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ON LORENTZIAN TRANS-SASAKIAN MANIFOLDS

U.C. DE AND KRISHNENDU DE

Abstract. The object of the present paper is to study the Trans-Sasakian structure on a manifold with Lorentzian metric. Several interesting results are obtained on the manifold. Also conformally flat Lorentzian Trans-Sasakian manifolds have been studied. Next, in three- dimensional Lorentzian Trans-Sasakian manifolds, explicit formulae for Ricci operator, Ricci tensor and curvature tensor are obtained. Also it is proved that a three-dimensional Lorentzian Trans-Sasakian manifold of type (α, β) is locally ϕ- symmetric if and only if the scalar curvature r is constant provided α and β are constants. Finally, we give some examples of three-dimensional Lorentzian Trans-Sasakian manifold.

1. Introduction

Let M be an odd dimensional manifold with Riemannian metric g. It is well known that an almost contact metric structure (ϕ, ξ, η) (with respect to g) can be defined on M by a tensor field ϕ of type $(1, 1)$, a vector field ξ and a 1- form η. If M has a Sasakian structure (Kenmotsu structure), then M is called a Sasakian manifold (Kenmotsu manifold). Sasakian manifolds and Kenmotsu manifolds have been studied by several authors.
In the classification of Gray and Hervella [8] of almost Hermitian manifolds there appears a class, W_4, of Hermitian manifolds which are closely related to locally conformally Kaehler manifolds. An almost contact metric structure (ϕ, ξ, η, g) on M is Trans-Sasakian [17] if $(M \times R, J, G)$ belongs to the class W_4, where J is the almost complex structure on $M \times$R defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}),$$

for all vector fields X on M, f is a smooth function on $M \times \mathbb{R}$ and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [2]

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{1.1}$$

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for smooth functions α and β on M. Hence we say that the Trans-Sasakian structure is of type (α,β). In particular, it is normal and it generalizes both α-Sasakian and β-Kenmotsu structures. From the formula (1.1) one easily obtains

$$\nabla_X \xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi). \tag{1.2}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{1.3}$$

In 1981, Janssens and Vanhecke introduced the notion of α- Sasakian and β-Kenmotsu manifolds where α and β are non zero real numbers. It is known that [6] Trans-Sasakian structures of type (0,0) , (0,β) and (α,0) are cosymplectic ([1], [2]), β - Kenmotsu ([6]) and α- Sasakian ([6]) respectively. The local structure of Trans-Sasakian manifolds of dimension $n \geq 5$ has been completely characterized by Marrero [10]. He proved that a Trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or α-Sasakian or β -Kenmotsu manifold. Trans-Sasakian manifolds have been studied by several authors ([3], [4], [5],[11], [18]).

Let (x, y, z) be cartesian co-ordinates in $\mathbb{R}^3$, then (ϕ, ξ, η, g) given by

$$\xi = \frac{\partial}{\partial z}, \eta = dz - ydx,$$

$$\phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -y & 0 \\ & & \end{pmatrix}, \quad g = \begin{pmatrix} e^z + y^2 & 0 & -y \\ 0 & e^z & 0 \\ -y & 0 & 1 \\ & & \end{pmatrix}$$

is a Trans-Sasakian structure of type $(\frac{-1}{2e^z}, \frac{1}{2})$ in $\mathbb{R}^3$ [2]. In general, in a three-dimensional K-contact manifold with structure tensors (ϕ, ξ, η, g) for a non-constant function f, if we define $\tilde{g} = fg + (1-f)\eta \otimes \eta$; then $(\phi, \xi, \eta, \tilde{g})$ is a Trans-Sasakian structure of type $(\frac{1}{f}, \frac{1}{2}\xi(lnf))$ [10].

Let M be a differentiable manifold. When M has a Lorentzian metric g, that is, a symmetric non degenerate $(0, 2)$ tensor field of index 1, then M is called a Lorentzian manifold. Since the Lorentzian metric is of index 1, Lorentzian manifold M has not only spacelike vector fields but also timelike and lightlike vector fields. This difference with the Riemannian case give interesting properties on the Lorentzian manifold. A differentiable manifold M has a Lorentzian metric if and only if M has a 1- dimensional distribution. Hence odd dimensional manifold is able to have a Lorentzian metric.
Therefore, it is very natural and interesting idea to define both a Trans-Sasakian structure and a Lorentzian metric on an odd dimensional manifold.

The paper is organized as follows. In Section 1, we give a brief account of Lorentzian Trans-Sasakian manifolds. After preliminaries, some basic results are given. In Section 4, we study conformally flat Lorentzian Trans-Sasakian manifolds.

In the next section, explicit formulae for Ricci operator, Ricci tensor and curvature tensor are obtained for three-dimensional Trans-Sasakian manifolds. Also it is proved that a three-dimensional Lorentzian Trans-Sasakian manifold of type (α, β) is locally ϕ- symmetric if and only if the scalar curvature r is constant provided α and β are constants. Finally we construct some examples of three-dimensional Lorentzian Trans-Sasakian manifolds.

2. Lorentzian Trans-Sasakian manifolds

A differentiable manifold M of dimension $(2n+1)$ is called a Lorentzian Trans-Sasakian manifold if it admits a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and the Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2 = I + \eta \otimes \xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \phi\xi = 0, \eta(\phi X) = 0, \tag{2.4}$$

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.5}$$

for all $X, Y \epsilon T(M)$.

Also a Lorentzian Trans-Sasakian manifold M satisfies

$$\nabla_X \xi = -\alpha(\phi X) - \beta(X + \eta(X)\xi), \tag{2.6}$$

$$(\nabla_X \eta)Y = \alpha g(\phi X, Y) + \beta g(\phi X, \phi Y), \tag{2.7}$$

where ∇ denotes the operator of covariant differentation with respect to the Lorentzian metric g .

If $\alpha = 0$ and $\beta \varepsilon \mathbb{R}$, the set of real numbers, then the manifold reduces to a Lorentzian β-Kenmotsu manifold studied by Funda Yaliniz, Yildiz, and Turan [20]. If $\beta = 0$ and $\alpha \varepsilon \mathbb{R}$, then the manifold reduces to a Lorentzian α- Sasakian manifold studied by Yildiz, Turan and Murathan [21]. If $\alpha = 0$ and $\beta = 1$, then the manifold reduces to a Lorentzian Kenmotsu manifold introduced by Mihai, Oiaga and Rosca [15]. Furthermore, if $\beta = 0$ and $\alpha = 1$, then the manifold reduces to a Lorentzian Sasakian manifold studied by Ikawa and Erdogan [15]. Also Lorentzian para contact manifolds were introduced by Matsumoto [12] and further studied by the authors ([13],[14],[16]). Trans Lorentzian para Sasakian manifolds have been used by Gill and Dube [7].

3. SOME BASIC RESULTS

In this section, we prove some Lemmas which are needed in the rest of the sections.

Lemma 3.1. *In a Lorentzian Trans-Sasakian manifold, we have*

$$\begin{aligned} R(X,Y)\xi &= (\alpha^2+\beta^2)(\eta(Y)X-\eta(X)Y) \\ &\quad +2\alpha\beta(\eta(Y)\phi X-\eta(X)\phi Y)+(Y\alpha)\phi X \\ &\quad -(X\alpha)\phi Y+(Y\beta)\phi^2 X-(X\beta)\phi^2 Y, \end{aligned} \tag{3.1}$$

where R is the curvature tensor.

Proof. We have

$$\begin{aligned} \nabla_X\nabla_Y\xi &= \nabla_X(-\alpha(\phi Y)-\beta(Y+\eta(Y)\xi)) \\ &= -(X\alpha)\phi Y-\alpha\nabla_X(\phi Y)-(X\beta)\phi^2 Y \\ &\quad -\beta\nabla_X Y-\beta(X\eta(Y))\xi+\alpha\beta\eta(Y)\phi X \\ &\quad +\beta^2\eta(Y)X+\beta^2\eta(X)\eta(Y)\xi, \end{aligned}$$

where (2.2) and (2.6) have been used. Hence, in view of the above equation and (2.6), we get

$$\begin{aligned} R(X,Y)\xi &= \nabla_X\nabla_Y\xi-\nabla_Y\nabla_X\xi-\nabla_{[X,Y]}\xi \\ &= -(X\alpha)\phi Y+(Y\alpha)\phi X-\alpha((\nabla_X\phi Y)-(\nabla_Y\phi X)) \\ &\quad -(X\beta)\phi^2 Y+(Y\beta)\phi^2 X-\beta((\nabla_X\eta)Y-(\nabla_Y\eta)X)\xi \\ &\quad +\alpha\beta(\eta(Y)\phi X-\eta(X)\phi Y)+\beta^2(\eta(Y)X-\eta(X)Y), \end{aligned}$$

which in view of (2.5) and (2.7) gives (3.1). □

Lemma 3.2. *For a Lorentzian Trans-Sasakian manifold, we have*

$$\eta(R(X,Y)Z)=(\alpha^2+\beta^2)(g(X,Z)\eta(Y)-g(Y,Z)\eta(X)). \tag{3.2}$$

Proof. We have from (3.1),

$$\begin{aligned} g(R(X,Y)\xi,Z) &= (\alpha^2+\beta^2)(\eta(Y)g(X,Z)-\eta(X)g(Y,Z)) \\ &\quad +2\alpha\beta(\eta(Y)g(\phi X,Z)-\eta(X)g(\phi Y,Z))+(Y\alpha)g(\phi X,Z) \\ &\quad -(X\alpha)g(\phi Y,Z)+(Y\beta)g(\phi^2 X,Z)-(X\beta)g(\phi^2 Y,Z), \end{aligned}$$

Now interchanging ξ and Z in the above equation, we get

$$\begin{aligned} -g(R(X,Y)Z,\xi) &= (\alpha^2+\beta^2)(g(Y,Z)\eta(X)-g(X,Z)\eta(Y)) \\ &\quad +2\alpha\beta(\eta(Y)g(\phi X,\xi)-\eta(X)g(\phi Y,\xi))+(Y\alpha)g(\phi X,\xi) \\ &\quad -(X\alpha)g(\phi Y,\xi)+(Y\beta)g(\phi^2 X,\xi)-(X\beta)g(\phi^2 Y,\xi). \end{aligned}$$

After simplification, we find,

$$g(R(X,Y)Z,\xi)=(\alpha^2+\beta^2)(g(X,Z)\eta(Y)-g(Y,Z)\eta(X)),$$

which gives (3.2). □

Lemma 3.3. *For a Lorentzian Trans-Sasakian manifold, we have*

$$R(\xi, Y)\xi = (\alpha^2 + \beta^2 - \xi\beta)\phi^2 Y + (2\alpha\beta - \xi\alpha)\phi Y. \tag{3.3}$$

Proof. Replacing X by ξ in (3.1), we get (3.3). □

Lemma 3.4. *In a $(2n+1)$- dimensional Lorentzian Trans-Sasakian manifold, we have*

$$\begin{aligned} S(X, \xi) &= (2n(\alpha^2 + \beta^2) - \xi\beta)\eta(X) + (2n-1)(X\beta) \\ &\quad -(\phi X)\alpha + \psi(2\alpha\beta\eta(X) + X\alpha), \end{aligned} \tag{3.4}$$

$$\begin{aligned} Q\xi &= (2n(\alpha^2 + \beta^2) - \xi\beta)\xi + (2n-1)grad\beta \\ &\quad -\phi(grad\alpha) + \psi(2\alpha\beta\xi + grad\alpha), \end{aligned} \tag{3.5}$$

where S is the Ricci curvature and Q is the Ricci operator given by

$$S(X, Y) = g(QX, Y) \quad and \quad \psi = \sum_{i=1}^{2n+1} \epsilon_i g(\phi e_i, e_i).$$

Proof. Let M be an $(2n+1)-$ dimensional Lorentzian Trans-Sasakian manifold. Then the Ricci tensor S of the manifold M is defined by

$$S(X, Y) = \sum_{i=1}^{2n+1} \epsilon_i g(R(e_i, X)Y, e_i),$$

where $\epsilon_i = g(e_i, e_i), \epsilon_i = \pm 1$. From (3.1), we have

$$\begin{aligned} S(X, \xi) &= (\alpha^2 + \beta^2)[\eta(X) \sum_{i=1}^{2n+1} g(e_i, e_i)g(e_i, e_i) - \sum_{i=1}^{2n+1} \eta(e_i)g(e_i, e_i)g(X, e_i)] \\ &\quad +2\alpha\beta[\eta(X) \sum_{i=1}^{2n+1} g(e_i, e_i)g(\phi e_i, e_i) - \sum_{i=1}^{2n+1} \eta(e_i)g(e_i, e_i)g(\phi X, e_i)] \\ &\quad - \sum_{i=1}^{2n+1} (e_i\alpha)g(e_i, e_i)g(\phi X, e_i) + \sum_{i=1}^{2n+1} (X\alpha)g(e_i, e_i)g(\phi e_i, e_i) \\ &\quad - \sum_{i=1}^{2n+1} (e_i\beta)g(e_i, e_i)g(\phi^2 X, e_i) + \sum_{i=1}^{2n+1} (X\beta)g(e_i, e_i)g(\phi^2 e_i, e_i) \\ &= (2n(\alpha^2 + \beta^2) - \xi\beta)\eta(X) + (2n-1)(X\beta) \\ &\quad -(\phi X)\alpha + \psi(2\alpha\beta\eta(X) + X\alpha) \end{aligned}$$

and hence from (3.4), we get (3.5). □

Remark 3.5. If in a $(2n+1)$- dimensional Lorentzian Trans-Sasakian manifold of type (α,β) we consider $\phi(grad\alpha)=(2n-1)grad\beta$, then

$$\begin{aligned}\xi\beta &= g(\xi, grad\beta)=\frac{1}{2n-1}g(\xi,\phi(grad\alpha))\\ &= \frac{1}{2n-1}\eta(\phi(grad\alpha))=0\end{aligned}$$

and

$$\begin{aligned}X\beta &= g(X, grad\beta)=\frac{1}{2n-1}g(X,\phi(grad\alpha))\\ &= \frac{1}{2n-1}g(\phi X,(grad\alpha))=\frac{1}{2n-1}(\phi X)\alpha\end{aligned}$$

and hence (3.4) and (3.5) are reduced to

$$S(X,\xi)=2n(\alpha^2+\beta^2)\eta(X)+\psi(2\alpha\beta\eta(X)+X\alpha) \tag{3.6}$$

and

$$Q\xi=(2n(\alpha^2+\beta^2)-\xi\beta)\xi+\psi(2\alpha\beta\xi+grad\alpha), \tag{3.7}$$

respectively.

4. Conformally Flat Lorentzian Trans-Sasakian manifolds

In this section we consider conformally flat Lorentzian Trans-Sasakian manifold M^{2n+1} (ϕ,ξ,η,g) $(n>1)$. The conformal curvature tensor C is given by

$$\begin{aligned}C(X,Y)Z &= R(X,Y)Z-\frac{1}{2n-1}[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX\\ &\quad -g(X,Z)QY]+\frac{r}{(2n)(2n-1)}[g(Y,Z)X-g(X,Z)Y],\end{aligned} \tag{4.1}$$

where r is the scalar curvature of M.
For conformally flat manifold, we have $C(X,Y)Z=0$ for $n>1$ and hence from (4.1) we have

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &= \frac{1}{2n-1}[S(Y,Z)g(X,W)-S(X,Z)g(Y,W)\\ &\quad +g(Y,Z)S(X,W)-g(X,Z)S(Y,W)]\\ &\quad -\frac{r}{(2n)(2n-1)}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)],\end{aligned} \tag{4.2}$$

where $g(R(X,Y)Z,U)=\widetilde{R}(X,Y,Z,U)$. Setting $W=\xi$ in (4.2) we get

$$\begin{aligned}\eta(R(X,Y)Z) &= \frac{1}{2n-1}[S(Y,Z)\eta(X)-S(X,Z)\eta(Y)\\ &+g(Y,Z)S(X,\xi)-g(X,Z)S(Y,\xi)]\\ &-\frac{r}{(2n)(2n-1)}[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]. \qquad (4.3)\end{aligned}$$

Replacing Y by ξ in (4.3) and using (3.2) and (3.6), we get

$$\begin{aligned}S(X,Z) &= [(\alpha^2+\beta^2)(1-4n)+\frac{r}{2n}+(\xi\alpha-2\alpha\beta)\psi]g(X,Z)\\ &+[(\alpha^2+\beta^2)(1-6n)+\frac{r}{2n}-4\alpha\beta\psi]\eta(X)\eta(Z)\\ &-[\eta(Z)(X\alpha)+\eta(X)(Z\alpha)]\psi. \qquad (4.4)\end{aligned}$$

This leads to the following:

Theorem 4.1. *A conformally flat Lorentzian Trans Sasakian manifold* M^{2n+1} (ϕ,ξ,η,g) $(n>1)$ *is an* $\eta-$ *Einstein manifold provided* $\psi = trace\phi = o$ *and* $\phi(grad\alpha) = (2n-1)grad\beta$.

Corollary 1. *A conformally flat Lorentzian* $\beta-$ *Kenmotsu manifold* M^{2n+1} (ϕ,ξ,η,g) $(n>1)$ *is an* $\eta-$ *Einstein manifold.*

5. Three- dimensional Lorentzian Trans- Sasakian manifolds

Since the conformal curvature tensor vanishes in a three-dimensional Riemannian manifold, therefore we get

$$\begin{aligned}R(X,Y)Z &= g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X-S(X,Z)Y\\ &-\frac{r}{2}[g(Y,Z)X-g(X,Z)Y], \qquad (5.1)\end{aligned}$$

where Q is the Ricci operator, that is, $g(QX,Y)=S(X,Y)$ and r is the scalar curvature of the manifold.

From Lemma 2.4, in a three- dimensional Lorentzian Trans-Sasakian manifold we have

$$\begin{aligned}S(X,\xi) &= (2(\alpha^2+\beta^2)-\xi\beta)\eta(X)+(X\beta)\\ &-(\phi X)\alpha+\psi(2\alpha\beta\eta(X)+X\alpha), \qquad (5.2)\end{aligned}$$

$$\begin{aligned}Q\xi &= (2(\alpha^2+\beta^2)-\xi\beta)\xi+grad\beta\\ &-\phi(grad\alpha)+\psi(2\alpha\beta\xi+grad\alpha). \qquad (5.3)\end{aligned}$$

Now, in the following theorem, we obtain an expression for Ricci operator in a three-dimensional Lorentzian Trans-Sasakian manifold.

Theorem 5.1. *In a three- dimensional Lorentzian Trans Sasakian manifold, the Ricci operator is given by*

$$\begin{aligned} QX &= (\frac{r}{2} + \xi\beta - (\alpha^2 + \beta^2) + \psi(\xi\alpha - 2\alpha\beta))X \\ &\quad +(\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2) - 4\alpha\beta\psi)\eta(X)\xi \\ &\quad -\eta(X)(grad\beta - \phi(grad\alpha) + \psi(grad\alpha)) - (X\beta - (\phi X)\alpha + \psi(X\alpha))\xi \\ &\quad +(2\alpha\beta - \xi\alpha)\phi X. \end{aligned} \tag{5.4}$$

Proof. For a three- dimensional Lorentzian Trans Sasakian manifold, from (5.1) and (5.2), we have

$$\begin{aligned} R(X,Y)\xi &= \eta(Y)QX - \eta(X)QY \\ &\quad -(\frac{r}{2} + \xi\beta - 2(\alpha^2 + \beta^2) - 2\alpha\beta\psi)[X\eta(Y) - Y\eta(X)] \\ &\quad +(Y\beta - (\phi Y)\alpha + (Y\alpha)\psi)X - (X\beta - (\phi X)\alpha + (X\alpha)\psi)Y. \end{aligned} \tag{5.5}$$

In view of (3.1) and (5.5), we obtain

$$\begin{aligned} 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) &+ (Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y \\ &= \eta(Y)QX - \eta(X)QY - (\frac{r}{2} + \xi\beta - (\alpha^2 + \beta^2) - 2\alpha\beta\psi) \\ &\quad [\eta(Y)X - \eta(X)Y] + (Y\beta - (\phi Y)\alpha + (Y\alpha)\psi)X \\ &\quad -(X\beta - (\phi X)\alpha + (X\alpha)\psi)Y. \end{aligned}$$

Putting $Y = \xi$ in the above equation, we get (5.4) . □

Corollary 2. *In a three- dimensional Lorentzian Trans Sasakian manifold, Ricci tensor and curvature tensor are given respectively by*

$$\begin{aligned} S(X,Y) &= (\frac{r}{2} + \xi\beta - (\alpha^2 + \beta^2) + \psi(\xi\alpha - 2\alpha\beta))g(X,Y) \\ &\quad +(\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2) - 4\alpha\beta\psi)\eta(X)\eta(Y) \\ &\quad +\eta(X)[-Y\beta + (\phi Y)\alpha - \psi(Y\alpha)] - \eta(Y)(X\beta - (\phi X)\alpha + \psi(X\alpha)) \\ &\quad +(2\alpha\beta - \xi\alpha)g(\phi X, Y). \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2\xi\beta - 2(\alpha^2+\beta^2) + 2\psi(\xi\alpha - 2\alpha\beta))[g(Y,Z)X - g(X,Z)Y] \\ &+g(Y,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2+\beta^2) - 4\alpha\beta\psi)\eta(X)\xi \\ &+\eta(X)(\phi(grad\alpha) - \psi(grad\alpha) - grad\beta) - (X\beta - (\phi X)\alpha + \psi(X\alpha))\xi] \\ &+g(X,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2+\beta^2) - 4\alpha\beta\psi)\eta(Y)\xi \\ &+\eta(Y)(\phi(grad\alpha) - \psi(grad\alpha) - grad\beta) - (Y\beta - (\phi Y)\alpha + \psi(Y\alpha))\xi] \\ &+[(\frac{r}{2} + \xi\beta - 3(\alpha^2+\beta^2) - 4\alpha\beta\psi)\eta(Y)\eta(Z) \\ &+\eta(Y)(-Z\beta + (\phi Z)\alpha - \psi(Z\alpha)) - \eta(Z)(Y\beta - (\phi Y)\alpha + \psi(Y\alpha))]X \\ &-[(\frac{r}{2} + \xi\beta - 3(\alpha^2+\beta^2) - 4\alpha\beta\psi)\eta(X)\eta(Z) \\ &+\eta(X)(-Z\beta + (\phi Z)\alpha - \psi(Z\alpha)) - \eta(Z)(X\beta - (\phi X)\alpha + \psi(X\alpha))]Y \\ &+(2\alpha\beta - \xi\alpha)[g(\phi Y,Z)X - g(\phi X,Z)Y]. \end{aligned} \tag{5.7}$$

*Equation (*5.6*) follows from* (5.4)*. Using* (5.4) *and* (5.6) *in* (5.1)*, the curvature tensor in a three- dimensional Lorentzian Trans-Sasakian manifold is given by* (5.7)*.*

6. Locally ϕ- symmetric three-dimensional Lorentzian Trans-Sasakian manifolds with *trace* $\phi = \psi = 0$

The notion of locally ϕ-symmetry was first introduced by T.Takahashi [19] on a Sasakian manifold. In this paper we study locally ϕ- symmetric three-dimensional Lorentzian Trans-Sasakian manifolds.

Definition 6.1. A three-dimensional Lorentzian Trans-Sasakian manifold is said to be locally ϕ- symmetric if

$$\phi^2(\nabla_W R)(X,Y)Z = 0, \tag{6.1}$$

where W,X,Y,Z are horizontal vector fields, that is W,X,Y,Z are orthogonal to ξ.

Let M be a three- dimensional Lorentzian Trans-Sasakian manifold with $trace\phi = \psi = 0$. Then its curvature tensor is given by

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2\xi\beta - 2(\alpha^2+\beta^2))[g(Y,Z)X - g(X,Z)Y] \\ &+g(Y,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2+\beta^2))\eta(X)\xi \\ &+\eta(X)(\phi(grad\alpha) - grad\beta) - (X\beta - (\phi X)\alpha)\xi] \\ &+g(X,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2+\beta^2))\eta(Y)\xi \\ &+\eta(Y)(\phi(grad\alpha) - grad\beta) - (Y\beta - (\phi Y)\alpha)\xi] \\ &+[(\frac{r}{2} + \xi\beta - 3(\alpha^2+\beta^2))\eta(Y)\eta(Z) \\ &+\eta(Y)(-Z\beta + (\phi Z\alpha)) - \eta(Z)(Y\beta - (\phi Y)\alpha)]X \\ &-[(\frac{r}{2} + \xi\beta - 3(\alpha^2+\beta^2))\eta(X)\eta(Z) \\ &+\eta(X)(-Z\beta + (\phi Z\alpha)) - \eta(Z)(X\beta - (\phi X)\alpha)]Y \\ &+(2\alpha\beta - \xi\alpha)[g(\phi Y,Z)X - g(\phi X,Z)Y]. \end{aligned} \tag{6.2}$$

Differentiating (6.2) we get

$$\begin{aligned} (\nabla_W R)(X,Y)Z &= [\frac{dr(W)}{2} + 2(\nabla_W(\xi\beta)) - 4(d\alpha(W) + d\beta(W))] \\ &[g(Y,Z)X - g(X,Z)Y] + g(Y,Z)[(\frac{dr(W)}{2} + (\nabla_W(\xi\beta)) \\ &-6(d\alpha(W) + d\beta(W)))\eta(X)\xi + (\frac{r}{2} + \xi\beta - 3(\alpha^2 \\ &+\beta^2))((\nabla_W\eta)(X)\xi + \eta(X)(\nabla_W\xi)) \\ &+(\nabla_W\eta)(X)(\phi(grad\alpha) - grad\beta) + \eta(X)(\nabla_W(\phi(grad\alpha) - grad\beta)) \\ &+(\nabla_W(X\beta - (\phi X)\alpha))\xi + (X\beta - (\phi X)\alpha)\nabla_W\xi] \\ &-g(X,Z)[(\frac{dr(W)}{2} + (\nabla_W(\xi\beta)) - 6(d\alpha(W) + d\beta(W)))\eta(Y)\xi \\ &+(\frac{r}{2} + \xi\beta - 3(\alpha^2+\beta^2))((\nabla_W\eta)(Y)\xi + \eta(Y)(\nabla_W\xi)) \\ &+(\nabla_W\eta)(Y)(\phi(grad\alpha) - grad\beta) + \eta(Y)(\nabla_W(\phi(grad\alpha) - grad\beta)) \\ &+(\nabla_W(Y\beta - (\phi Y)\alpha))\xi + (Y\beta - (\phi Y)\alpha)\nabla_W\xi] \\ &-Y[(\nabla_W(Y\beta - (\phi Y)\alpha))\eta(Z) + (Y\beta - (\phi Y)\alpha)(\nabla_W\eta)Z \\ &+\nabla_W(Z\beta - (\phi Z)\alpha))\eta(Y) + (Z\beta - (\phi Z)\alpha)(\nabla_W\eta)Y \\ &-(\frac{dr(W)}{2} + (\nabla_W(\xi\beta)) - 6(d\alpha(W) + d\beta(W)))\eta(Y)\eta(Z) \end{aligned}$$

$$-(\frac{r}{2}+\xi\beta-3(\alpha^2+\beta^2))$$
$$((\nabla_W\eta)Y\eta(Z)+\eta(Y)(\nabla_W\eta)Z)]$$
$$+X[(\nabla_W(X\beta-(\phi X)\alpha))\eta(Z)+(X\beta-(\phi X)\alpha)(\nabla_W\eta)Z$$
$$+\nabla_W(Z\beta-(\phi Z)\alpha))\eta(X)+(Z\beta-(\phi Z)\alpha)(\nabla_W\eta)X$$
$$-(\frac{dr(W)}{2}+(\nabla_W(\xi\beta))-6(d\alpha(W)+d\beta(W)))\eta(X)\eta(Z)$$
$$-(\frac{r}{2}+\xi\beta-3(\alpha^2+\beta^2))$$
$$((\nabla_W\eta)X\eta(Z)+\eta(X)(\nabla_W\eta)Z)]$$
$$+(2(\nabla_W(\alpha\beta))-(\nabla_W(\xi\alpha)))[g(\phi Y,Z)X-g(\phi X,Z)Y]. \tag{6.3}$$

Suppose that α and β are constants and X,Y,Z,W are orthogonal to ξ. Then using $\phi\xi=0$ and (6.1), we get

$$\phi^2(\nabla_W R)(X,Y)Z=(\frac{dr(W)}{2})[g(Y,Z)X-g(X,Z)Y]. \tag{6.4}$$

Thus we can state the following:

Theorem 6.2. *A three-dimensional Lorentzian Trans-Sasakian manifold of type (α,β) is locally ϕ- symmetric if and only if the scalar curvature r is constant provided α and β are constants.*

7. Examples

Example 7.1: We consider the three-dimensional manifold $M=\{(x,y,z)\varepsilon\mathbb{R}^3, z\neq 0\}$, where (x,y,z) are standard co-ordinate of $\mathbb{R}^3$.

The vector fields

$$e_1=z(\frac{\partial}{\partial x}+y\frac{\partial}{\partial z}),\quad e_2=z\frac{\partial}{\partial y},\quad e_3=\frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Riemannian metric defined by

$$g(e_1,e_3)=g(e_1,e_2)=g(e_2,e_3)=0,$$
$$g(e_1,e_1)=g(e_2,e_2)=1, g(e_3,e_3)=-1.$$

Let η be the 1-form defined by $\eta(Z)=g(Z,e_3)$ for any $Z\varepsilon\chi(M)$.

Let ϕ be the $(1,1)$ tensor field defined by

$$\phi(e_1)=-e_2,\quad \phi(e_2)=-e_1,\quad \phi(e_3)=0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_3)=-1,$$
$$\phi^2 Z=Z+\eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W),$$

for any $Z, W \varepsilon \chi(M)$.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an Lorentzian structure on M.

Let ∇ be the Levi-Civita connection with respect to metric g and R be the curvature tensor of g. Then we have

$$[e_1, e_2] = ye_2 - z^2 e_3 \quad , [e_1, e_3] = -\frac{1}{z} e_1 \quad and \quad [e_2, e_3] = -\frac{1}{z} e_2.$$

Taking $e_3 = \xi$ and using Koszul formula for the Riemannian metric g, we can easily calculate

$$\nabla_{e_1} e_3 = -\frac{1}{z} e_1 + \frac{1}{z^2} e_2, \quad \nabla_{e_1} e_2 = -\frac{1}{2} z^2 e_3,$$
$$\nabla_{e_1} e_1 = -\frac{1}{z} e_3, \nabla_{e_2} e_3 = -\frac{1}{z} e_2 + \frac{1}{2} z^2 e_1,$$
$$\nabla_{e_2} e_2 = ye_1 - \frac{1}{z} e_3, \quad \nabla_{e_2} e_1 = \frac{1}{2} z^2 e_3 - ye_2,$$
$$\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = \frac{1}{2} z^2 e_1, \quad \nabla_{e_3} e_1 = -\frac{1}{2} z^2 e_2.$$

From the above it can be easily seen that (ϕ, ξ, η, g) is an Lorentzian Trans-Sasakian structure on M.Consequently$M^3(\phi, \xi, \eta, g)$ is an Lorentzian Trans-Sasakian manifold with $\alpha = \frac{1}{2} z^2 \neq 0$ and $\beta = \frac{1}{z} \neq 0$.

Example 7.2: We consider the three-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are standard co-ordinate of $\mathbb{R}^3$.

The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = z\frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$$
$$g(e_1, e_1) = g(e_2, e_2) = 1, g(e_3, e_3) = -1$$

that is, the form of the metric becomes

$$g = \frac{dx^2 + dy^2 - dz^2}{z^2}.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \varepsilon \chi(M)$.

Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_3) = -1,$$
$$\phi^2 Z = Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an Lorentzian structure on M.

Let ∇ be the Levi-Civita connection with respect to metric g. Then we have

$$\begin{aligned} [e_1, e_3] &= e_1 e_3 - e_3 e_1 \\ &= z\frac{\partial}{\partial x}(z\frac{\partial}{\partial z}) - z\frac{\partial}{\partial z}(z\frac{\partial}{\partial x}) \\ &= z^2\frac{\partial^2}{\partial x \partial z} - z^2\frac{\partial^2}{\partial z \partial x} - z\frac{\partial}{\partial x} \\ &= -e_1. \end{aligned}$$

Similarly

$$[e_1, e_2] = 0 \quad \textit{and} \quad [e_2, e_3] = -e_2.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned} \tag{7.1}$$

which known as Koszul's formula.

Using (7.1) we have

$$\begin{aligned} 2g(\nabla_{e_1} e_3, e_1) &= -2g(e_1, e_1) \\ &= 2g(-e_1, e_1). \end{aligned} \tag{7.2}$$

Again by (7.1)

$$2g(\nabla_{e_1} e_3, e_2) = 0 = 2g(-e_1, e_2) \tag{7.3}$$

and

$$2g(\nabla_{e_1} e_3, e_3) = 0 = 2g(-e_1, e_3). \tag{7.4}$$

From (7.2), (7.3) and (7.4) we obtain

$$2g(\nabla_{e_1} e_3, X) = 2g(-e_1, X),$$

for all $X \varepsilon \chi(M)$.

Thus

$$\nabla_{e_1} e_3 = -e_1.$$

Therefore, (7.1) further yields

$$\begin{aligned} &\nabla_{e_1} e_3 = -e_1, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_1 = -e_3, \\ &\nabla_{e_2} e_3 = -e_2, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_1 = 0, \\ &\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_1 = 0. \end{aligned} \tag{7.5}$$

(7.5) tells us that the manifold satisfies (1.3) for $\alpha = 0$, $\beta = 1$ and $\xi = e_3$. Hence the manifold is a Lorentzian Trans-Sasakian manifold of type $(0, 1)$. It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{7.6}$$

With the help of the above results and using (7.6) it can be easily verified that

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_1, e_3)e_3 = -e_1,$$

$$R(e_1, e_2)e_2 = -e_1, \quad R(e_2, e_3)e_2 = -e_3, \quad R(e_1, e_3)e_2 = 0,$$

$$R(e_1, e_2)e_1 = e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_1 = -e_3.$$

From the expression of the curvature tensor it follows that the manifold is of constant curvature -1. Hence the manifold is locally ϕ-symmetric. Also from the above expressions of the curvature tensor, we obtain

$$\begin{aligned} S(e_1, e_1) &= g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) \\ &= -2. \end{aligned}$$

Similarly, we have

$$S(e_2, e_2) = -2, S(e_3, e_3) = 2.$$

Therefore,

$$r = S(e_1, e_1) + S(e_2, e_2) - S(e_3, e_3) = -6.$$

Thus the scalar curvature r is constant. Hence Theorem 6.1 is verified.

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Current address: Uday Chand De,, Department of Pure Mathematics,, Calcutta University ,, 35 Ballygunge Circular Road, Kol 700019,West Bengal, INDIA. , Krishnendu De, Konnagar High School(H.S.),, 68 G.T. Road (West),Konnagar,Hooghly, Pin.712235, West Bengal, INDIA.

E-mail address: `uc_de@yahoo.com, krishnendu_de@yahoo.com`

URL: `http://communications.science.ankara.edu.tr/index.php?series=A1`