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On Lorentzian Para-Sasakian Manifolds Satisfying W$_2$-Curvature Tensor

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Abstract: *The object of the present paper is to study some properties of W$_2$- curvature tensor in an Lorentzian para-Sasakian manifolds.*
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I. Introduction

In 1989, K. Matsumoto [5] introduced the notion of Lorentzian para-Sasakian manifold. Then, these manifolds have been studied by many geometers like K. Matsumoto and I. Mihai [6], I. Mihai and R. Rosca [8], I. Mihai, A.A. Shaikh and U.C. De [7], Venkatesha and C.S. Bagewadi [14], etc., obtained several results on this manifold.

In the present paper, we study flat W_2-curvature tensor, irrotational W_2-curvature tensor and conservative W_2-curvature tensor in an Lorentzian para-Sasakian manifolds. Also we have obtained results on Einstein Lorentzian para-Sasakian manifold satisfying $R(X,Y)\cdot W_2 = 0$.

II. Preliminaries

An n-dimensional differentiable manifold M is called an Lorentzian para-Sasakian manifold ([5], [8]) if it admits a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

(2.1) $$\eta(\xi) = -1,$$
(2.2) $$\phi^2 X = X + \eta(X)\xi,$$
(2.3) $$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y),$$
(2.4) $$g(X,\xi) = \eta(X),$$
(2.5) $$\nabla_X \xi = \phi X,$$
(2.6) $$(\nabla_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$

where ∇ denotes the operator of covariant differentiation.

It can be easily seen that in a Lorentzian para-Sasakian manifold, the following relations hold:

(2.7) $$\phi\xi = 0,\ \eta(\phi X) = 0,\ rank\phi = n-1.$$

Again if we put

(2.8) $$\Phi(X,Y) = g(X,\phi Y),$$

for any vector fields X, Y, then the tensor field $\Phi(X,Y)$ is a symmetric $(0,2)$ tensor field [5].
Also, since the 1-form η is closed in an Lorentzian para-Sasakian manifold, we have ([5], [7])

(2.9) $$(\nabla_X \eta)(Y) = \Phi(X,Y), \quad \Phi(X,\xi) = 0,$$

for any vector fields X and Y.
Also in an Lorentzian para-Sasakian manifold, the following relations hold ([6], [7]):

(2.10) $$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y),$$
(2.11) $$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X,$$
(2.12) $$R(X,Y)\xi = \eta(Y)X - \eta(X)Y,$$
(2.13) $$S(X,\xi) = (n-1)\eta(X),$$
(2.14) $$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y),$$

for any vector fields X, Y and Z, where R is the Riemannian curvature tensor and S is the Ricci tensor of M.
An Lorentzian para-Sasakian manifold M is said to be Einstein if its Ricci tensor S is of the form

(2.15) $$S(X,Y) = ag(X,Y),$$

for any vector fields X and Y, where a is a function on M.
In [10], Pokhariyal and Mishra have defined the curvature tensor W_2, given by

(2.16) $$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\{g(X,Z)QY - g(Y,Z)QX\}.$$

III. Flat W_2-Curvature Tensor in an Lorentzian Para-Sasakian Manifolds

If the Lorentzian para-Sasakian manifold has flat W_2-curvature tensor, then

(3.1) $g(W_2(X,Y)Z,\phi W) = 0,$

(3.2) $g(R(X,Y)Z,\phi W) + \frac{1}{n-1}\{g(X,Z)S(Y,\phi W) - g(Y,Z)S(X,\phi W)\} = 0.$

Putting Y = Z = ξ in (3.2), we have

(3.3) $g(R(X,\xi)\xi,\phi W) + \frac{1}{n-1}\{\eta(X)S(\xi,\phi W) + S(X,\phi W)\} = 0.$

Using (2.12) and (2.13) in (3.3), we get

(3.4) $-g(X,\phi W) + \frac{1}{n-1}S(X,\phi W) = 0,$

on simplification, we have

(3.5) $S(X,\phi W) = (n-1)g(X,\phi W),$

replacing W by ϕW in (3.5), we have

(3.6) $S(X,W) = (n-1)g(X,W).$

On contracting the above relation, we obtain

(3.7) $r = n(n-1).$

Thus we can state:

Theorem 3.1. In a Lorentzian para-Sasakian manifold the W_2-curvature tensor is flat then it is an Einstein manifold and also a space of constant scalar curvature.

IV. Irrotational W2-Curvature Tensor in an Lorentzian Para-Sasakian Manifolds

Definition 4.1. Let ∇ be a Riemannian connection, then the rotation (curl) of W_2-curvature tensor in a Lorentzian para-Sasakian manifold M is defined as

(4.1) $RotW_2 = (\nabla_U W_2)(X,Y)Z + (\nabla_X W_2)(U,Y)Z + (\nabla_Y W_2)(X,U)Z - (\nabla_Z W_2)(X,Y)U.$

In consequence of Bianchi's second identity for Riemannian connection ∇, (4.1) becomes

(4.2) $RotW_2 = -(\nabla_Z W_2)(X,Y)U.$

If the W_2-curvature tensor is irrotational, then $RotW_2 = 0$ and therefore

(4.3) $(\nabla_Z W_2)(X,Y)U = 0,$

Which gives

(4.4) $\nabla_Z(W_2(X,Y)U) = W_2(\nabla_Z X,Y)U + W_2(X,\nabla_Z Y)U + W_2(X,Y)\nabla_Z U.$

Replacing $U = \xi$ in (4.4), we have

(4.5) $\nabla_Z(W_2(X,Y)\xi) = W_2(\nabla_Z X,Y)\xi + W_2(X,\nabla_Z Y)\xi + W_2(X,Y)\nabla_Z\xi.$

Now substituting $Z = \xi$ in (2.16) and using (2.12) and (2.13), we obtain

(4.6) $W_2(X,Y)\xi = k[\eta(Y)X - \eta(X)Y],$

Where

(4.7) $k = \left[1 - \frac{1}{n-1}\left\{\frac{r}{n-1} - 1\right\}\right].$

Using (4.6) in (4.5),we obtain

(4.8) $W_2(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y].$

Also equations (2.16) and (4.8) gives

(4.9) $S(Y,Z) = (n-1)g(Y,Z),$

which gives

(4.10) $r = n(n-1).$

In consequence of (2.16), (4.7), (4.8), (4.9) and (4.10), we find

(4.11) $R(X,Y)Z = g(Y,Z)X - g(X,Z)Y.$

Hence we can state:

Theorem 4.2. If the W_2-curvature tensor in a Lorentzian para-Sasakian manifold is irrotational then the manifold is a space of constant curvature.

V. Conservative W_2-Curvature Tensor in an Lorentzian Para-Sasakian Manifolds

Differentiating (2.16) with respect to U, we have

(5.1) $(\nabla_U W_2)(X,Y)Z = (\nabla_U R)(X,Y)Z + \frac{1}{(n-1)}[g(X,Z)(\nabla_U Q)(Y) - g(Y,Z)(\nabla_U Q)(X)].$

On contracting (5.1), we get

(5.2) $(divW_2)(X,Y)Z = [(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] + \frac{1}{2(n-1)}[g(X,Z)dr(X) - g(Y,Z)dr(Y)].$

If W_2-curvature tensor is conservative $(divW_2 = 0)$, then (5.2) can be written as

(5.3) $[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] = \frac{1}{2(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)].$

Putting $X = \xi$ in (5.3), we have

(5.4) $\left[(\nabla_\xi S)(Y,Z) - (\nabla_Y S)(\xi,Z)\right] = \frac{1}{2(n-1)}[g(Y,Z)dr(\xi) - g(\xi,Z)dr(Y)].$

Since ξ is a killing vector, r remains invariant under it, that is, $\mathcal{L}_\xi r = 0$, where $\mathcal{L}$ denotes the Lie derivative. But then the relation,

$$(\nabla_\xi S)(Y,Z) = \xi S(Y,Z) - S(\nabla_\xi Y,Z) - S(Y,\nabla_\xi Z)$$

(5.5) $$= (\mathcal{L}_\xi S)(Y,Z) - S(\nabla_Y \xi,Z) - S(Y,\nabla_Z \xi).$$

Yields

(5.6) $(\nabla_\xi S)(Y,Z) = 0.$

Now by substituting (5.6) in (5.4), we have

$$-[\nabla_Y S(\xi,Z) - S(\nabla_Y \xi,Z) - S(\xi,\nabla_Y Z)]$$

(5.7) $$= \frac{1}{2(n-1)}[g(Y,Z)dr(\xi) - \eta(Z)dr(Y)].$$

By using (2.5), (2.13) and $dr(\xi) = 0$ in (5.7), we get

$$[-\nabla_Y\{(n-1)\eta(Z)\} + S(\phi Y,Z) + (n-1)\eta(\nabla_Y Z)]$$

(5.8) $$=-\frac{1}{2(n-1)}[\eta(Z)dr(Y)].$$

Simplifying (5.8), we get

(5.9) $[-(n-1)g(\phi Y,Z) + S(\phi Y,Z)] = -\frac{1}{2(n-1)}[\eta(Z)dr(Y)].$

Putting $Z = \phi Z$ in (5.9), we obtain

(5.10) $(n-1)g(\phi Y,\phi Z) = S(\phi Y,\phi Z).$

It implies that

(5.11) $S(Y,Z) = (n-1)g(Y,Z).$

On contracting (5.11), we obtain

(5.12) $r = n(n-1).$

Thus we state:

Theorem 5.3. If W_2-curvature tensor in a Lorentzian para-Sasakian manifold is conservative then it is an Einstein manifold and also of constant scalar curvature.

VI. Einstein Lorentzian Para-Sasakian Manifolds Satisfying $R(X,Y)\cdot W_2 = 0$

In consequence of $QX = hX$, (2.16) becomes

(6.1) $W_2(X,Y)Z = R(X,Y)Z + \frac{h}{(n-1)}\{g(X,Z)Y - g(Y,Z)X\}.$

In view of (2.10) and (6.1), we obtain

(6.2) $\eta(W_2(X,Y)Z) = (1 - \frac{h}{n-1})\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}.$

Replacing Z by ξ in (6.2), we have

(6.3) $\eta(W_2(X,Y)\xi) = 0.$

Now

(6.4) $$(R(X,Y)\cdot W_2)(Z,U)V = R(X,Y)W_2(Z,U)V - W_2(R(X,Y)Z,U)V$$
$$-W_2(Z,R(X,Y)U)V - W_2(Z,U)R(X,Y)V.$$

Using $R(X,Y)\cdot W_2 = 0$ in the above equation, we obtain

$$R(X,Y)W_2(Z,U)V - W_2(R(X,Y)Z,U)V - W_2(Z,R(X,Y)U)V - W_2(Z,U)R(X,Y)V = 0.$$

By taking the inner product of the above relation with ξ, we get

(6.5) $$g(R(X,Y)W_2(Z,U)V,\xi) - g(W_2(R(X,Y)Z,U)V,\xi)$$
$$-g(W_2(Z,R(X,Y)U)V,\xi) - g(W_2(Z,U)R(X,Y)V,\xi) = 0.$$

Putting $X = \xi$ in (6.5) and then using (2.11), we obtain

$$-W_2(Z,U,V,Y) - \eta(Y)\eta(W_2(Z,U)V) + \eta(Z)\eta(W_2(Y,U)V) + \eta(U)\eta(W_2(Z,Y)V) + \eta(V)\eta(W_2(Z,U)Y)$$
$$- g(Y,Z)\eta(W_2(\xi,U)V) - g(Y,U)\eta(W_2(Z,\xi)V) - g(Y,V)\eta(W_2(Z,U)\xi) = 0.$$

In consequence of (6.2), the above equation gives

(6.6) $$-W_2(Z,U,V,Y) + \eta(V)\left[\left(1 - \frac{h}{n-1}\right)\eta(Z)g(U,Y) - \eta(U)g(Y,Z)\right]$$
$$+g(Y,Z)\left[\left(1 - \frac{h}{n-1}\right)g(U,V) + \eta(U)\eta(V)\right]$$
$$-g(Y,U)\left[\left(1 - \frac{h}{n-1}\right)\eta(Z)\eta(V) + g(Z,V)\right]$$
$$-g(Y,V)\left[\left(1 - \frac{h}{n-1}\right)\eta(Z)\eta(U) - \eta(U)\eta(Z)\right] = 0,$$

Which on simplification we obtain

(6.7) $W_2(Z,U,V,Y) = \left(1 - \frac{h}{n-1}\right)[g(Y,Z)g(U,V) - g(Y,U)g(Z,V).$

And so

(6.8) $$W_2(Z,U)V = \left(1 - \frac{h}{n-1}\right)[g(U,V)Z - g(Z,V)U].$$

Thus in view of (6.1) and (6.8), we obtain

(6.9) $$R(Z,U)V = g(U,VZ - g(Z,V)U.$$

Thus we have the following:

Theorem 6.4. A Lorentzian para-Sasakian manifold satisfying $R(X,Y)\cdot W_2 = 0$ is a space of constant curvature.

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