension property [2].

Let $x \in X$ be given. Take a complex number such that the equation $(\lambda - T) f(\lambda) =$ $= x$ has a holomorphic solution in some neighbourhood of this number; denote by $\delta_T(x)$ the set of all complex numbers with this property. Further, set $\gamma_T(x) = \mathbf{C} \setminus \delta_T(x)$, $\varrho_T(x) = \delta_T(x) \cap \Omega_T$ and $\sigma_T(x) = \mathbf{C} \setminus \varrho_T(x) = \gamma_T(x) \cup S_T$. Clearly $\gamma_T(x) \subset \sigma_T(x) \subset \sigma(T)$. It is an easy application of the open mapping theorem that $\bigcup_{x\in X} \sigma_T(x) = \sigma(T)$ (or equivalently $\bigcap_{x\in X} \varrho_T(x) = \varrho(T)$). See [7].

Remark 1. *The set S_T is closed, $S_T = (\text{Int } S_T)^-$. Moreover, S_T is the closure of the set of all points λ such that there exists an $h_\lambda \neq 0$ with the property $(\lambda - T)\, h_\lambda = 0$, $\gamma_T(h_\lambda) = \emptyset$.*

Proof. The set S_T consists of all λ for which in every neighbourhood U_λ there is an open set $\omega \subset U_\lambda$ and a non-zero holomorphic function f satisfying

$$(\mu - T) f(\mu) = 0$$

in ω.

It follows that $\omega \subset \text{Int } S_T$ and consequently $S_T = (\text{Int } S_T)^-$. Further, if $\lambda \in S_T$ we may suppose that ω is such that f has only a finite number of zeros in ω. Take arbitrary $\mu \in \omega$ such that $f(\mu) \neq 0$. Then $\gamma_T(f(\mu)) = \emptyset$. Indeed, $(\mu - T) f(\mu) = 0$ so that the function $h(\xi) = (\xi - \mu)^{-1} f(\mu)$ is holomorphic for $\xi \neq \mu$ and

$$(\xi - T)\, h(\xi) = (\mu - \omega)\, h(\xi) + (\xi - \mu)\, h(\xi) = f(\mu)$$

hence $\gamma_T(f(\mu)) \subset (\mu)$. On the other hand, the function $g(\xi) = (\mu - \xi)^{-1} (f(\xi) - f(\mu))$ for $\xi \neq \mu$, $g(\mu) = -f'(\mu)$ is holomorphic in ω and

$$(\xi - T)\, g(\xi) = (\xi - T)(\mu - \xi)^{-1} f(\xi) - (\mu - T)(\mu - \xi)^{-1} f(\mu) + f(\mu) = f(\mu)$$

for $\xi \neq \mu$ and by continuity everywhere in ω. Thus $\mu \in \delta_T(f(\mu))$ as well and $\gamma_T(f(\mu)) = \emptyset$. To prove the assertion it is sufficient to prove that every λ with the property $(\lambda - T)\, h = 0$ for some $h \neq 0$ and $\gamma_T(h) = \emptyset$ belongs to S_T. Let f be holomorphic in a neighbourhood U_λ of λ such that $h = (\mu - T) f(\mu)$ in U_λ. The function $g(\mu) = (\lambda - T) f(\mu)$ is non-zero in U_λ. Moreover

$$(\mu - T)\, g(\mu) = (\lambda - T)(\mu - T) f(\mu) = (\lambda - T)\, h = 0$$

in U_λ. Thus $U_\lambda \subset \text{Int } S_T$. The proof is complete.

1.1. *Let $x \in X$ be given. The following conditions are equivalent:*

1° $\lambda_1, \ldots, \lambda_m \in \delta_T(x)$,

2° *there exist a positive k and a sequence $(x_n) \subset X$ such that*
$(\lambda_1 - T) \ldots (\lambda_m - T)\, x_{n+1} = x_n$, $|x_n| \leqq k^n$ *for* $n \geqq 1$ *and*
$(\lambda_1 - T) \ldots (\lambda_m - T)\, x_1 = x$.

Proof. Suppose that 1° is satisfied. Then there exist an $\varepsilon > 0$ and a holomorphic function f defined in $M = \bigcup_{i=1}^{m} D(\lambda_i, 2\varepsilon)$ ($D(\lambda_i, 2\varepsilon)$ is the open disk with center λ_i and radius 2ε; the sets $D(\lambda_i, 2\varepsilon)$ are supposed to be pairwise disjoint) such that $(\mu - T) f(\mu) = x$ in M. Form an infinite sequence (μ_i) setting $\mu_n = \lambda_k$ for every natural n in the form $ml + k$, $1 \leqq k \leqq m$. Define by induction the sequence of functions g_i on M as follows:

$$g_1 = f,\ g_n(\mu) = (\mu - \mu_{n-1})^{-1} (g_{n-1}(\mu_{n-1}) - g_{n-1}(\mu)) \quad \text{for} \quad \mu \neq \mu_{n-1}\,,$$
$$g_n(\mu_{n-1}) = -g'_{n-1}(\mu_{n-1})\ (n \geqq 2)\,.$$

Then all g_n are holomorphic in M. Using continuity of g_{n+1} and the equality

$$(\mu - T)\, g_{n+1}(\mu) = (\mu - T)(\mu - \mu_n)^{-1} g_n(\mu_n) - (\mu - T)(\mu - \mu_n)^{-1} g_n(\mu) =$$
$$= g_n(\mu_n) + (\mu_n - T)(\mu - \mu_n)^{-1} g_n(\mu_n) - (\mu - T)(\mu - \mu_n)^{-1} g_n(\mu)$$

for $\mu \neq \mu_n$ we obtain by induction $(\mu - T)\, g_{n+1}(\mu) = g_n(\mu_n)$ in M, $n \geqq 1$.

Denote by $(g_n)^i_k$ the k-th coefficient of the expansion of g_n in a power series in $D(\lambda_i, \varepsilon)$, i.e. $g_n(\mu) = \sum_{k=0}^{\infty} (g_n)^i_k (\mu - \lambda_i)^k$ in $D(\lambda_i, \varepsilon)$. We shall prove by induction that $|(g_n)^i_k| \leqq K(1/\varepsilon)^{k+n}$ for some K. Indeed, we have $\limsup_{k\to\infty} |(f)^i_k|^{1/k} \leqq (2\varepsilon)^{-1}$ and consequently $|(g_1)^i_k| \leqq K\varepsilon^{-k-1}$ for suitable K and $1 \leqq i \leqq m$. Further, we have, for $\lambda_i = \mu_n$, $g_{n+1}(\mu) = -\sum_{k=0}^{\infty} (g_n)^i_{k+1} (\mu - \lambda_i)^k$ in $D(\lambda_i, 2\varepsilon)$. If $\lambda_i \neq \mu_n$, then

$$-g_{n+1}(\mu) = (\mu - \mu_n)^{-1} \Big(\sum_{k=0}^{\infty} (g_n)^i_k (\mu - \lambda_i)^k - g_n(\mu_n)\Big) =$$
$$= \Big(\sum_{k=0}^{\infty} (-1)^k (\mu - \lambda_i)^k (\lambda_i - \mu_n)^{-k-1}\Big)\Big(\sum_{k=0}^{\infty} (g_n)^i_k (\mu - \lambda_i)^k - g_n(\mu_n)\Big) =$$
$$= \sum_{k=0}^{\infty} \big((-1)^k (\lambda_i - \mu_n)^{-k-1} (g_n)^i_0 + \ldots + (\lambda_i - \mu_n)^{-1} (g_n)^i_k -$$
$$- (-1)^k (\lambda_i - \mu_n)^{-k-1} g_n(\mu_n)\big) (\mu - \lambda_i)^k \quad \text{in} \quad D(\lambda_i, 2\varepsilon)\,.$$

By the induction assumption we have

$$|(g_{n+1})^i_k| = |(g_n)^i_{k+1}| \leqq K\varepsilon^{-k-n-1} \quad \text{for} \quad \lambda_i = \mu_n$$

and

$$|(g_{n+1})^i_k| \leqq K\Big(\sum_{j=0}^{k} (2\varepsilon)^{-k-1+j}\, \varepsilon^{-j-n} + (2\varepsilon)^{-k-1}\, \varepsilon^{-n}\Big) =$$
$$= K\varepsilon^{-k-n-1}\Big(\sum_{j=0}^{k} 2^{-k-1+j} + 2^{-k-1}\Big) = K\varepsilon^{-k-n-1} \quad \text{for} \quad \lambda_i \neq \mu_n\,.$$

It follows that $|g_n(\mu_n)| \leqq K\varepsilon^{-n}$ for $n \geqq 1$. Set $x_n = g_{mn}(\mu_{mn})$.

Then we have $|x_n| \leqq K\varepsilon^{-m.n}$ and the equality $(\mu - T)\, g_{n+1}(\mu) = g_n(\mu_n)$ implies $(\lambda_1 - T)\dots(\lambda_m - T)\, x_{n+1} = x_n$ for $n \geqq 1$, $(\lambda_1 - T)\dots(\lambda_m - T)\, x_1 = x$. This proves 2°.

To prove the implication 2° → 1° it suffices to show that the function $f_i(\lambda) = = \sum_{n=0}^{\infty}(-1)^n \prod_{j\neq i,j=1}^{m} (\lambda_j - T)^{n+1} x_{n+1}(\lambda - \lambda_i)^n$ satisfies $(\lambda - T)\, f_i(\lambda) = x$ in $D(\lambda_i, \varepsilon_i)$, where $\varepsilon_i = (|\prod_{j\neq i,j=1}^{m} (\lambda_j - T)|\, k)^{-1}$. Indeed, we have

$$(\lambda - T) f_i(\lambda) = (\lambda - \lambda_i) f_i(\lambda) + (\lambda_i - T) f_i(\lambda) =$$
$$= \sum_{n=0}^{\infty}(-1)^n \prod_{j\neq i,j=1}^{m} (\lambda_j - T)^{n+1}\, x_{n+1}(\lambda - \lambda_i)^{n+1} + x +$$
$$+ \sum_{n=1}^{\infty}(-1)^n \prod_{j\neq i,j=1}^{m} (\lambda_j - T)^n\, x_n(\lambda - \lambda_i)^n = x.$$

This completes the proof.

1.2. *Suppose that* 0 *does not belong to* S_T. *If* $TX = X$ *then* $T^{-1} \in B(X)$.

Proof. Denote by X_1 the unit ball of X. According to the open mapping theorem $X_1 \subset kTX_1$ for some positive k. Given an $x \in X$ we can define a sequence of elements $x_n \in X$ such that $Tx_1 = x$, $Tx_{n+1} = x_n$ and $|x_n| \leqq k^n|x|$ for $n \geqq 1$. It follows that $0 \in \delta_T(x)$ by 1.1. Together with the assumption $0 \notin S_T$ it gives $0 \in \varrho_T(x)$ for every x and consequently, $0 \in \varrho(T)$. The proof is complete.

1.3. $\sigma(T) = S_T \cup \{\lambda : (\lambda - T)\, B(X) \neq B(X)\} = S_T \cup \{\lambda : (\lambda - T)\, X \neq X\}$.

Proof. Clearly $S_T \cup \{\lambda : (\lambda - T)\, X \neq X\} \subset S_T \cup \{\lambda : (\lambda - T)\, B(X) \neq B(X)\} \subset \subset \sigma(T)$. To prove the assertion it suffices to show that

$$\sigma(T) \subset S_T \cup \{\lambda : (\lambda - T)\, X \neq X\}$$

or equivalently,

$$\Omega_T \cap \{\lambda : (\lambda - T)\, X = X\} \subset \varrho(T)\,.$$

If $\lambda \in \Omega_T$ then $0 \in \Omega_{\lambda - T}$ and it follows that $(\lambda - T)^{-1} \in B(X)$ by 1.2. The proof is complete.

The following collorary is related to [1].

1.4. *Let* H *be a Hilbert space, let* $T \in B(H)$. *If* T^* *has the single-valued extension property then* $\sigma(T) = \sigma_a(T)$ *(the approximate point spectrum).*

Proof. Take an $S \in B(H)$. A complex number λ does not belong to $\sigma_a(S)$ if and only if $\inf_{|x|=1} |(\lambda - S)\, x| > 0$. The last inequality is equivalent to the fact that $(\lambda - S)$

is one-to-one and its range is closed. This is, in its turn, equivalent to the existence of an operator R such that $R(\lambda - S) = I$. Indeed, denote by P the projection on $(\lambda - S)H$ then $(\lambda - S \mid (\lambda - S)H)^{-1} P(\lambda - S) = I$. On the other hand if $R(\lambda - S) = I$ then $|x| \leqq |R|\,|(\lambda - S)x|$ for every $x \in H$ and $\inf_{|x|=1} |(\lambda - S)x| \geqq |R|^{-1}$. Hence, $\sigma_a(S) = \{\lambda : B(H)(\lambda - S) \neq B(H)\}$. Since T^* has the single-valued extension property we obtain by 1.3 $\sigma(T) = \sigma(T^*)^* = \{\lambda : (\lambda - T^*)B(H) \neq B(H)\}^* = \{\lambda : B(H)(\lambda - T) \neq B(H)\} = \sigma_a(T)$.

1.5. Theorem. *Let $T \in B(X)$. Then each of the sets $\{x : \gamma_T(x) \supset \operatorname{bd} \sigma(T)\}$, $\{x : \sigma_T(x) = \sigma(T)\}$ is nonmeagre (i.e. it is not of the first category) in X.*

Proof. Denote by L_1 (resp. L_2) a countable dense subset of $\operatorname{bd} \sigma(T)$ (resp. $\sigma(T) \setminus S_T$). Since $L_1 \subset \operatorname{bd} \sigma(T)$ it follows $(\lambda - T)X \neq X$ for every $\lambda \in L_1$. Moreover, the set $(\lambda - T)X$ is of the first category in X by the open mapping theorem and so is the set $\bigcup_{\lambda \in L_1} (\lambda - T)X$. Taking an $x \notin \bigcup_{L_1} (\lambda - T)X$ we obtain $L_1 \subset \gamma_T(x)$ and consequently, $\operatorname{bd} \sigma(T) = L_1^- \subset \gamma_T(x)$.

To prove the theorem it remains to prove that the set $\{x : \sigma_T(x) = \sigma(T)\}$ is nonmeagre in X. If $\sigma(T) = S_T$ then $\sigma_T(x) \supset S_T$ by its definition so that $\sigma_T(x) = \sigma(T)$ for each $x \in X$. Hence assume that $\sigma(T) \setminus S_T \neq \emptyset$. Given $\lambda \in \sigma(T) \setminus S_T$ we have $(\lambda - T)X \neq X$ by 1.2. Again, the set $\bigcup_{\lambda \in L_2} (\lambda - T)X$ is of the first category in X and we have $\gamma_T(x) \supset L_2$ for every $x \notin \bigcup_{\lambda \in L_2} (\lambda - T)X$. It follows that $\sigma_T(x) = \gamma_T(x) \cup S_T \supset L_2^- \cup S_T = \sigma(T)$. The proof is complete.

Remark. On the other hand, let us mention that the set of those elements $x \in X$ for which $\sigma_T(x) \neq \sigma(T)$ may consist only of zero element. Take, for instance, the example presented by R. C. SINE [6] (See also [4]), i.e. the isometric shift in l_2, $T(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots)$. The operator T has the single-valued extension property and $\sigma_T(x) = \sigma(T)$ for every $0 \neq x \in l_2$. Indeed, since T is isometric, we have $|(\lambda - T)x| \geqq (1 - |\lambda|)|x| > 0$ for $|\lambda| < 1$ and $0 \neq x \in l_2$. It follows that T has no eigenvalue in the open unit disc and clearly it has the single-valued extension property. Further, suppose that a complex number λ_0, $|\lambda_0| < 1$ belongs to $\varrho_T(x)$ for some $x = (x_n)_1^\infty \in l_2$. Then $x \in (\lambda - T)X$ and $x \in \operatorname{Ker}(\lambda^* - T^*)^\perp$ for λ in a neighbourhood U_0 of λ_0. The only eigenvector corresponding to T^* and to λ^* is $x_\lambda = (\lambda^{*n-1})_1^\infty$ so that $\sum_{n=1}^{\infty} \lambda^{n-1} x_n = 0$ for all $\lambda \in U_0$. It follows that the power series $\sum_{n=1}^{\infty} \mu^{n-1} x_n$ converges also for all $|\mu| < |\lambda_0|$ and $\sum_{n=1}^{\infty} \mu^{n-1} x_n = 0$. Thus $x_n = 0$ for $n \geqq 1$ and $x = 0$. Since $|T| = 1$ it follows that $\sigma(T) = D(0, 1)^- = \sigma_T(x)$ for every $x \in l_2$.

The operator T^* has not the single-valued extension property and $\gamma_{T^*}(x)$ is empty or $\gamma_{T^*}(x) \subseteq \operatorname{bd} \sigma(T^*) = \{\lambda : |\lambda| = 1\}$ for every $x \in l_2$ [10].

The following theorem (with the exception of 1°) is included in [9]. We shall give now an elementary proof.

Denote by $\mathscr{F}(T)$ the set of all functions holomorphic in a neighbourhood of $\sigma(T)$.

1.6. *Let $f \in \mathscr{F}(T)$ which is non-constant in each component of definition. Then*

1° $f(\gamma_T(x)) = \gamma_{f(T)}(x)$ *for every* $x \in X$
2° $f(\sigma_T(x)) = \sigma_{f(T)}(x)$ *for every* $x \in X$
3° $S_{f(T)} = f(S_T)$.

Proof. Let $x \in X$ be given. The inclusion $f(\gamma_T(x)) \subset \gamma_{f(T)}(x)$ being trivial for $\gamma_T(x) = \emptyset$ we shall suppose that $\gamma_T(x) \neq \emptyset$. Given λ, let $g \in \mathscr{F}(T)$ be such that $f(z) - f(\lambda) = (z - \lambda)\, g(z)$. Suppose that $f(\lambda) \in \delta_{f(T)}(x)$. Let (x_n) be a sequence corresponding to $f(\lambda)$ and to the operator $f(T)$ by 1.1. Set $y_n = g^n(T)\, x_n$ for $n \geqq 1$. Then the sequence (y_n) satisfies

$$(T - \lambda)\, y_{n+1} = (T - \lambda)\, g^{n+1}(T)\, x_{n+1} = g^n(T)\,(T - \lambda)\, g(T)\, x_{n+1} = g^n(T)\, x_n = y_n\,,$$
$$(T - \lambda)\, y_1 = (T - \lambda)\, g(T)\, x_1 = x\,.$$

It follows that $\lambda \in \delta_T(x)$ and consequently, $f(\gamma_T(x)) \subset \gamma_{f(T)}(x)$.

On the other hand, take $\lambda \in \gamma_{f(T)}(x)$. The function f is non-constant so that $\lambda - f$ has only a finite number of zeros in some neighbourhood of $\sigma(T)$ so there exists a $g \in \mathscr{F}(T)$, $g(z) \neq 0$ for all z and $\lambda - f(z) = (\lambda_1 - z) \dots (\lambda_m - z)\, g(z)$. Suppose that all $\lambda_i \in \delta_T(x)$. By 1.1 we can choose elements x_n such that

$$(\lambda_1 - T) \dots (\lambda_m - T)\, x_{n+1} = x_n\,,$$

$|x_n| \leqq k^n$ for $n \geqq 1$ and some positive k, $(\lambda_1 - T) \dots (\lambda_m - T)\, x_1 = x$. Put

$$y_n = (1/g)^n\, (T)\, x_n\,.$$

Then $(\lambda - f(T))\, y_{n+1} = y_n$ and $|y_n| \leqq (|(1/g)\,(T)|\, k)^n$ for $n \geqq 1$, $(\lambda - f(T))\, y_1 = x$. Thus $\lambda \in \delta_{f(T)}(x)$ which is a contradiction. Hence $\lambda_i \in \gamma_T(x)$ for some i and

$$\lambda = f(\lambda_i) \in f(\gamma_T(x))\,.$$

This completes the proof of 1°.

Now we shall prove $f(S_T) = S_{f(T)}$. Take λ such that $(\lambda - T)\, h = 0$ with a suitable $h \neq 0$, $\gamma_T(h) = \emptyset$. Then also $(f(\lambda) - f(T))\, h = 0$ and by 1° $\gamma_{f(T)}(h) = \emptyset$ as well. This proves the inclusion $f(S_T) \subset S_{f(T)}$. Further, let λ be such that $(\lambda - f(T))\, h = 0$ with an $h \neq 0$, $\gamma_{f(T)}(h) = \emptyset$. To prove the inclusion $S_{f(T)} \subset f(S_T)$ it is sufficient to show that $f(\lambda) \in f(S_T)$ for each such λ.

By 1° $\gamma_T(h) = \emptyset$ as well. Let $\lambda - f(z) = (z - \lambda_1) \dots (z - \lambda_m)\, g(z)$ with $g \in \mathscr{F}(T)$

and g has no zeros. We have $0 = (\lambda - f(T))\, h = (T - \lambda_1) \ldots (T - \lambda_m)\, g(T)\, h$. Suppose that all $\lambda_i \in \Omega_T$. Let j be the greatest index such that

$$(T - \lambda_j) \ldots (T - \lambda_m)\, g(T)\, h = 0 .$$

Denote by $y = (T - \lambda_{j+1}) \ldots (T - \lambda_n)\, g(T)\, h \neq 0$. If $j = n$ set $y = g(T)\, h$. Since $1/g \in \mathscr{F}(T)$ we have $g(T)\, h \neq 0$.

Since $(\lambda_j - T)\, y = 0$ we have $(\lambda - T)\, (\lambda - \lambda_j)^{-1}\, y = y$ for $\lambda \neq \lambda_j$. Since $\lambda_j \in \Omega_T$ it follows that $\gamma_T(y) = (\lambda_j)$. On the other hand, $\emptyset = \gamma_T(h) \supset \gamma_T(y)$ which is a contradiction. It follows that at least one $\lambda_i \in S_T$ and $\lambda = f(\lambda_i) = f(S_T)$. We proved 3°.

We obtain condition 2° as follows:

$$\sigma_{f(T)}(x) = \gamma_{f(T)}(x) \cup S_{f(T)} = f(\gamma_T(x)) \cup f(S_T) = f(\gamma_T(x) \cup S_T) = f(\sigma_T(x)) .$$

2. Denote by P_n^1 the set of all monic polynomials (i.e. the leading coefficient is 1) of degree n, by P^1 the set of all monic polynomials. By $d(p)$ we shall denote the degree of $p \in P^1$.

Definition. Let A be a Banach algebra with a unit over the complex field, let $a \in A$ be given. Denote by $\operatorname{cap}_n a = \inf_{p \in P_n^1} |p(a)|$ and by $\operatorname{cap} a = \lim_n (\operatorname{cap}_n a)^{1/n}$. The number cap a is called the capacity of a.

The norm in A is submultiplicative so that $\operatorname{cap}_{n+m} a \leqq \operatorname{cap}_n a \,.\, \operatorname{cap}_m a$ and thus $\lim_n (\operatorname{cap}_n a)^{1/n}$ exists and $\operatorname{cap} a = \inf_n (\operatorname{cap}_n a)^{1/n}$.

Let K be a compact set in the complex plane. Consider the algebra $C(K)$ with norm $|f|_K = \sup_{z \in K} |f(z)|$. Then the capacity of the identical function $a(\lambda) = \lambda$, i.e. $\lim_{n \to \infty} (\inf_{p \in P_r^1} |p|_K)^{1/n}$ is known as the Čebyšev constant of the set K and we shall denote it by cap K. Set cap $\emptyset = 0$.

The present definition of capacity differs slightly from that given in the paper of P. R. Halmos [5]. We shall give now a shorter proof of some results included in [5].

2.1. $\operatorname{cap} a = \inf_{p \in P^1} |p(a)|^{1/d(p)} = \inf_{p \in P^1} |p(a)|_\sigma^{1/d(p)}$.

Proof. The relation $\operatorname{cap} a = \inf_{p \in P^1} |p(a)|^{1/d(p)}$ follows immediately from the definition of capacity. Further, the inequality $|p(a)|_\sigma \leqq |p(a)|$ gives

$$\inf_{p \in P^1} |p(a)|_\sigma^{1/d(p)} \leqq \inf_{p \in P^1} |p(a)|^{1/d(p)} .$$

On the other hand, we have

$$\operatorname{cap} a \leqq \inf_{p \in P_{nk}^1} |p(a)|^{1/nk} \leqq \inf_{p \in P_n^1} |p^k(a)|^{1/nk}$$

for every natural k, n and this implies

$$\text{cap } a \leqq \inf_{k,n} \inf_{p \in P_n^1} |p^k(a)|^{1/nk} = \inf_n \inf_{p \in P_n^1} \inf_k |p^k(a)|^{1/nk} = \inf_n \inf_{p \in P_n^1} |p(a)|_\sigma^{1/n} = \inf_{p \in P^1} |p(a)|_\sigma^{1/d(p)} .$$

This completes the proof.

2.2. $$\text{cap } a = \text{cap } \sigma(a) .$$

Proof. According to 2.1 and to the spectral mapping theorem

$$\text{cap } \sigma(a) = \inf_{p \in P^1} |p|_{\sigma(a)}^{1/d(p)} = \inf_{p \in P^1} |p(a)|_\sigma^{1/d(p)} = \text{cap } a .$$

Definition. An element $a \in A$ is said to be *algebraic* if $p(a) = 0$ for some $p \in P^1$. An element a is said to be *quasialgebraic* if there exists a sequence of polynomials $p_n \in P^1$ such that $|p_n(a)|^{1/d(p_n)} \to 0$. It follows from 2.1 that

2.3. *The following assertions are equivalent*:

1° *there exists a sequence of monic polynomials* (p_n) *such that* $|p_n(a)|^{1/d(p_n)} \to 0$,

2° $\text{cap } a = 0$,

3° *there exists a sequence of monic polynomials* (p_n) *such that* $|p_n(a)|_\sigma^{1/d(p_n)} \to 0$.

2.4. *Let $a \in A$ be quasialgebraic, let f be a function holomorphic in a neighbourhood of $\sigma(a)$. Then $f(a)$ is quasialgebraic.*

Proof. According to 2.2 we have $\text{cap } \sigma(a) = 0$. According to a classical theorem [3] (285–291) there exist monic polynomials t_n of degree n with roots in $\sigma(a)$ such that $\lim_n |t_n|_{\sigma(a)}^{1/n} = \text{cap } \sigma(a) = 0$. Write t_n in the form $t_n(z) = (z - \lambda_1) \dots (z - \lambda_n)$. Define functions f_n^i $(i = 1, 2, \dots, n)$ by the following two postulates: $f_n^i(z) = (f(z) - f(\lambda_i))(z - \lambda_i)^{-1}$ for $z \neq \lambda_i$, $f_n^i(z) = f'(\lambda_i)$ for $z = \lambda_i$. All functions f_n^i are holomorphic in the same neighbourhood of $\sigma(a)$ as f is and we have $|f_n^i|_{\sigma(a)} \leqq 2|f'|_{\sigma(a)} d(\sigma(a))$. Set $f_n(z) = \prod_{i=1}^n f_n^i(z)$. Then we have $|f_n|_{\sigma(a)} \leqq [2|f'|_{\sigma(a)} \mathrm{d}(\sigma(a))]^n$. Further, take the polynomial $q_n(z) = \prod_{i=1}^n (z - f(\lambda_i))$. The order of q_n is n and $q_n(f(z)) = f_n(z)\, t_n(z)$ for $z \neq \lambda_i$, by continuity equality holds everywhere. Further, the following estimate is true

$$|q_n|_{f(\sigma(a))}^{1/n} = |q_n \circ f|_{\sigma(a)}^{1/n} = |f_n t_n|_{\sigma(a)}^{1/n} \leqq |f_n|_{\sigma(a)}^{1/n} |t_n|_{\sigma(a)}^{1/n} \leqq 2|f'|_{\sigma(a)} d(\sigma(a)) \,.\, |t_n|_{\sigma(a)}^{1/n} .$$

The last term tends to zero as $n \to \infty$ and this implies that $0 = \text{cap } f(\sigma(a)) = \text{cap } \sigma(f(a)) = \text{cap } f(a)$.

Let X be a Banach space. 2.3 provides different methods how to define locally quasialgebraic operators in the algebra $B(X)$. In his paper [8] F. H. Vasilescu took the local version of 3° which turned out to be the local version of 2°. We shall begin with some definitions: Given an $x \in X$, denote by $r(T, x) = \limsup_n |T^n x|^{1/n}$. Clearly $r(T, x)$ is the radius of the minimal disk outside of which $x = (\mu - T) f_x(\mu)$ for a holomorphic function f_x.

Denote by $\operatorname{cap}_n(T, x) = \inf_{p \in P_n^1} r(p(T), x)$ and by $\operatorname{cap}(T, x) = \inf_{p \in P^1} r(p(T), x)^{1/d(p)}$.

2.5. $\operatorname{cap} \gamma_T(x) \leqq \operatorname{cap}(T, x) \leqq \operatorname{cap} \sigma_T(x)$ *for every* $x \in X$.

Proof. Clearly $\gamma_{p(T)}(x) \subset \{\lambda : |\lambda| \leqq r(p(T), x)\} \subset \sigma_{p(T)}(x)$. It follows $|a|_{\gamma_{p(T)}(x)} \leqq$ $\leqq r(p(T), x) \leqq |a|_{\sigma_{p(T)}(x)}$ (recall that a is the identical function). From 1.6 it follows that $|p|_{\gamma_T(x)} = |a|_{\gamma_{p(T)}(x)} \leqq r(p(T), x) \leqq |a|_{\sigma_{p(T)}(x)} = |p|_{\sigma_T(x)}$ for each $p \in P^1$ and consequently $\operatorname{cap} \gamma_T(x) \leqq \operatorname{cap}(T, x) \leqq \operatorname{cap} \sigma_T(x)$. The proof is complete.

Now, we shall give an example of an operator for which the inequalities in the preceding proposition are strict.

Let S be the left shift on the space l_1. Denote by $e_i = (\xi_1^i, \xi_2^i, \ldots)$ such that $\xi_k^i = \delta_{ik}$. Then $S \sum_{i=1}^{\infty} \xi_i e_i = \sum_{i=1}^{\infty} \xi_{i+1} e_i$. Denote by X the Banach space of all bounded sequences $x = (x_n)$, $x_n \in l_1$ with the supremum norm, $|x| = \sup |x_n|$. Define the operator $T \in B(X)$ by the formula $(Tx)_n = Sx_n$. Since $x_\lambda = \sum_1^\infty \lambda^{n-1} e_n$ satisfies $Sx_\lambda = \lambda x_\lambda$ for $|\lambda| < 1$ if follows that $|\lambda| < 1$ is an eigenvalue of T; $|T| = 1$ so that $\sigma(T) =$ $= D(0, 1)^-$. Further, $S_T = D(0, 1)^-$. Indeed, $(\mu - S) \sum_1^\infty \mu^{n-1} e_n = 0$ in $D(0, 1)$ and $|\sum_1^\infty \mu^{n-1} e_n| = 1/(1 - |\mu|)$. Thus $D(0, 1)^- \subseteq S_T \subseteq \sigma(T) = D(0, 1)^-$.

Take a sequence (λ_i) of complex numbers such that $(\lambda_1, \lambda_2, \ldots)^- = D(0, \varepsilon)^-$ for some $0 < \varepsilon < 1$. Form a sequence $x_0 = (x_n) \subset X$ such that $Sx_n = \lambda_n x_n$, $|x_n| = 1$. We shall show that $0 = \operatorname{cap} \gamma_T(x_0) < \operatorname{cap}(T, x_0) = \operatorname{cap} D(0, \varepsilon)^- < \operatorname{cap} \sigma_T(x_0) =$ $= \operatorname{cap} D(0, 1)^-$. Clearly $\sigma_T(x_0) = S_T = D(0, 1)^-$.

Observe that $|p^k(T) x_0|^{1/k} = (\sup_n |p^k(S) x_n|)^{1/k} = (\sup_n |p^k(\lambda_n)|)^{1/k} = \sup_n |p(\lambda_n)| =$ $= |p|_{D(0,\varepsilon)^-}$ and $r(p(T), x_0) = |p|_{D(0,\varepsilon)^-}$. From the last equality it follows that $\operatorname{cap}(T, x_0) = \operatorname{cap} D(0, \varepsilon)^-$.

Take $0 < \delta < 1$ arbitrary. Set $\hat{f}_\lambda(\mu) = \sum_1^\infty (\mu^n - \lambda^n)(\mu - \lambda)^{-1} e_{n+1}$ for $\mu \neq \lambda$, $\mu \in D(0, \delta)$, $\lambda \in D(0, 1)$. The function $\hat{f}_\lambda$ can be extended to an analytic function f_λ in $D(0, \delta)$ and $|f_\lambda|_{D(0,\delta)} \leqq |x_\lambda| \, [1/(1 - \delta)]$. Moreover, we have $(\mu - S) f_\lambda(\mu) = x_\lambda$ in $D(0, \delta)$. It follows that $D(0, 1) \subset \delta_T(x_0)$. Now, take ε', $0 < \varepsilon < \varepsilon' < 1$ and define $g_n(\mu) = (\mu - \lambda_n)^{-1} x_n$ for $|\mu| > \varepsilon'$. Then $(\mu - S) g_n(\mu) = x_n$ and $|g_n(\mu)| \leqq 1/(\varepsilon' - \varepsilon)$ for $|\mu| > \varepsilon'$. Thus $\delta_T(x_0) \supset (|\lambda| \geqq \varepsilon')$ and $\gamma_T(x_0) = \emptyset$.

Definition. An operator $T \in B(X)$ is said to be locally *quasialgebraic* if for every $x \in X$ there exists a sequence of monic polynomials p_n such that $r(p_n(T), x)^{1/d(p_n)} \to 0$.

2.6. *Every locally quasialgebraic operator is quasialgebraic.*

Proof. Let $T \in B(X)$ be locally quasialgebraic. By definition of cap (T, x) and by 2.6 it follows that cap $\gamma_T(x) = 0$ for every $x \in X$. Choose an $x \neq 0$ such that $\gamma_T(x) \supset \text{bd } \sigma(T)$ by 1.4. Then cap $\sigma(T) =$ cap bd $\sigma(T) \leqq$ cap $\gamma_T(x) = 0$. This completes the proof.

References

[1] *J. Bunce:* Characters on singly generated C* algebras, Proc. Amer. Math. Soc. *25* (1970), 297—303.

[2] *N. Dunford:* Spectral operators, Pacific J. Math. *4* (1954), 321—354.

[3] *G. M. Goluzin:* Геометрическая теория функций комплексного переменного, Moscow 1966.

[4] *J. D. Gray:* Local analytic extensions of the resolvent, Pacific J. Math. *27* (1968), 305—324.

[5] *P. R. Halmos:* Capacity in Banach algebras, Indiana Univ. Math. J. *20* (1971), 855—863.

[6] *R. C. Sine:* Spectral decomposition of a class of operators, Pacific J. Math. *14* (1964), 333—352.

[7] *F. H. Vasilescu:* Residually decomposable operators in Banach spaces, Tôhoku Math. Journ. *21* (1969), 509—522.

[8] *F. H. Vasilescu:* Local capacity of operators, Indiana Univ. Math. J. *21* (1972), 743—749.

[9] *F. H. Vasilescu:* Analytic functions and some residual spectral properties, Rev. Roum. Math. *15* (1970), 435—451.

[10] *F. H. Vasilescu:* Operatori rezidual decompozabili în spaţii Fréchet, Studii Cerc. Mat. *21* (1969), 1181—1248.

Author's address: 115 67 Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV v Praze).