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On Locally Normal States in Quantum Statistical Mechanics

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Abstract. A sufficient condition is given in order that a von Neumann algebra with cyclic vector is quasi-standard. With the help of this result it is proved that a locally normal state with a cyclic and separating vector in the representation space gives rise to a quasi-standard von Neumann algebra. Furthermore it is proved that the representation space determined by a locally normal state in the G.N.S. construction is separable.

I. Introduction

The results of this work have their origin in two recent papers [4, 9]. In [9] TOMITA proves the highly non-trivial result that a von Neumann algebra which has a cyclic and separating vector is quasistandard. In this paper an easy proof of this result will be given for a special case which is of particular interest for physical applications, where the von Neumann algebra considered is generated by the representation of a C^*-algebra of quasi-local observables determined by a locally normal state.

Following [4] we define an algebra $\mathfrak{A}$ of quasi-local observables as a norm-closed algebra of operators in Fock-space $\mathfrak{H}_F$: to every finite volume V is assigned the algebra $\mathfrak{A}(V)$ of all bounded operators operating in the sub-Fock space $\mathfrak{H}_F^V \subset \mathfrak{H}_F$, hence $\mathfrak{A}(V)$ is a norm closed and weakly closed algebra; then $\mathfrak{A}$ is defined in $\mathfrak{H}_F$ as the closure in the norm topology of the union of the $\mathfrak{A}(V)$ for all finite V, $\mathfrak{A} = \overline{\bigcup_V \mathfrak{A}(V)} = \overline{\mathfrak{A}}_L$, where the "local" algebra $\mathfrak{A}_L$ is defined by $\mathfrak{A}_L = \bigcup_V \mathfrak{A}(V)$.

In [4] the G.N.S. representation determined by a normal state

$$\omega_V(A) = \mathrm{Tr}_V \varrho_V A,\ A \in \mathfrak{A}(V)$$

defined over $\mathfrak{A}(V)$, with the additional property $\omega_V(A^*A) = 0$ implies $A = 0$ for every $A \in \mathfrak{A}(V)$ is constructed. Here ϱ_V is a density operator in $\mathfrak{H}_F^V$ with $\mathrm{Tr}_V \varrho_V = 1$ (the index V in Tr_V denotes that the trace is taken in the Hilbert space $\mathfrak{H}_F^V$). One of the results that can be inferred from that paper is that the von Neumann algebra generated by the representatives of the elements from $\mathfrak{A}(V)$ is quasi-standard [5], that means is the left representation of a certain quasi-unitary algebra $\tilde{\mathfrak{A}}(V) \subset \mathfrak{A}(V)$. (For a fuller discussion of quasi-standard von Neumann algebras see Section II.) We want to stress the fact that the normal state ω_V over $\mathfrak{A}(V)$ not only gives rise to a standard von Neumann algebra [4] but that this algebra is also quasi-standard in a non-trivial sense since the quasi-unitary algebra $\tilde{\mathfrak{A}}(V)$ is not a Hilbert algebra.

A discussion of quasi-standard von Neumann algebras is given in Section II together with a sufficient condition (Theorem 1) in order that a von Neumann algebra be quasi-standard. From Theorem 1 one concludes immediately that the von Neumann algebra in the case of a finite volume V is quasi-standard. A state over the C^*-algebra of quasi-local observables will be called locally normal if its restriction to any local algebra $\mathfrak{A}(V) \subset \mathfrak{A}$ is normal: let ω be a locally normal state over $\mathfrak{A}$ then there exists a density operator σ_V on Fock space $\mathfrak{H}_F^V$ with $\mathrm{Tr}_V \sigma_V = 1$ such that

$$\omega(A) = \mathrm{Tr}_V \sigma_V A \quad \text{for all} \quad A \in \mathfrak{A}(V) .$$

It can be proved [1] (see also [8]) that this condition is equivalent to the requirement that for every finite volume V there exists in the representation space a total particle number operator N_V. It is for this reason that for applications in statistical mechanics one deals almost exclusively with locally normal states.

In view of the above mentioned result from [4] it follows that a locally normal state ω over $\mathfrak{A}$, with the additional property that $\omega(A^*A) = 0$ implies $A = 0$ for all $A \in \mathfrak{A}_L$, is the pointwise limit of a sequence of (normal) states each generating in its representation space a quasi-standard von Neumann algebra. It is shown in Section III that this quasi-standard structure is conserved in the limit $V \to \infty$. This constitutes an easy proof of Theorem 2, that the von Neumann algebra generated by a locally normal state is quasi-standard provided there is a cyclic and separating vector. As already mentioned above this is a special case of a result obtained by TOMITA.

In Section III we will furthermore show that locally normal states have the attractive property that their representation space is separable (Theorem 3). Two lemmas which are needed in the proof of Theorem 2 are given in an appendix.

II. Quasi-Standard von Neumann Algebras

In this section quasi-standard von Neumann algebras will be discussed and a theorem concerning these algebras will be proved. The definitions as given in [2] will be used.

We first define a quasi-unitary algebra $\mathscr{A}$. This is an algebra over the complex number field with a scalar product (A, B) for its elements which makes $\mathscr{A}$ into a pre-Hilbert space; the completion of this pre-Hilbert space will be called $\mathfrak{H}$. Furthermore there are given an automorphism

$A \to A^{\vartheta}$ and an anti-automorphism $A \to A^{j}$ such that the following five conditions are fulfilled:

A1: $(A, A^{\vartheta}) \geqq 0, A \in \mathscr{A}$;

A2: $(A, A) = (A^{j}, A^{j}), A \in \mathscr{A}$;

A3: $(A B, C) = (B, A^{j\vartheta} C), A, B, C \in \mathscr{A}$;

A4: the mapping $B \to A B$ is continuous for all $A \in \mathscr{A}$;

A5: the set of elements $A B + (A B)^{\vartheta}$, where $A, B \in \mathscr{A}$ is dense in $\mathscr{A}$.

The left representation $A \to U_A$ of $\mathscr{A}$ into the bounded operators of $\mathfrak{H}$, where U_A is the extension to $\mathfrak{H}$ of the mapping $B \to A B$ generates a von Neumann algebra $\mathfrak{R}$. We can now give a definition of a quasi-standard von Neumann algebra: a von Neumann algebra is quasi-standard if it is (spatially isomorphic to) the left representation of a certain quasi-unitary algebra. In particular $\mathfrak{R}$ is quasi-standard.
When the mapping $A \to A^{\vartheta}$ is the identity mapping one speaks of a Hilbert algebra resp. a standard von Neumann algebra.
N.B. A quasi-unitary algebra need not have an identity. If, however, $\mathscr{A}$ has an identity E then condition A5 may be replaced by:

A5': The set of elements $A + A^{\vartheta}$ is dence in $\mathscr{A}$.

In that case the vector E will be a cyclic and separating vector for $\mathfrak{R}$.

Theorem 1. *Let $\mathfrak{R}$ be a von Neumann algebra in a Hilbert space $\mathfrak{H}$ with cyclic vector Ω such that*

(i) *there exists in $\mathfrak{H}$ an involution J with $J\Omega = \Omega$, $J \mathfrak{R} J = \mathfrak{R}'$;*

(ii) *there exists in $\mathfrak{H}$ a positive self-adjoint operator T with self-adjoint inverse T^{-1} and*

$$T R \Omega = J I^{R} R \Omega, \; R \in \mathfrak{R} ,$$

where I^R is defined by $I^R R \Omega = R^* \Omega$;

(iii) *the unitary operator U_τ defined by $U_\tau = T^{-2 i \tau}$ satisfies*

$$U_\tau \mathfrak{R} U_\tau^{-1} = \mathfrak{R} .$$

Then $\mathfrak{R}$ is quasi-standard.

Proof. To prove Theorem 1 it is sufficient to prove the existence of a subalgebra $\tilde{\mathfrak{R}} \subset \mathfrak{R}$ which is quasi-unitary and has the property that $\tilde{\mathfrak{R}} \Omega$ is dense in $\mathfrak{H}$. We construct $\tilde{\mathfrak{R}}$ as follows:

Let $\Phi \in \mathfrak{H}$, $R \in \mathfrak{R}$ and $R_\tau = U_\tau R U_\tau^{-1}$ then $R_\tau \Phi$ is strongly continuous. If $f(\tau) \in \mathscr{S}$ (the class of infinitely differentiable rapidly decreasing functions) then $f(\tau) R_\tau \Phi$ is Bochner integrable.

We define the linear operator $R(f)$ by

$$R(f) \Phi = \int d\tau \, f(\tau) \, R_\tau \Phi . \tag{2.1}$$

One proves without difficulty that $R(f) \in \mathfrak{R}$. Eq. (2.1) defines a tempered distribution with values in $\mathfrak{H}$.

Definition 1. $\tilde{\mathfrak{R}}$ is the set of all $R \in \mathfrak{R}$ such that $R_\tau \Phi$ has a Fourier transform with compact support independent of Φ.

It can be shown [10] that this definition is equivalent with

Definition 1'. $\tilde{\mathfrak{R}}$ is the set of all $R(f)$ with $R \in \mathfrak{R}$ and $\hat{f}$ compact support.

Lemma 1. *$\tilde{\mathfrak{R}}$ is a *-algebra.*

Proof. a) Let $R \in \tilde{\mathfrak{R}}$, then $R = R(f)$ where the Fourier-transform $\hat{f}$ of f has compact support. $(R(f)^* \Psi, \Phi) = (\Psi, R(f)\Phi) = \int d\tau f(\tau)(\Psi, R_\tau \Phi)$ $= \int d\tau f(\tau)(R_\tau^* \Psi, \Phi) = (R^*(f^*)\Psi, \Phi)$ for all $\Psi, \Phi \in \mathfrak{H}$. Hence $R(f)^*$ $= R^*(f^*) \in \tilde{\mathfrak{R}}$.

b) Let the functions $f_1(\tau)$ and $f_2(\tau)$ have Fourier-transforms $\hat{f}_1(\varepsilon)$ and $\hat{f}_2(\varepsilon)$ of compact support and consider $R_3 = R_1(f_1)R_2(f_2)$. We shall prove that $R_3 \in \tilde{\mathfrak{R}}$.

$$R_3(f_3) = \int d\tau_3 \iint d\tau_1 d\tau_2 R_1(\tau_1 + \tau_3) R_2(\tau_2 + \tau_3) f_1(\tau_1) f_2(\tau_2) f_3(\tau_3)$$
$$= \iint d\tau_1 d\tau_2 R_1(\tau_1) R_2(\tau_2) f(\tau_1, \tau_2),$$

where $f(\tau_1, \tau_2) = \int d\tau_3 f_1(\tau_1 - \tau_3) f_2(\tau_2 - \tau_3) f_3(\tau_3)$. One proves easily that $f(\tau_1, \tau_2) \equiv 0$ if the support of $\hat{f}_3(\varepsilon)$ is completely outside the sum of the supports of $\hat{f}_1(\varepsilon)$ and $\hat{f}_2(\varepsilon)$. Indeed,

$$\hat{f}(\varepsilon_1, \varepsilon_2) = \hat{f}_1(\varepsilon_1)\hat{f}_2(\varepsilon_2)\hat{f}_3(\varepsilon_1 + \varepsilon_2). \tag{2.2}$$

This proves that $R_3 \in \tilde{\mathfrak{R}}$.

Definition 2. Let $R \in \tilde{\mathfrak{R}}$, then $R = R(f)$ where $\hat{f}(\varepsilon)$ has compact support; we define $R^\vartheta = R(f^\vartheta)$, where $\hat{f}^\vartheta(\varepsilon) = \hat{f}(\varepsilon) e^{\frac{1}{2}\varepsilon}$

Lemma 2. *The mapping $R \to R^\vartheta$ is an automorphism of $\tilde{\mathfrak{R}}$ and $R^{\vartheta *} = R^{*(-\vartheta)}$.*

Proof. We must prove first that $(R_1 R_2)^\vartheta = R_1^\vartheta R_2^\vartheta$. This can be inferred from the proof of b) in Lemma 1. Indeed, the change in $\hat{f}(\varepsilon_1, \varepsilon_2)$ in (2.2) due to the replacement $f_1 \to f_1^\vartheta$ and $f_2 \to f_2^\vartheta$ is the same as the change due to the replacement $f_3 \to f_3^\vartheta$. The proof of the second part is immediate.

Lemma 3[1]. *For $R \in \tilde{\mathfrak{R}}$ we have $TR\Omega = R^{-\vartheta}\Omega$.*

Lemma 4[2]. *The set of all $(R + R^{-\vartheta})\Omega$ for $R \in \tilde{\mathfrak{R}}$ is dense in $\mathfrak{H}$.*

Definition 3. $R^j = R^{*-\vartheta}$ for all $R \in \tilde{\mathfrak{R}}$.

Lemma 5. *$JR\Omega = R^j\Omega$ for all $R \in \tilde{\mathfrak{R}}$.*

Proof. $R^j\Omega = TR^*\Omega = JR\Omega$.

Definition 4. We define in $\tilde{\mathfrak{R}}$ the scalar product $(R_1, R_2) = (R_1\Omega, R_2\Omega)$.

Using the Lemmas 1, 2, 3, 4, 5 one proves without difficulty that $\tilde{\mathfrak{R}}$ with the automorphism $R \to R^\vartheta$ and anti-automorphism $R \to R^j$ satisfies all axioms A1 to A5 of a quasi-unitary algebra.

Remark. TOMITA [9] has shown that the conditions of Theorem 1 are also necessary.

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[1] For a proof compare Ref. [10], Appendix.

[2] For a proof see e.g. [4], Theorem 3A.

III. A Special Case of Tomita's Result

1. Von Neumann Algebras with Cyclic and Separating Vector

In this subsection some introductory results obtained by TOMITA [9], which will be needed in the following, are discussed. Consider a von Neumann algebra $\mathfrak{R}$ operating in a Hilbert-space $\mathfrak{H}$ with cyclic and separating vector $\Omega \in \mathfrak{H}$. Hence there exists a one-to-one correspondence $R \in \mathfrak{R} \leftrightarrow R\Omega \in \mathfrak{R}\Omega$ and $S \leftrightarrow S\Omega$ for $S \in \mathfrak{R}'$. One says that a vector $\Phi \in \mathfrak{H}$ has an R-adjointive Φ^R when $(\Phi, S\Omega) = (S^*\Omega, \Phi^R)$ for all $S \in \mathfrak{R}'$, and $\Phi \in \mathfrak{H}$ has an S-adjointive Φ^S when the relation $(\Phi, R\Omega) = (R^*\Omega, \Phi^S)$ is fulfilled for all $R \in \mathfrak{R}$. It is easy to see for instance, that all vectors from $\mathfrak{R}\Omega$ have an R-adjointive, namely $\Phi^R = R^*\Omega$ if $\Phi = R\Omega$. Now the space $\mathfrak{H}^R \subset \mathfrak{H}$ is defined as the set $\{\Phi \in \mathfrak{H} : \Phi^R \text{ exists}\}$, and analogously $\mathfrak{H}^S \subset \mathfrak{H}$ is the set $\{\Phi \in \mathfrak{H} : \Phi^S \text{ exists}\}$. In $\mathfrak{H}^R$ the following inner product $(\Phi, \Psi)_R$ and norm $\|\Phi\|_R$ can be introduced:

$$(\Phi, \Psi)_R = (\Phi, \Psi) + (\Psi^R, \Phi^R)\,, \tag{3.1}$$

$$\|\Phi\|_R^2 = \|\Phi\|^2 + \|\Phi^R\|^2\,. \tag{3.2}$$

It can be shown that $\mathfrak{H}^R$ is a Hilbert-space and that the set $\{R\Omega : R \in \mathfrak{R}\}$ is dense in $\mathfrak{H}^R$ (in the topology of $\mathfrak{H}^R$). The operator I^R defined by

$$I^R\Phi = \Phi^R, \ \Phi \in \mathfrak{H}^R \tag{3.3}$$

is an involution in $\mathfrak{H}^R$. Similar properties hold for $\mathfrak{H}^S$.

In $\mathfrak{H}^R$ is defined a self-adjoint positive definite bounded operator $\hat{\Gamma}$ by its matrixelements

$$(\Phi, \hat{\Gamma}\Psi)_R = (\Phi, \Psi),\ \Phi, \Psi \in \mathfrak{H}^R\,. \tag{3.4}$$

Since $\|\hat{\Gamma}^{1/2}\Psi\|_R = \|\Psi\|$ the mapping $\Psi \in \mathfrak{H}^R \to \hat{\Gamma}^{1/2}\Psi$ can be extended to an isometric mapping $\Gamma^{1/2}$ of $\mathfrak{H}$ onto $\mathfrak{H}^R : \Gamma^{1/2}\mathfrak{H} = \mathfrak{H}^R$. This isometry maps the involution $\Phi \to I^R\Phi$ in $\mathfrak{H}^R$ onto an involution J' in $\mathfrak{H}$, where

$$J'\Phi = \Gamma^{-1/2}\, I^R\, \Gamma^{1/2}\Phi,\ \Phi \in \mathfrak{H}\,. \tag{3.5}$$

It follows that J' has the properties

$$J'\Omega = \Omega\,, \qquad J'\Gamma J' = 1 - \Gamma\,. \tag{3.6}$$

The operator T' defined by

$$(T')^2 = \Gamma^{-1}(1 - \Gamma) \tag{3.7}$$

is a positive definite invertible operator in $\mathfrak{H}$ with domain $\mathfrak{H}^R$ and range $\mathfrak{H}^S$ and which maps $\mathfrak{H}^R$ isometrically onto $\mathfrak{H}^S$. From (3.6) and (3.7) one derives without difficulty the equation

$$T' = J'I^R\,. \tag{3.8}$$

Eqs. (3.3) and (3.8) imply that the involution J' maps $\mathfrak{H}^R$ onto $\mathfrak{H}^S$:

$$J'\mathfrak{H}^R = \mathfrak{H}^S\,. \tag{3.9}$$

2. *Representations Induced by Normal States*

In this subsection we shall discuss in some detail the G.N.S. representation of a normal state over the set of all bounded operators $\mathfrak{A}$ in a Hilbert space $\mathfrak{H}$

$$\omega(A) = \mathrm{Tr}\,\varrho A, \qquad A \in \mathfrak{A}, \tag{3.10}$$

where ϱ has the additional property that ϱ^{-1} is a positive self-adjoint operator. We define the unitary operator U_τ by the equation

$$U_\tau = \varrho^{-i\tau}. \tag{3.11}$$

The mapping

$$A \to A_\tau = U_\tau A U_\tau^{-1}, \qquad A \in \mathfrak{A}, \tag{3.12}$$

is an inner automorphism of $\mathfrak{A}$, since $\mathfrak{A}$ consists of all bounded operators on $\mathfrak{H}$. With respect to this automorphism the state ω satisfies the same analyticity condition and boundary condition as the equilibrium states discussed by HAAG, HUGENHOLTZ and WINNINK [4]. We conclude therefore that the representation determined by the state ω has the same structure as that discussed in [4]. In particular

a) In the representation space $\mathfrak{K}$, whose vectors $\varkappa$ are the Hilbert-Schmidt operators on $\mathfrak{H}$, exist two cyclic representations $A \in \mathfrak{A} \to R(A)$ and $A \to S(A)$, respectively defined by

$$R(A)\,\varkappa = A\varkappa, \qquad S(A)\,\varkappa = \varkappa A^*, \tag{3.13}$$

and with cyclic vector $\varkappa_0 = \varrho^{1/2}$.

These two sets of operators are von Neumann algebras[3] $R(\mathfrak{A})$ resp. $S(\mathfrak{A})$ with the property $R(\mathfrak{A}) = S(\mathfrak{A})'$. Since $\varkappa_0$ is cyclic for $S(\mathfrak{A})$ this implies that $\varkappa_0$ is separating for $R(\mathfrak{A})$.

b) There exists in $\mathfrak{K}$ a conjugation J with the properties

$$J R(A)\, J = S(A), \qquad J\varkappa_0 = \varkappa_0. \tag{3.14}$$

c) In $\mathfrak{K}$ exists a positive, self-adjoint and invertible operator T which satisfies

$$T R(A)\,\varkappa_0 = S(A^*)\,\varkappa_0 = J I^R R(A)\,\varkappa_0. \tag{3.15}$$

The automorphism $A \to A_\tau$, $A \in \mathfrak{A}$, is implemented by the operator $\hat{U}_\tau$, which is of the form

$$\hat{U}_\tau = R(U_\tau)\, S(U_\tau), \tag{3.16}$$

and

$$\hat{U}_\tau = T^{-2i\tau}. \tag{3.17}$$

From Theorem 1 one immediately concludes that the von Neumann algebra $R(\mathfrak{A})$ is quasi-standard.

[3] See [3] p. 57, Prop. 1.

It can be shown[4] that the domain of T coincides with the set $\mathfrak{H}^R$, introduced in Section III.1. Consequently from (3.15) follows

$$J\,T\,\Phi = I^R\,\Phi, \quad \text{for all} \quad \Phi \in \mathfrak{H}^R . \tag{3.18}$$

As remarked above we have a cyclic and separating vector $\varkappa_0$ and can therefore apply the results of Section III.1, in particular

$$J'\,T'\,\Phi = I^R\,\Phi, \quad \text{for all} \quad \Phi \in \mathfrak{H}^R . \tag{3.19}$$

Combining (3.18) and (3.19) it follows that

$$J\,T = J'\,T' . \tag{3.20}$$

Since the product of two conjugations in a Hilbert-space is a unitary mapping one can write $J' = J\,U$ where U is a unitary operator. Hence from (3.20) it follows that

$$T = U\,T' ,$$

and because the polar decomposition is unique we have $U = 1$ and

$$T = T', \qquad J = J' . \tag{3.21}$$

3. Statement and Proof of the Main Theorem

Before stating our main result we prove two lemmas that are needed. The first is a sharper form of a result of Tomita already mentioned above.

Lemma 6. *Let again* $\mathfrak{A}_L = \bigcup_V \mathfrak{A}(V)$ *then there exists a sequence* $R_i \in R(\mathfrak{A}_L)$ *such that*

$$\|R_i\Omega - \Phi\|_R \to 0 \quad \text{for} \quad i \to \infty$$

for any given vector $\Phi \in \mathfrak{H}^R$.

Proof. Since the weak closure of $R(\mathfrak{A}_L)$ is equal to $\mathfrak{R} = R(\mathfrak{A})''$, it is known that the set of hermitian elements from $R(\mathfrak{A}_L)$ is strongly dense in the set of hermitian elements from $\mathfrak{R}$. Hence for a given $R \in \mathfrak{R}$ there exists a sequence $A_n \in \mathfrak{A}_L$ such that $R(A_n) \xrightarrow{s} R$ and $R(A_n)^* \xrightarrow{s} R^*$. Therefore $\|R(A_n)\,\Omega - R\Omega\|_R \to 0$ for $n \to \infty$. Because the set $\{\mathfrak{R}\Omega\}$ is dense in $\mathfrak{H}^R$ as mentioned already in Section III.1 the lemma follows.

Lemma 7. *There exists a countable open covering of* R^3 *by volumes* V_n *such that*

$$\mathfrak{A}_L = \bigcup_n \mathfrak{A}(V_n)\,, \qquad \mathfrak{A} = \overline{\bigcup_n \mathfrak{A}(V_n)}\,.$$

Proof. Recall that $\mathfrak{A}_L$ was defined as the union over all finite $V \subset R^3$ of the algebras $\mathfrak{A}(V)$ (see introduction). Now take the V_n for instance as open spheres with the origin as centre and radius $n = 1, 2, \ldots$ Using the fact that $V \subset V'$ implies $\mathfrak{A}(V) \subset \mathfrak{A}(V')$ the lemma is easily established.

We now state and prove our main theorem which is a special case of Tomita's general result by restricting ourselves to the G.N.S. represen-

[4] Compare also [10], Lemma 1, p. 31.

tation determined by a locally normal state ω (see Section I). Denoting by $\mathfrak{R}$ the von Neumann algebra generated by the representatives of the elements of $\mathfrak{A}$ we have

Theorem 2. *When ω is a locally normal state over $\mathfrak{A}$ and Ω is a cyclic and separating vector for $\mathfrak{R}$, then $\mathfrak{R}$ is quasi-standard.*

The proof will be divided in several lemmas.

From the definition of locally normal states it follows that for every finite volume V there exists a positive self-adjoint trace-class operator ϱ_V with the property $\omega(A) = \mathrm{Tr}_V \varrho_V A$ for $A \in \mathfrak{A}(V)$.

Lemma 8. *$\omega(A^*A) = 0$ implies $A = 0$ for all $A \in \mathfrak{A}_L$.*

Proof. Suppose there is an $A \neq 0 \in \mathfrak{A}(V)$ such that $\omega(A^* A) = 0$. The set of all elements of $\mathfrak{A}(V)$ with that property forms a closed two-sided ideal of $\mathfrak{A}(V)$. Since $\mathfrak{A}(V) = \mathfrak{B}(\mathfrak{H}_F^V)$ this ideal is either the set of compact operators or $\mathfrak{A}(V)$ itself. Due to the continuity of the normal state $\omega(A)$ over $\mathfrak{A}(V)$ we conclude that $\omega(A) = 0$ for all $A \in \mathfrak{A}(V)$. Since $\mathfrak{A}(V)$ contains the unity, this would imply that ω vanishes identically which gives a contradiction.

Lemma 9. *ϱ_V has a self-adjoint unbounded inverse.*

Proof. Since $\mathfrak{A}(V)$ consists of all bounded operators in $\mathfrak{H}_F^V$ we may take in $\omega(A^*A)$ the projection P_Ψ onto any vector $\Psi \in \mathfrak{H}_F^V$. Then $\omega(P_\Psi) = \mathrm{Tr}_V \varrho_V P_\Psi = (\Psi, \varrho_V \Psi) \neq 0$. This implies that ϱ_V^{-1} exists and is a positive s. a. operator.

Lemma 10. *Let the operator U_τ^V, τ real, be defined by $U_\tau^V = \varrho_V^{-i\tau}$. Then U_τ^V is a unitary operator from $\mathfrak{A}(V)$ which is weakly continuous in τ and hence strongly continuous in τ.*

Proof. This follows without difficulty from the spectral decomposition of ϱ_V.

Corollary. *The mapping $A \to A_\tau^V$, $A \in \mathfrak{A}(V)$ where A_τ^V is defined by $A_\tau^V = U_\tau^V A U_\tau^{V-1}$ is an inner automorphism of $\mathfrak{A}(V)$.*

Let us now consider the G.N.S. representation determined by the state ω. This defines up to unitary equivalence a Hilbert space $\mathfrak{H}$, a cyclic (and separating in our case) vector Ω and a representation $A \in \mathfrak{A} \to R(A)$ into the bounded operators of $\mathfrak{H}$. Define the Hilbert subspace $\mathfrak{H}_V \subset \mathfrak{H}$ as the norm closure of the set of vectors $\{R(A)\,\Omega : A \in \mathfrak{A}(V)\}$. If Q_V is the projection onto $\mathfrak{H}_V$ then $[Q_V, R(A)] = 0$ for all $A \in \mathfrak{A}(V)$, hence we may define the operators $R_V(A)$ as the restriction of $R(A)$ to $\mathfrak{H}_V$ for all $A \in \mathfrak{A}(V)$.

Then the mapping $A \in \mathfrak{A}(V) \to R_V(A)$ into the bounded operators of $\mathfrak{H}_V$ is (up to unitary equivalence) the G.N.S. representation of $\mathfrak{A}(V)$ determined by the restriction ω_V of ω to $\mathfrak{A}(V)$, with cyclic vector Ω. The relation $\omega_V(A) = (\Omega, R_V(A)\,\Omega)$ is also trivially satisfied.

We now have in $\mathfrak{H}_V$ the structure as discussed in Sections III.1 and III.2. That means in $\mathfrak{H}_V$ exists a conjugation J_V,

$$J_V = \Gamma_V^{-1/2}\, I_V^R\, \Gamma_V^{1/2} \quad \text{with} \quad J_V\, R_V(A)\, J_V = S_V(A)\,, \qquad A \in \mathfrak{A}(V)\,,$$

a unitary operator $\hat{U}_\tau^V = T_V^{-2i\tau}$ which implements the automorphism $A \to A_\tau^V$

$$\hat{U}_\tau^V R_V(A) \hat{U}_\tau^{V-1} = R_V(A_\tau^V) ,$$

where

$$T_V^2 = \Gamma_V^{-1}(1 - \Gamma_V) .$$

The operator $\hat{\Gamma}_V$ originally defined on $\mathfrak{H}_V^R$ can in many ways be extended to an operator on $\mathfrak{H}^R$ (we repeate that the connection between $\mathfrak{H}^R$ and $\mathfrak{H}$ as discussed in Section III.1 is exactly the same as the connection between $\mathfrak{H}_V^R$ and $\mathfrak{H}_V$ for every finite V). For the extension $\bar{\hat{\Gamma}}_V$ of $\hat{\Gamma}_V$ to $\mathfrak{H}^R$ we choose:

$\bar{\hat{\Gamma}}_V \Phi = \hat{\Gamma}_V \Phi$ if $\Phi \in \mathfrak{H}_V^R$, and $\bar{\hat{\Gamma}}_V \Phi = \Phi$ if $\Phi \in \mathfrak{H}_V^{R\perp}$, where $\mathfrak{H}_V^{R\perp}$ is the orthogonal complement of $\mathfrak{H}_V^R$ with respect to $\mathfrak{H}^R$ on which $\hat{\Gamma}$ is defined (note that $\mathfrak{H}_V^R$ is a closed subspace of $\mathfrak{H}^R$ with respect to the topology deduced from the norm $\| \ \|_R$). From this definition it follows that $(\Phi, \hat{\Gamma}\Psi)_R = (\Phi, \Psi) = (\Phi, \bar{\hat{\Gamma}}_V \Psi)_R$ if $\Phi, \Psi \in \mathfrak{H}_V^R$. Then one can write $\bar{\hat{\Gamma}}_V = P_V \hat{\Gamma} P_V + 1 - P_V$, where P_V is the projection onto $\mathfrak{H}_V^R$ defined in $\mathfrak{H}^R$. $\bar{\hat{\Gamma}}_V$ is a positive definite self-adjoint operator and one has the following

Lemma 11. $\bar{\hat{\Gamma}}_{V_n}^{1/2} \xrightarrow{s} \hat{\Gamma}^{1/2}$ *on* $\mathfrak{H}^R$, *that means with respect to the strong operator topology deduced from the norm* $\| \ \|_R$.

Proof. From the Lemmas 6 and 7 it follows that there exists a sequence of volumes V_n with $V_n \subset V_m$ if $n < m$ such that the set of vectors $\{\Phi : \Phi \in \bigcup_n \mathfrak{H}_{V_n}^R\}$ is dense in $\mathfrak{H}^R$ with respect to the norm $\| \ \|_R$. Let now $\Phi \in \bigcup_n \mathfrak{H}_{V_n}^R$, that means there exists an integer n_0 such that $\Phi \in \mathfrak{H}_{V_{n_0}}^R$, hence $\Phi \in \mathfrak{H}_{V_n}^R$ for all $n \geqq n_0$, thus $P_{V_n} \Phi = \Phi$ for all $n \geqq n_0$. We therefore have that the uniformly bounded sequence $\{P_{V_n}\}$ converges strongly to 1 on a dense set in $\mathfrak{H}^R$, hence $P_{V_n} \xrightarrow{s} 1$ in $\mathfrak{H}^R$[5].

For $\Phi \in \mathfrak{H}_{V_{n_0}}^R$ one has $(\bar{\hat{\Gamma}}_{V_n} - \hat{\Gamma})\Phi = (P_{V_n} - 1)\hat{\Gamma}\Phi$ if $n \geqq n_0$ and therefore $\|(\bar{\hat{\Gamma}}_{V_n} - \hat{\Gamma})\Phi\|_R^2 = (\hat{\Gamma}\Phi, (1 - P_{V_n})\hat{\Gamma}\Phi)_R \to 0$ for $n \to \infty$. Again because the set $\{\Phi : \Phi \in \bigcup_n \mathfrak{H}_{V_n}^R\}$ is dense in $\mathfrak{H}^R$ it follows that $\|(\bar{\hat{\Gamma}}_{V_n} - \hat{\Gamma})\Psi\|_R \to 0$ for all $\Psi \in \mathfrak{H}^R$, thus $\bar{\hat{\Gamma}}_{V_n} \xrightarrow{s} \hat{\Gamma}$ on $\mathfrak{H}^R$ and Lemma A of the appendix now gives $\bar{\hat{\Gamma}}_{V_n}^{1/2} \xrightarrow{s} \hat{\Gamma}^{1/2}$ on $\mathfrak{H}^R$, which proves the lemma.

From the above proof we know $\bar{\hat{\Gamma}}_{V_n} \xrightarrow{s} \hat{\Gamma}$ on $\mathfrak{H}^R$, a fact that also may be written in the form $s - \lim \hat{\Gamma}_{V_n} = \hat{\Gamma}$ on $\mathfrak{H}^R$. For later use we now proof.

Lemma 12. $\hat{\Gamma}_{V_n} \xrightarrow{s} \hat{\Gamma}$ *on* $\mathfrak{H}^R$ *implies* $\Gamma_{V_n} \xrightarrow{s} \Gamma$ *on* $\mathfrak{H}$.

Proof. This is an immediate consequence of the fact that the topology defined by the scalar product $(\Psi, \Phi)_R$ is stronger than that defined by the scalar product (Ψ, Φ).

[5] See e.g. [6], Lemma 3.5, p. 151.

The conjugation J_{V_n} defined on $\mathfrak{H}_{V_n}$ can be extended (not uniquely) to a conjugation $\overline{J_{V_n}}$ defined on $\mathfrak{H}$. $\overline{J_{V_n}}$ we define as follows: $\overline{J_{V_n}}\Phi = J_{V_n}\Phi$ if $\Phi \in \mathfrak{H}_{V_n}$; choosing a basis $\{\Phi_\iota\}_{\iota\in I}$ of $\mathfrak{H}_V^\perp$ (the orthogonal complement of $\mathfrak{H}_V$ with respect to $\mathfrak{H}$) we set $\overline{J_V}\Phi_\iota = \Phi_\iota$ for all $\iota \in I$ and $\overline{J_{V_n}}\Phi = \sum_\iota \overline{C}_\iota \Phi_\iota$ if $\Phi = \sum_\iota C_\iota \Phi_\iota$ (here $\overline{C}_\iota$ is the complex conjugate of C_ι).

Lemma 13. *$\overline{J_{V_n}} \xrightarrow{s} J'$ in $\mathfrak{H}$.*

Proof. Let $\Psi \in \bigcup_n \mathfrak{H}_{V_n}^R$ where $\mathfrak{H}_{V_n}^R$ is considered as a subspace of $\mathfrak{H}_{V_n}$ thus $\Psi \in \mathfrak{H}_{V_{n_0}}^R$ for some integer n_0 and therefore $\Psi \in \mathfrak{H}_{V_n}^R$ for $n \geqq n_0$. For $n \geqq n_0$ we now have

$$\begin{aligned}(\Psi, \overline{J_{V_n}}\Psi) &= (\Psi, J_{V_n}\Psi) = (\Psi, \Gamma_{V_n}^{-1/2} I_{V_n}^R \Gamma_{V_n}^{1/2}\Psi) = (\Psi, \hat{\Gamma}_{V_n}^{-1/2} I_{V_n}^R \hat{\Gamma}_{V_n}^{1/2}\Psi) \\ &= (\Psi, \hat{\Gamma}_{V_n}^{-1/2} I^R \hat{\Gamma}_{V_n}^{1/2}\Psi) = (\Psi, \hat{\Gamma}_{V_n}^{1/2} I^R \hat{\Gamma}_{V_n}^{1/2}\Psi)_R \\ &= (\hat{\Gamma}_{V_n}^{1/2}\Psi, I^R \hat{\Gamma}_{V_n}^{1/2}\Psi)_R = (\overline{\hat{\Gamma}}_{V_n}^{1/2}\Psi, I^R \overline{\hat{\Gamma}}_{V_n}^{1/2}\Psi)_R ,\end{aligned}$$

where we have used $\overline{\hat{\Gamma}_{V_n}^{1/2}} = \overline{\hat{\Gamma}}_{V_n}^{1/2}$, which is correct if one defines

$$\overline{\hat{\Gamma}_{V_n}^{1/2}}\Phi = \hat{\Gamma}_{V_n}^{1/2}\Phi,\ \Phi \in \mathfrak{H}_{V_n}^R \quad \text{and} \quad \overline{\hat{\Gamma}_{V_n}^{1/2}}\Phi = \Phi \quad \text{if} \quad \Phi \in \mathfrak{H}_{V_n}^{R\perp} .$$

Now using Lemma 11 it follows that for $n \to \infty$

$$\begin{aligned}(\overline{\hat{\Gamma}}_{V_n}^{1/2}\Psi, I^R \overline{\hat{\Gamma}}_{V_n}^{1/2}\Psi)_R &\to (\hat{\Gamma}^{1/2}\Psi, I^R \hat{\Gamma}^{1/2}\Psi)_R = (\Psi, \hat{\Gamma}^{-1/2} I^R \hat{\Gamma}^{1/2}\Psi) \\ &= (\Psi, \Gamma^{-1/2} I^R \Gamma^{1/2}\Psi) = (\Psi, J'\Psi) .\end{aligned}$$

Since the set $\{\Psi : \Psi \in \bigcup_n \mathfrak{H}_{V_n}^R\}$ is dense in $\mathfrak{H}^R$ with respect to the topology defined by the norm $\| \ \|_R$ this set is also dense in $\mathfrak{H}^R$, now considered as a subspace of $\mathfrak{H}$, with respect to the less fine topology defined by $\| \ \|$. And since $\mathfrak{H}^R$ is dense in $\mathfrak{H}$ with respect to the norm $\| \ \|$ one concludes that the set $\{\Psi : \Psi \in \bigcup_n \mathfrak{H}_{V_n}^R\}$ considered as a subspace of $\mathfrak{H}$ is dense in $\mathfrak{H}$. We conclude that $\overline{J_{V_n}} \xrightarrow{w} J'$. Since both $\overline{J_{V_n}^2} = 1$ and $J'^2 = 1$ this implies that $\overline{J_{V_n}} \xrightarrow{s} J'$.

We define the operators $S(A)$ on $\mathfrak{H}$ for $A \in \mathfrak{A}$ by $S(A) = J' R(A) J'$. Then one has the following

Lemma 14. *$S(A) \in \mathfrak{R}'$ for all $A \in \mathfrak{A}$.*

Proof. Take first $A \in \mathfrak{A}_L$ and $\Phi, \Psi \in \bigcup_n \mathfrak{H}_{V_n}$, then there exists an index n_0 such that $A \in \mathfrak{A}(V_{n_0})$ and $\Phi, \Psi \in \mathfrak{H}_{V_{n_0}}$. For $n > n_0$ we then know from the beginning of this section $J_{V_n} R_{V_n}(A) J_{V_n} = S_{V_n}(A)$ and $[S_{V_n}(A), R_{V_n}(B)] = 0$ for all $B \in \mathfrak{A}(V_n)$.

Let now $\Phi, \Psi \in \mathfrak{H}$ then there exists according to Lemma 7 an index n_0 and vectors $\Phi_{n_0}, \Psi_{n_0} \in \mathfrak{H}_{V_{n_0}}$ such that $\|\Phi_{n_0} - \Phi\| < \varepsilon$, $\|\Psi_{n_0} - \Psi\| < \varepsilon$.

Hence for $n \geqq n_0$

$$\begin{aligned}
&|(\Phi, [\overline{J_{V_n}} R(A) \overline{J_{V_n}}, R(B)] \Psi)| \\
&\quad \leqq |((\Phi - \Phi_{n_0}), [\overline{J_{V_n}} R(A) \overline{J_{V_n}}, R(B)] \Psi)| \\
&\quad + |(\Phi_{n_0}, [\overline{J_{V_n}} R(A) \overline{J_{V_n}}, R(B)] (\Psi - \Psi_{n_0}))| \\
&\quad + |(\Phi_{n_0}, [J_{V_n} R_{V_n}(A) J_{V_n}, R_{V_n}(B)] \Psi_{n_0})| \\
&\quad \leqq 2\varepsilon \|A\| \|B\| \|\Psi\| + 2\varepsilon \|A\| \|B\| (\varepsilon + \|\Phi\|) ,
\end{aligned}$$

since the last term on the right hand side is equal to zero because of the first part of the proof. Therefore we have for $n \to \infty$

$$(\Phi, [\overline{J_{V_n}} R(A) \overline{J_{V_n}}, R(B)] \Psi) \to 0 \text{ for all } \Phi, \Psi \in \mathfrak{H} \text{ and } A, B \in \mathfrak{A}_L .$$

Hence $[J' R(A) J', R(B)] = 0$ for all A, $B \in \mathfrak{A}_L$.

We conclude that $S(A) = J' R(A) J' \in \mathfrak{R}'$ for all $A \in \mathfrak{A}_L$ and thus, by continuity for all $A \in \mathfrak{A}$.

In order to proof that $J'\mathfrak{R}J' = \mathfrak{R}'$ let $\mathfrak{S} = J'\mathfrak{R}J'$. Then one has to proof the following

Lemma 15. $\mathfrak{S} = \mathfrak{R}'$.

Proof. From Eq. (3.9) we know $J'\mathfrak{H}^R = \mathfrak{H}^S$.

Since $\mathfrak{R}\Omega$ is dense in $\mathfrak{H}^R$ with respect to the topology generated by $\| \ \|_R$ $J'\mathfrak{R}\,\Omega = \mathfrak{S}\Omega$ is dense in $\mathfrak{H}^S$ with respect to the topology generated by $\| \ \|_S$. That means if $R' \in \mathfrak{R}'$, there exists a sequence $S_n \in \mathfrak{S}$ such that $\|R'\Omega - S_n\Omega\|_S \to 0$ for $n \to \infty$, hence $\|R'\Omega - S_n\Omega\| \to 0$ and

$$\|R'^*\Omega - S_n^*\,\Omega\| \to 0 .$$

Let $S' \in \mathfrak{S}'$, then for arbitrary R_1, $R_2 \in \mathfrak{R}$ one has

$$\begin{aligned}
(R_1\Omega, R'S'R_2\Omega) &= (R'^*\,\Omega, R_1^*\, S'R_2\Omega) = \lim_{n\to\infty} (S_n^*\,\Omega, R_1^*\, S'R_2\Omega) \\
&= \lim_{n\to\infty} (\Omega, R_1^*\, S'R_2S_n\Omega) = (\Omega, R_1^*\, S'R_2R'\Omega) \\
&= (R_1\Omega, S'R'R_2\Omega) ,
\end{aligned}$$

that means $[R', S'] = 0$, and hence $\mathfrak{R}' \subset \mathfrak{S}$. On the other hand, from Lemma 14, $\mathfrak{S} \subset \mathfrak{R}'$ and thus $\mathfrak{S} = \mathfrak{R}'$.

Lemma 16. $\hat{U}_\tau^V \xrightarrow{s} \hat{U}_\tau$.

Proof. We already know $T'^{-2} = \Gamma(1 - \Gamma)^{-1}$ and $T_V^{-2} = \Gamma_V(1 - \Gamma_V)^{-1}$. Then it follows that $\hat{U}_\tau = (T')^{-2i\tau} = \Gamma^{i\tau}(1 - \Gamma)^{-i\tau}$ and $\hat{U}_\tau^V = T_V^{-2i\tau} = \Gamma_V^{i\tau}(1 - \Gamma_V)^{-i\tau}$, τ real.

From the analysis given in the beginning of this section it follows that the domain of $\Gamma^{-1/2}$ is $\mathfrak{H}^R$ and the domain of $(1 - \Gamma)^{-1/2}$ is $\mathfrak{H}^S$. Analogously the domain of $\Gamma_{V_n}^{-1/2}$ is $\mathfrak{H}_{V_n}^R$ and that of $(1 - \Gamma_{V_n})^{-1/2}$ is $\mathfrak{H}_{V_n}^S$.

As already pointed out the set $\{\Phi : \Phi \in \bigcup_n \mathfrak{H}_{V_n}^R\}$ is dense in $\mathfrak{H}$. Furthermore $\|\Gamma_{V_n}\| \leqq 1$, $\|1 - \Gamma_{V_n}\| \leqq 1$ for all n, and $\|\Gamma\| \leqq 1$, $\|1 - \Gamma\| \leqq 1$.

Together with Lemma 12 this means that the conditions of Lemma B in the appendix are fulfilled so that $\Gamma_{V_n}^{i\tau} \xrightarrow{s} \Gamma^{i\tau}$ and $(1 - \Gamma_{V_n})^{i\tau} \xrightarrow{s} (1 - \Gamma)^{i\tau}$. Hence[6] $T_{V_n}^{-2i\tau} \xrightarrow{s} (T')^{-2i\tau}$ for all real τ.

From Lemma 16 now easily follows $R_{V_n}(A_\tau^{V_n}) \xrightarrow{s} \hat{U}_\tau R(A)\, \hat{U}_\tau^{-1} \equiv R(A)_\tau$ for $A \in \mathfrak{A}_L$ since $R_{V_n}(A_\tau^{V_n})$ can be written as $R_{V_n}(A_\tau^{V_n}) = \hat{U}_\tau^{V_n} Q_{V_n} R(A)\, \hat{U}_\tau^{V_n-1}$, where Q_{V_n} is the projection onto $\mathfrak{H}_{V_n}$. Thus $R(A)_\tau \in \mathfrak{R}$. Let now $R \in \mathfrak{R}$, then there exists a sequence $R(A_m)$, $A_m \in \mathfrak{A}_L$ which tends strongly to R and therefore $\hat{U}_\tau R(A_m)\, \hat{U}_\tau^{-1} \xrightarrow{s} \hat{U}_\tau R\, \hat{U}_\tau^{-1} \equiv R_\tau$. Hence $R_\tau \in \mathfrak{R}$ for all τ; this proves

Lemma 17. *$\hat{U}_\tau \mathfrak{R}\, \hat{U}_\tau^{-1} = \mathfrak{R}$, and $\hat{U}_\tau \mathfrak{R}'\, \hat{U}_\tau^{-1} = \mathfrak{R}'$.*

With the proof of the Lemmas 15 and 17 we have shown that all conditions of Theorem 1 are satisfied, and Theorem 2 has been demonstrated.

4. Separability of $\mathfrak{H}$

Lemma 18. *Let ω_V be a state over $\mathfrak{A}(V)$ such that*

$$\omega_V(A) = \mathrm{Tr}_V \varrho_V A\,, \qquad A \in \mathfrak{A}(V)\,,$$

where ϱ_V is a trace class operator in $\mathfrak{H}_F^V$ with $\mathrm{Tr}_V \varrho_V = 1$. Let $\mathfrak{H}_V$ be the Hilbert space determined (up to equivalence) by ω_V in the G.N.S. construction. Then $\mathfrak{H}_V$ is separable.

Proof[7]. As shown in Section III.2 $\mathfrak{H}_V$ is isometric with the Hilbert space $\mathfrak{K}_V$ consisting of all Hilbert-Schmidt operators on $\mathfrak{H}_F^V$. It is well-known that the Hilbert-Schmidt operators with a metric defined by the norm $|||\varkappa||| = \sqrt{\mathrm{Tr}\varkappa^*\varkappa}$ are a seperable set provided $\mathfrak{H}_F^V$ is separable.

Theorem 3. *Let ω be a locally normal state defined over $\mathfrak{A}$. If $\mathfrak{H}$ is the representation space determined by ω in the G.N.S. construction, then $\mathfrak{H}$ is separable.*

Proof. From Lemma 2 it follows that there exists a denumerable set of Hilbert spaces $\mathfrak{H}_{V_n}$, $n = 1, 2, \ldots$ such that the set of vectors $\{\Phi : \Phi \in \bigcup_n \mathfrak{H}_{V_n}\}$ is dense in $\mathfrak{H}$ (see for the definition of the spaces $\mathfrak{H}_{V_n}$ the proof of Theorem 2). According to Lemma 18 each $\mathfrak{H}_{V_n}$ is separable since ω is locally normal. Thus a countable set of vectors is dense in $\mathfrak{H}$, that means $\mathfrak{H}$ is separable.

This completes the proof of Theorem 3.

IV. Concluding Remarks

As already mentioned in the introduction a state which is the thermodynamical limit ($V \to \infty$) of Gibbs states satisfies the so-called K.M.S. condition. A question of considerable interest is whether any state

[6] See [6], Lemma 3.8, p. 151.

[7] A proof of this fact is also given in [7], Corol. 6.4.

satisfying the K.M.S. condition is the thermodynamical limit of Gibbs states. To analyse the precise meaning of this question let us recall that a Gibbs state of a finite volume V, temperature T ($\beta = (kT)^{-1}$) and chemical potential μ is determined by a density operator $\varrho_V = C_V e^{-\beta(H_V - \mu N_V)}$, where H_V and N_V are the hamiltonian and particle number operator corresponding to this volume V. We want to stress that H_V is not uniquely determined, due to boundary effects. We shall say that H_V is a hamiltonian for the system in volume V if the automorphism $A \to A_t^V = e^{iH_V t} A e^{-iH_V t}$ of $\mathfrak{A}(V)$ leads to time-translation in the thermodynamical limit.

It is not a priori clear in which topology this should hold but one should at least require that

$$A_t^V \to A_t \tag{4.1}$$

in a topology independent of the representation e.g. in the norm topology.

Let us now consider a locally normal state satisfying the K.M.S. condition. On the basis of our results in Section III we have in the representation space $\mathfrak{H}$:

$$\operatorname*{strong\,lim}_{V\to\infty} R(A_t^V) = R(A_t)\,, \tag{4.2}$$

which is a direct consequence of Lemma 16. Since condition (4.2) is clearly weaker than (4.1) we cannot yet conclude that any locally normal state which satisfies the K.M.S. condition is the limit of Gibbs states.

Appendix

In this appendix two lemmas will be proved which to our knowledge do not exist in the literature.

Lemma A. *Let A and A_n, $n = 1, 2, \ldots$ be self-adjoint operators in a Hilbert space $\mathfrak{H}$ such that $\|A\| \leqq 1$ and $\|A_n\| \leqq 1$ for $n = 1, 2, \ldots$ Let furthermore $A_n \xrightarrow{s} A$, then $f(A_n) \xrightarrow{s} f(A)$ for every continuous function $f(\lambda)$ defined on $[-1, 1]$.*

Proof. From $A_n \xrightarrow{s} A$ follows $A_n^p \xrightarrow{w} A^p$ for any integer p ([6], p. 151, Lemma 3.9). Hence if $P(\lambda)$ is any polynominal in λ one has $P(A_n) \xrightarrow{w} P(A)$. From the Weierstrass' theorem it follows that there exists a polynomial $P(\lambda)$ on $[-1, 1]$ such that $|P(\lambda) - f(\lambda)| < \varepsilon$, $\lambda \in [-1, 1]$. Using the spectral decomposition $A = \int_{-1}^{1} \lambda dE(\lambda)$ of A one gets $P(A) - f(A) = \int_{-1}^{1} (P(\lambda) - f(\lambda))\, dE(\lambda)$, and $|(\Phi, (P(A) - f(A))\,\Phi)| < \varepsilon$ for arbitrary vector $\Phi \in \mathfrak{H}$ such that $(\Phi, \Phi) = 1$. Then $|(\Phi, f(A_n)\,\Phi) - (\Phi, f(A)\,\Phi)| \leqq |(\Phi, f(A_n)\,\Phi) - (\Phi, P(A_n)\,\Phi)| + |(\Phi, P(A_n)\,\Phi) - (\Phi, P(A)\,\Phi)| + |(\Phi, P(A)\,\Phi) - (\Phi, f(A)\,\Phi)| < 3\varepsilon$ for n sufficiently large.

Hence $f(A_n) \xrightarrow{w} f(A)$. Replacing f by $|f|^2$ one analogously finds $\bar{f}(A_n)\, f(A_n) \xrightarrow{w} \bar{f}(A)\, f(A)$, where $\bar{f}$ is the complex conjugate of f. Then it easily follows that $f(A_n) \xrightarrow{s} f(A)$.

Lemma B. *Let A and A_n, $n = 1, 2, \ldots$, be positive self-adjoint operators in a Hilbert space $\mathfrak{H}$ such that $\|A\| \leqq 1$, $\|A_n\| \leqq 1$, $n = 1, 2, \ldots$ and $A_n \xrightarrow{s} A$. Let furthermore A^{-1} and A_n^{-1}, $n = 1, 2, \ldots$ be self-adjoint (not necessarily bounded) operators which have a common dense domain D in $\mathfrak{H}$. Then $A_n^i \xrightarrow{s} A^i$, where $i = \sqrt{-1}$.*

Proof. Define the function $f_\varepsilon(\lambda)$ on $[0, 1]$ by: $f_\varepsilon(\lambda) = \lambda^i$ for $\varepsilon \leqq \lambda \leqq 1$, and $f_\varepsilon(\lambda) = \varepsilon^i$ for $0 \leqq \lambda \leqq \varepsilon$. Then $f_\varepsilon(\lambda)$ is a continuous function defined on $[0, 1]$. If $\Phi \in D$, $(\Phi, A^{-1}\Phi) = \int_0^1 \lambda^{-1}\, d(\Phi, E(\lambda)\,\Phi) = C(A, \Phi) < \infty$, where $A = \int_0^1 \lambda\, dE(\lambda)$ is the spectral decomposition of A and $C(A, \Phi)$ is a finite constant dependent on A and Φ.

Hence also $\int_0^\varepsilon \lambda^{-1}\, d(\Phi, E(\lambda)\,\Phi) < C(A, \Phi)$, thus $\int_0^\varepsilon d(\Phi, E(\lambda)\,\Phi) < \varepsilon C(A, \Phi)$. Now $|(\Phi, A^i\,\Phi) - (\Phi, f_\varepsilon(A)\,\Phi)| \leqq \int_0^\varepsilon |\lambda^i - \varepsilon^i|\, d(\Phi, E(\lambda)\,\Phi)$.

Since there exists a continuous function $g_\varepsilon(\lambda)$ on $[0, 1]$ such that $|\lambda^i - \varepsilon^i| < g_\varepsilon(\lambda)$ and $g_\varepsilon(\lambda) < 2$ for $0 \leqq \lambda \leqq \varepsilon$, $g_\varepsilon(\lambda) = 0$ for $\varepsilon \leqq \lambda \leqq 1$, it follows that $|(\Phi, A^i\,\Phi) - (\Phi, f_\varepsilon(A)\,\Phi)| \leqq \int_0^1 g_\varepsilon(\lambda)\, d(\Phi, E(\lambda)\,\Phi)$. Completely analogously one has

$$|(\Phi, A_n^i\,\Phi) - (\Phi, f_\varepsilon(A_n)\,\Phi)| \leqq \int_0^1 g_\varepsilon(\lambda)\, d(\Phi, E_n(\lambda)\,\Phi)\,.$$

Since by Lemma A

$$\lim_{n\to\infty} \int_0^1 g_\varepsilon(\lambda)\, d(\Phi, E_n(\lambda)\,\Phi) = \int_0^1 g_\varepsilon(\lambda)\, d(\Phi, E(\lambda)\,\Phi) < 2\,\varepsilon\, C(A, \Phi)$$

we have from a certain index n

$$|(\Phi, A_n^i\,\Phi) - (\Phi, f_\varepsilon(A_n)\,\Phi)| \leqq 2\,\varepsilon\, C(A, \Phi)\,,$$

$$|(\Phi, A^i\,\Phi) - (\Phi, f_\varepsilon(A)\,\Phi)| \leqq 2\,\varepsilon\, C(A, \Phi).$$

Thus

$$|(\Phi, A_n^i\,\Phi) - (\Phi, A^i\,\Phi)| \leqq 4\,\varepsilon\, C(A, \Phi) + |(\Phi, f_\varepsilon(A_n)\,\Phi) - (\Phi, f_\varepsilon(A)\,\Phi)|\,.$$

Hence for all $\Phi \in D$ $\lim_{n\to\infty} (\Phi, A_n^i\,\Phi) = (\Phi, A^i\,\Phi)$.

Since both A_n^i, $n = 1, 2, \ldots$ and A^i are unitary this equation is valid for all $\Phi \in \mathfrak{H}$. Hence $A_n^i \xrightarrow{w} A^i$, and thus $A_n^i \xrightarrow{s} A^i$. This completes the proof of Lemma B.

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