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On Lovelock analogs of the Riemann tensor

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Abstract It is possible to define an analog of the Riemann tensor for Nth order Lovelock gravity, its characterizing property being that the trace of its Bianchi derivative yields the corresponding analog of the Einstein tensor. Interestingly there exist two parallel but distinct such analogs and the main purpose of this note is to reconcile both formulations. In addition we will introduce a simple tensor identity and use it to show that any pure Lovelock vacuum in odd $d = 2N + 1$ dimensions is Lovelock flat, i.e. any vacuum solution of the theory has vanishing Lovelock–Riemann tensor. Further, in the presence of cosmological constant it is the Lovelock–Weyl tensor that vanishes.

1 Introduction

In order to write an equation of motion for Einstein gravity, one has to obtain a divergence free second rank symmetric tensor constructed solely from the metric and the Riemann curvature – the Einstein tensor. This is usually done by varying the Einstein–Hilbert action, the scalar curvature R, relative to the metric tensor. Alternatively one can obtain the same result by invoking the differential geometric property that the Bianchi derivative of the Riemann tensor identically vanishes – the Bianchi identity. From the trace of this identity we can then extract the required Einstein tensor. This is a very neat and elegant purely geometric way to get to the equation of motion.

The most natural generalization of Einstein gravity in higher dimensions is Lovelock gravity, whose equation of motion inherits the basic property of being second order, though polynomial in curvature. The natural question then arises, could the same geometric method used for Einstein gravity also work for Lovelock theories? The answer is yes. In [1], one of the authors defined an Nth order Lovelock analog of the Riemann curvature, which is a homogeneous polynomial in the Riemann tensor. Even though this tensor does not satisfy the Bianchi identity, the trace of its Bianchi derivative vanishes yielding the Nth order Lovelock analog of the Einstein tensor. This tensor agrees with the one obtained by varying Nth order Lovelock Lagrangian and is divergence free.

There is an alternative formulation due to Kastor [2] that also leads to the definition of a different higher order analog of the Riemann tensor. His construction has a much richer geometric structure as it involves a 4Nth rank tensor as its basic object. This higher rank tensor does satisfy the Bianchi identity, i.e. its Bianchi derivative vanishes, and again the trace of this identity leads to the corresponding Einstein analog. Interestingly the analog tensors obtained in Dadhich's and Kastor's formulations agree and therefore lead to the same equation of motion. Both descriptions are dynamically equivalent. This had to be the case as the Lovelock Lagrangian is unique at each order. The main aim of this note is to reconcile these two parallel formulations and also to illuminate a universal property of pure Lovelock gravity that distinguishes between odd $d = 2N + 1$ (the critical dimension) and even $2N + 2$ (or higher) dimensions. Dadhich et al. [4] considered pure Lovelock static vacuum solutions and established that pure Lovelock gravity in odd $d = 2N + 1$ dimensions is *kinematic*, i.e. whenever the Love-lock-Ricci vanishes so does the corresponding Riemann. That is, pure Lovelock vacuum in critical odd dimension is Lovelock flat, as is the case for $N = 1$ Einstein gravity in 3 dimension. Based on this result, Dadhich [5] conjectured that this should be true not only for static vacuum solutions but in general for all vacuum spacetimes. It would be a universal gravitational property. However, it later turned out that this is not true in general for Dadhich's Lovelock–Riemann tensor while it actually holds for Kastor's analog [2]. This is a purely algebraic property due to the fact that we can write Kastor's 4Nth rank ten-

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sor[1] ${}_{(N)}\mathbb{R}^{(4N)}$ (therefore also all its contractions) in terms of the Lovelock–Ricci (or equivalently the corresponding Lovelock–Einstein, ${}_{(N)}\mathcal{E}^{a}_{\ b}$). As we will describe below, in $d = 2N + 1$ we can write

$$ {}_{(N)}\mathbb{R}^{b_1 \ldots b_{2N}}_{a_1 \ldots a_{2N}} = \frac{1}{(2N)!} \varepsilon^{b_1 \ldots b_{2N+1}} \, \varepsilon_{a_1 \ldots a_{2N+1}} \, {}_{(N)}\mathcal{E}^{a_{2N+1}}_{\phantom{a_{2N+1}} b_{2N+1}}. \tag{1} $$

Notice that this also fixes completely the form of ${}_{(N)}\mathbb{R}^{(4N)}$ in the presence of a cosmological constant. Drawing again parallels with three dimensional general relativity we will show that there is also a higher order analog of the Weyl tensor that vanishes in that case. In order to prove these properties, we will introduce an interesting set of tensorial identities. These imply a set of identities previously proposed by Kastor, even though are implemented in a simpler and more geometrically transparent way.

Lovelock gravities are a very interesting set of theories and have been used in many contexts with very diverse applications (see for instance [3] for a recent review). They can be viewed as a model of ghost free higher curvature/derivative gravity as Lovelock gravity captures many of the defining features of those theories while avoiding some of their problems, in particular the existence of higher derivative ghosts. Lovelock gravities have also been instrumental in exploring the role of higher curvature corrections in the holographic context.

The paper is organized as follows: we will first review Kastor's formulation that can be suitably described in the language of differential forms. This will be particularly convenient for the derivation of the Bianchi identities for the new higher order tensors, and also to introduce some simple tensor identities. Using these we will find a much more direct route to show the kinematicity of pure Lovelock gravity in odd critical dimensions. Next we reconcile the two formulations showing their equivalence and we end with a discussion.

2 Kastor's formulation

The starting point of Kastor's construction [2] is a $(2N, 2N)$-rank tensor product of N Riemann tensors, completely antisymmetric, both in its upper and lower indices,

$$ {}_{(N)}\mathbb{R}^{b_1 b_2 \ldots b_{2N}}_{a_1 a_2 \ldots a_{2N}} = R^{[b_1 b_2}_{ [a_1 a_2} \cdots R^{b_{2N-1} b_{2N}]}_{\phantom{b_{2N-1} b_{2N}]} a_{2N-1} a_{2N}]}. \tag{2} $$

With all indices lowered, this tensor is also symmetric under the exchange of both groups of indices, $a_i \leftrightarrow b_i$. In a similar way we will denote the contractions of ${}_{(N)}\mathbb{R}^{(4N)}$ simply as

$$ {}_{(N)}\mathbb{R}^{b_1 b_2 \ldots b_J}_{a_1 a_2 \ldots a_J} = {}_{(N)}\mathbb{R}^{b_1 b_2 \ldots b_J c_{J+1} \ldots c_{2N}}_{a_1 a_2 \ldots a_J c_{J+1} \ldots c_{2N}}; \quad \forall J < 2N. \tag{3} $$

We will use Latin indices generally. When the difference between tangent space and coordinate frames is needed, the latter indices will be denoted with Greek letters. In what follows we will omit the index (N) indicating the Lovelock or curvature order except when not clear from the context.

Rather than working with tensors, in some cases it is convenient to use the language of differential forms and the exterior algebra. This is particularly useful for Lovelock gravities as it makes the expressions much more compact and simplifies many manipulations. Recalling the relation between the curvature 2-form and the Riemann tensor,

$$ R^{ab} = \frac{1}{2} R^{ab}_{\mu\nu} \, \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}, $$

we can use (2) as components of a $2N$ form which is the antisymmetrized wedge product of N curvature 2-forms,

$$ \mathbb{R}^{b_1 b_2 \ldots b_{2K}}_{(N)} = R^{[b_1 b_2} \wedge \cdots \wedge R^{b_{2N-1} b_{2N}]}. \tag{4} $$

From this $2N$-form it is trivial to construct both the Lovelock action and its corresponding equation of motion. We just need to complete a d-form with vielbeins and contract with the antisymmetric symbol,

$$ \begin{aligned} \mathcal{L} &= \frac{2^N}{(2N)!(d-2N)!} \varepsilon_{a_1 a_2 \ldots a_d} \\ &\quad \times \mathbb{R}^{a_1 a_2 \cdots a_{2N}} \wedge e^{a_{2N+1}} \wedge \cdots \wedge e^{a_d} \,, \end{aligned} \tag{5} $$

$$ \begin{aligned} \mathcal{E}^{b}_{\ c} &= \frac{2^N (d-2N)}{(2N)!(d-2N)!} \, \varepsilon_{a_1 a_2 \ldots a_{d-1} c} \\ &\quad \times \mathbb{R}^{a_1 a_2 \cdots a_{2N}} \wedge e^{a_{2N+1}} \wedge \cdots \wedge e^{a_{d-1}} \wedge e^{b}. \end{aligned} \tag{6} $$

Notice that the antisymmetrization of the upper indices of $\mathbb{R}$ is completely irrelevant for constructing the action or the equation of motion. We could have defined an analogous tensor without antisymmetrizing the upper indices and the above formulas would have remained unchanged. However, all extra terms introduced this way would be irrelevant as they are zero upon contraction with the antisymmetric symbol. Thus the $\mathbb{R}^{(4N)}$ tensor encodes all the relevant dynamical information with the minimal number of independent components.

The expressions and derivation of Bianchi identities are also simpler in differential form language. This will also be true for the Bianchi identities associated with these new tensors that, due to their high degree of symmetry, have a very simple form. Let us first step back a bit and explain how Bianchi identities arise in the case of Einstein–Hilbert gravity. In differential form language, the torsion and curvature forms are introduced via Cartan's structure equations,

$$ \begin{aligned} T^{a} &= De^{a} = de^{a} + \omega^{a}_{\ b} \wedge e^{b}, \\ R^{a}_{\ b} &= d\omega^{a}_{\ b} + \omega^{a}_{\ c} \wedge \omega^{c}_{\ b} \,, \end{aligned} \tag{7} $$

[1] The lower index indicates the curvature order of the Lovelock term in the action, whereas the upper index corresponds to the tensor rank. Thus contractions of ${}_{(N)}\mathbb{R}^{(4N)}$ will be denoted ${}_{(N)}\mathbb{R}^{(2n)}$, for $n < 2N$.

for which we have introduced a covariant exterior derivative, D, with the corresponding connection 1-form $\omega^a{}_b$, in addition to the usual exterior operator d. In the Lovelock case, we may take the usual metric variation of the action to obtain the equation of motion or, instead, take independent variations with respect to vielbein and spin connection. The two approaches yield the exact same result, the equation coming from the ω-variation being proportional to the torsion and thus zero by assumption.

From Eq. (7) it is easy to derive the corresponding Bianchi identities just using the nilpotency of the exterior derivative, i.e. $d^2 = 0$ identically. Notice, however, that the covariant derivative D is not nilpotent. The Bianchi identities can be written simply as

$$DT^a = R^a{}_b \wedge e^b,$$
$$DR^{ab} = 0. \tag{8}$$

For vanishing torsion, expressing the above in components, we recover the well-known expressions

$$R^a{}_{[bcd]} = 0,$$
$$R^{ab}{}_{[\mu\nu;\alpha]} = 0. \tag{9}$$

These expressions have a very easy generalization for Nth order Lovelock gravity. In the same way as for the curvature 2-form, we can take the exterior covariant derivative of $\mathbb{R}$ and write

$$D\mathbb{R}^{a_1a_2\ldots a_{2N}} = NDR^{[a_1a_2} \wedge R^{a_3a_4} \wedge \cdots \wedge R^{a_{2K-1}a_{2K}]} = 0 \tag{10}$$

or again in components

$$\mathbb{R}^{a_1a_2\ldots a_{2N}}_{[\mu_1\mu_2\ldots\mu_{2N};\nu]} = 0. \tag{11}$$

On taking the trace of this identity we get $\mathcal{E}^a_{b;a} = 0$ from where we can then extract the required divergence free Lovelock–Einstein tensor (6). In terms of contractions of $\mathbb{R}^{(4N)}$, namely the Lovelock–Ricci tensor $\mathbb{R}^a{}_b$ and the respective scalar $\mathbb{R}$, it can be written as

$$\mathcal{E}^a{}_b = -(2N+1)\delta^{aa_1\ldots a_{2N}}_{bb1\ldots b_{2N}}\mathbb{R}^{b_1\ldots b_{2N}}_{a_1\ldots a_{2N}} = 2N\mathbb{R}^a{}_b - \delta^a{}_b\mathbb{R}. \tag{12}$$

In order to obtain the other Bianchi identity we need to contract $\mathbb{R}^{(4N)}$ with a vielbein to get

$$\mathbb{R}^{a_1a_2\ldots a_{2N}} \wedge e_{a_1} = 0 \tag{13}$$

or equivalently in components,

$$\mathbb{R}^{a_1a_2\ldots a_{2N-1}}_{[a_{2N}b_1b_2\ldots b_{2N}]} = 0. \tag{14}$$

These generalized Bianchi identities trivially reduce to the usual ones for $N = 1$.

In deriving the form of $\mathcal{E}^a_b$ we have made use of a couple of very handy identities. The first one allows us to rewrite a contraction of antisymmetric symbols in terms of antisymmetrized products of δ-functions,

$$\varepsilon^{a_1\ldots a_k c_{k+1}\ldots c_d}\,\varepsilon_{b_1\ldots b_k c_{k+1}\ldots c_d} = -k!(d-k)!\delta^{a_1\ldots a_k}_{b_1\ldots b_k}, \tag{15}$$

where

$$\delta^{a_1a_2\ldots a_n}_{b_1b_2\ldots b_n} = \delta^{a_1}_{[b_1}\delta^{a_2}_{b_2}\ldots\delta^{a_n}_{b_n]} = \delta^{[a_1}_{b_1}\delta^{a_2}_{b_2}\ldots\delta^{a_n]}_{b_n}. \tag{16}$$

Equation (15) is valid for Lorentzian signature, having opposite sign in the Euclidean case. The second useful expression involves contractions of the latter

$$\delta^{aa_1\ldots a_{2N}}_{bb_1\ldots b_{2N}} = \frac{1}{2N+1}\left(\delta^a_b\delta^{a_1\ldots a_{2N}}_{b_1\ldots b_{2N}} - 2N\delta^a_{[b_1}\delta^{a_1\ldots a_{2N}}_{|b|b_2\ldots b_{2N}]}\right). \tag{17}$$

This formulas will prove extremely useful to carry out the computations contained in the rest of this note.

3 Tensor identities and kinematicity

Three dimensional Einstein gravity is kinematic, it does not have local degrees of freedom. This is due to the fact that in three dimensions the Riemann tensor is completely fixed by the Ricci and vice versa. This can already be seen at the level of the number of independent components, both having six, and we can explicitly write

$$R^{ab}{}_{cd} = 4\delta^{[a}_{[c}R^{b]}{}_{d]} - \delta^{[a}_{[c}\delta^{b]}_{d]}R. \tag{18}$$

Equivalently we can just say that the Weyl tensor identically vanishes in three dimensions, this being also true in the presence of cosmological constant. The equation of motion in vacuum thus fixes completely the form of the Riemann curvature that in turn fixes the metric up to diffeomorphism. The metric itself has to be that of a maximally symmetric space and no local degrees of freedom can propagate.

Higher order analogs of Eq. (18) have been derived in [2]. There, a new set of dimensional dependent identities has been used to write the 4Nth rank tensor $\mathbb{R}$ in terms of its contractions in dimensions $d < 4N$. The way these identities appear is easy to understand. Notice that the expression of $\mathbb{R}^{(4N)}$ in Eq. (2) is antisymmetrized over sets of $2N$ indices. Thus this tensor vanishes for dimensions $d < 2N$ and that is the reason for the corresponding Lovelock term to become trivial. We can now antisymmetrize over bigger sets of indices respecting the symmetry properties of the above tensor, i.e. we may define

$$\mathbb{A}^{b_1b_2\ldots b_{2N}}_{a_1a_2\ldots a_{2N}} = \delta^{b_1b_2\ldots b_{2N}c_1c_2\ldots c_{2N}}_{a_1a_2\ldots a_{2N}d_1d_2\ldots d_{2N}} R^{d_1d_2}{}_{c_1c_2} \ldots R^{d_{2N-1}d_{2N}}{}_{c_{2N-1}c_{2N}} = \delta^{[b_1}_{[a_1}\delta^{b_2}_{a_2} \ldots \delta^{b_{2N}}_{a_{2N}}\mathbb{R}^{c_1\ldots c_{2N}]}_{c_1\cdots c_{2N}]} \tag{19}$$

antisymmetrizing over sets of $4N$ indices. This new tensor, which can be written explicitly in terms of $\mathbb{R}^{(4N)}$ and its contractions, vanishes for dimensions $d < 4N$. Interestingly, we get a way of writing $\mathbb{R}^{(4N)}$ completely in terms of its contractions below that dimensionality. Further, we can define similar tensors reducing the number of free indices on each set,

$$\mathbb{A}^{b_1b_2\ldots b_{2N-J}}_{a_1a_2\ldots a_{2N-J}} = \delta^{[b_1}_{[a_1}\delta^{b_2}_{a_2} \ldots \delta^{b_J}_{a_J}\mathbb{R}^{c_1\ldots c_{2N}]}_{c_1\cdots c_{2N}]}, \tag{20}$$

for all $0 < J < 2N$. This tensor will vanish for $d < 4N - J$ and will allow us to write the Jth contraction of $\mathbb{R}^{(4N)}$ in terms of lower contractions below that threshold dimension. These identities were used by Kastor [2] in order to prove that in all odd $d = 2N + 1$, the 4Nth rank tensor $\mathbb{R}^{(4N)}$ can be written completely in terms of its corresponding Lovelock–Ricci. Therefore whenever the latter vanishes so does the former. This is, however, not the case in the next even $d = 2N + 2$ dimension or higher. Depending on the number of free indices these tensors ${}_{(N)}\mathbb{A}^{(2n)}$ correspond to various structures appearing in Lovelock gravity, for instance the equations of motion themselves, ${}_{(N)}\mathbb{A}^{(2)}$, or the tensor multiplying the linearized Riemann in the linearized equations of motion, ${}_{(N-1)}\mathbb{A}^{(6)}$.

The above construction based on identities looks simple, however, it may become very cumbersome when it comes to derive explicit expressions. In $d = 2N + 1$ dimensions, for example, in writing explicitly the 4Nth rank tensor in terms of the corresponding Lovelock–Ricci, one has to write down the explicit expressions of the $2N-1$ identities available for that dimensionality and combine them to get the desired expression. In the next few paragraphs we will describe a much more elegant and efficient way of deriving these expressions based on the use of the Hodge duality. In fact, we will introduce a single tensor identity, basically $(\star\star\mathbb{R}^{(4N)}\star\star) = \mathbb{R}^{(4N)}$ (see details below), that will imply all of those used by Kastor and has lots of potential applications in the context of Lovelock gravity. In particular, as stated in the introduction, this will lead directly to our Eq. (1).

Our starting point is the basic observation that antisymmetric tensors of ranks n and $(d - n)$ have the same number of independent components. In fact there is a reversible map between the two equivalent representations, namely the Hodge duality, which basically amounts to a contraction with the antisymmetric symbol,

$$(\star T)_{a_{n+1}\ldots a_d} = \frac{1}{n!}\varepsilon_{a_1a_2\ldots a_na_{n+1}\ldots a_d}T^{a_1a_2\ldots a_n}. \tag{21}$$

One important advantage of this transformation is that it is easily reversible. Applying it twice we get, up to sign, the original tensor,

$$(\star\star T)^{a_1\ldots a_n} = \frac{1}{n!(d-n)!}T^{b_1\ldots b_n}\varepsilon_{b_1\ldots b_d}\varepsilon^{b_{n-1}\ldots b_d a_1\ldots a_n} = (-1)^{1+n(d-n)}\, T^{a_1\ldots a_n}. \tag{22}$$

In the present case we are also dealing with antisymmetric sets of indices, the only difference being that the tensor of interest has two such sets instead of just one. Either way we can still apply the dual map to each set separately and get an equivalent $(d - 2N, d - 2N)$-tensor as

$$(\star\mathbb{R}\star)^{b_1b_2\ldots b_{d-2N}}_{a_1a_2\ldots a_{d-2N}} \times = \left(\frac{1}{(2N)!}\right)^2 \varepsilon^{b_1b_2\ldots b_{d-2N}c_1c_2\ldots c_{2N}} \varepsilon_{a_1a_2\ldots a_{d-2N}d_1d_2\ldots d_{2N}}\mathbb{R}^{d_1d_2\ldots d_{2N}}_{c_1c_2\ldots c_{2N}} \tag{23}$$

the new tensor verifying $\star(\star\mathbb{R}^{(4N)}\star)\star = \mathbb{R}^{(4N)}$. For $d < 4N$ the new tensor will be of lower rank as compared to the original $\mathbb{R}^{(4N)}$. In fact $(\star\mathbb{R}\star)^{(2d-4N)}$ will be given by a particular combination of contractions of $\mathbb{R}^{(4N)}$ in that case. Therefore, applying the Hodge star again we recover the original tensor $\mathbb{R}^{(4N)}$, now expressed in terms of its contractions. This is precisely the identity we were looking for. In particular we can recover all of Kastor's identities from this single one.

Notice that Eq. (23) is very similar in form to the Lovelock–Einstein tensor (6). In the critical dimension, $d = 2N + 1$, we actually get

$$(\star\mathbb{R}\star)^b_a = \frac{1}{(2N)!}\mathcal{E}^b{}_a = \frac{1}{(2N)!}\left(2N\mathbb{R}^a{}_b - \delta^a{}_b\mathbb{R}\right) \tag{24}$$

and vice versa,

$$\mathbb{R}^{b_1\ldots b_{2N}}_{a_1\ldots a_{2N}} = \frac{1}{(2N)!}\varepsilon^{b_1\ldots b_{2N+1}}\varepsilon_{a_1\ldots a_{2N+1}}\mathcal{E}^{a_{2N+1}}{}_{b_{2N+1}}, \tag{25}$$

making explicit what we wanted to prove. When the Lovelock–Ricci tensor vanishes (or equivalently $\mathcal{E}^a{}_b$) the whole tensor $\mathbb{R}^{(4N)}$ vanishes as well, along with all its contractions.

In dimensions above the critical one this does not directly apply. In particular, in $d = 2N+2$ dimensions in order for the tensor $\mathbb{R}^{(4N)}$ to be identically zero, not just the corresponding Ricci has to vanish but also the Lovelock–Riemann has to be zero. This can easily be guessed as $(\star\mathbb{R}\star)^{(2d-4N)}$ is a (2,2)-tensor in this case. More explicitly, we may write

$$(\star\mathbb{R}\star)^{ab}_{cd} = \frac{-1}{(2N)!}\Big[2\delta^{[a}_{[c}\delta^{b]}_{d]}\mathbb{R} - 8N\delta^{[a}_{[c}\mathbb{R}^{b]}_{d]} + 2N(2N-1)\mathbb{R}^{ab}_{cd}\Big], \tag{26}$$

and the whole tensor $\mathbb{R}^{(4N)}$ is given in terms of its Lovelock–Riemann and its contractions. The higher we go in dimension, the higher the rank of the contractions involved in the expressions for $(\star\mathbb{R}\star)^{(2d-4N)}$ and $\mathbb{R}^{(4N)}$. In Kastor's approach, this is reflected in the fact that we have less identities to play with.

As a check of our formulas we can compute the rank four Lovelock–Riemann tensor in terms of its double dual tensor in $d = 2N + 2$,

$$\mathbb{R}^{ab}_{cd} = -(2N-2)!\Big[\delta^{[a}_{[c}\delta^{b]}_{d]}(\star\mathbb{R}\star) - 4\delta^{[a}_{[c}(\star\mathbb{R}\star)^{b]}_{d]} + (\star\mathbb{R}\star)^{ab}_{cd}\Big] \tag{27}$$

then, plugging in the explicit expression of $(\star\mathbb{R}\star)^{(2d-4N)}$ and its contractions, we can see that in fact the right hand side yields $\mathbb{R}^{ab}_{cd}$. We can also check that the Lovelock–Einstein tensor can be written as

$$(\star\mathbb{R}\star)^{a}_{c} = (\star\mathbb{R}\star)^{ab}_{cb} = \frac{1}{(2N)!}\mathcal{E}^{a}{}_{c}, \tag{28}$$

a contraction of $(\star\mathbb{R}\star)^{(2d-4N)}$ in $d = 2N + 2$ (and also in higher dimensions). To make again contact with Kastor's approach, the non-vanishing tensor ${}_{(N)}\mathbb{A}^{(2n)}$ with the maximum number of free indices, $n = d - 2N$, is proportional to $(\star\mathbb{R}\star)^{(2d-4N)}$. From this, lower rank $\mathbb{A}$-tensors can be obtained taking traces.

The previous discussion can be trivially modified to include a nonzero cosmological constant, the above equations being still valid in that case. We just have to modify the equation of motion as

$$\mathcal{E}^{a}{}_{b} = \lambda\delta^{a}_{b}, \tag{29}$$

such that, instead of zero, the Lovelock–Ricci tensor is now proportional to the metric. In odd critical dimensions we can again use Eq. (25) and verify that the form of the tensor $\mathbb{R}$ is completely fixed to

$$\mathbb{R}^{a_1\ldots a_{2N}}_{b_1\ldots b_{2N}} = \lambda\delta^{a_1\ldots a_{2N}}_{b_1\ldots b_{2N}}. \tag{30}$$

Analogously to what happens for Einstein gravity we can define a Lovelock–Weyl tensor

$$\mathbb{W}^{ab}_{cd} = \mathbb{R}^{ab}_{cd} - \frac{4}{d-2}\delta^{[a}_{[c}\mathbb{R}^{b]}_{d]} + \frac{2}{(d-1)(d-2)}\delta^{[a}_{[c}\delta^{b]}_{d]}\mathbb{R} \tag{31}$$

that then vanishes in odd $d = 2N + 1$ dimensions whereas it is completely unconstrained in $d = 2N + 2$ or higher. In [6] other higher order analogs of the Weyl tensor have been constructed that vanish for $d < 4N$ [7]. A key feature of the Weyl tensor is that under a conformal transformation it transforms by an overall rescaling, and therefore conformally flat spacetimes have vanishing Weyl curvature. This property is also true for the higher order Weyl tensor of [6,7] but not for our Lovelock–Weyl tensor, even for $d = 2N+2$ dimensions. This can be checked simply by taking the double Hodge dual of Eq. 25 of [7] and realizing that the dual of the second term of the right hand side is not of a pure trace form, i.e. $\delta^{[a}_{[c}\tilde{\Lambda}^{b]}_{d]}$.

To sum up, in any dimension we can always write $\mathbb{R}^{(4N)} = \star(\star\mathbb{R}^{(4N)}\star)\star$ and thus express $\mathbb{R}^{(4N)}$ in terms of $(\star\mathbb{R}\star)^{(2d-4N)}$. For low enough dimension, $d < 4N$, the dual tensor itself will be a combination of contractions of $\mathbb{R}^{(4N)}$ with the same number of free indices. In $d = 2N$, $(\star\mathbb{R}\star)^{(0)} \sim \mathcal{L}$ is just a scalar, which means that $\mathbb{R}^{(4N)}$ can be written solely in terms of the Lovelock scalar, in $d = 2N + 1$ it can be written in terms of the Lovelock–Ricci and its trace, in $d = 2N + 2$ we need to include also the rank four contraction and so on.

The above discussion implies that, for pure Lovelock gravity in the critical dimensionality, the vacuum equation of motion (with or without cosmological constant) $\mathcal{E}^{a}{}_{b} = \lambda\delta^{a}_{b}$ completely fixes $\mathbb{R}^{(4N)}$ and all its contractions. Thus we have proved in a very direct way that pure Lovelock gravity is kinematic in all odd $d = 2N + 1$ dimensions, its Lovelock–Riemann tensor is completely fixed, generalizing the well-known three dimensional property. However, unlike in $d = 3$ this does not fix completely the Riemann curvature, thus our solutions are not locally maximally symmetric spaces and we have in general propagating degrees of freedom. For zero λ we can rephrase this by saying that any solution of pure Lovelock is Lovelock flat even though the Riemann curvature is not necessarily zero. For nonzero λ the Lovelock–Weyl tensor is zero but still the Weyl tensor does not necessarily vanish. The implications of this for the dynamics of pure Lovelock theories are still unclear.

4 Kastor–Dadhich reconciliation

As we have described in the previous section, Kastor's tensors contain all the relevant information from which we can reconstruct action and equation of motion in any pure Lovelock theory. In this way, any other formulation that cannot be obtained from this one would contain more information on the spacetime that does not enter either the action or the corresponding Lovelock–Einstein tensor. In particular, Kastor's 4Nth rank tensor is totally antisymmetric on each set of $2N$ indices and symmetric under exchange of both sets. In addition, the dualization procedure allowed us to write $\mathbb{R}^{(4N)}$ in terms of its contractions in low dimensions, $d < 4N$. This reduces the information contained in $\mathbb{R}^{(4N)}$ to the minimal possible amount still capturing the whole dynamics. Any extra information is thus irrelevant from this point of view. We can rephrase the dynamical information contained in $\mathbb{R}^{(4N)}$ in terms of its contraction of rank $(d-2N, d-2N)$, i.e. the Lovelock–Ricci tensor in the critical $d = 2N+1$, the fourth rank Lovelock–Riemann tensor in $d = 2N + 2$, and so on.

Dadhich [1] proposed a different set of tensors as a suitable tool for describing the dynamics of pure Lovelock gravity.

For it to be equivalent to Kastor's description, it should be possible to write everything in terms of $\mathbb{R}^{(4N)}$. We will see that this is actually not possible and that we will need to add a new tensor structure to the ones used by Dadhich in order to recover Kastor's Lovelock–Riemann. The basic object in Dadhich's formulation is a fourth rank Riemann analog tensor which is a Nth order homogeneous polynomial in the Riemann curvature. It is given by

$$\mathcal{R}_{abcd} = F_{abcd} - \frac{N-1}{N(d-1)(d-2)} F \left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) \tag{32}$$

where

$$\begin{aligned} F^{cd}_{\ \ ab} &= \left(\delta^{mna_1b_1\ldots a_{N-1}b_{N-1}}_{abc_1d_1\ldots c_{N-1}d_{N-1}} R^{c_1d_1}_{\ \ \ \ a_1b_1} \ldots R^{c_{N-1}d_{N-1}}_{\ \ \ \ a_{N-1}b_{N-1}}\right) R^{cd}_{\ \ mn} \\ &= R^{[c_1d_1}_{\ \ \ \ [c_1d_1} \cdots R^{c_{N-1}d_{N-1}]}_{\ \ \ \ c_{N-1}d_{N-1}} R^{cd}_{\ \ ab]}. \end{aligned} \tag{33}$$

Written in this way the tensor F looks very similar to the Riemann contraction of $\mathbb{R}$, except that upper indices are not completely antisymmetrized. The contracted indices can be considered as antisymmetrized as lower indices are, but not the whole set. We shall now extract the difference between the two classes of tensors. Comparing the above F with the fourth rank Lovelock–Riemann in Kastor's formulation,

$$\mathbb{R}^{cd}_{ab} = R^{[c_1d_1}_{\ \ \ \ [c_1d_1} \cdots R^{c_{N-1}d_{N-1}}_{\ \ \ \ c_{N-1}d_{N-1}} R^{cd]}_{\ \ \ ab]}, \tag{34}$$

we can easily see that F is not symmetric under the exchange of both pairs of indices (when all lowered), whereas this contraction of $\mathbb{R}^{(4N)}$ is. By repeatedly using Eq. (17), $\mathbb{R}^{cd}_{ab}$ can be rewritten as

$$\begin{aligned} \mathbb{R}^{cd}_{ab} = \frac{1}{N(2N-1)} &\Big(\left[1 + (N-1)(2N-3)\right] F^{cd}_{\ \ ab} \\ &- 4(N-1)\, R^{ca_1}_{\ \ \ [ab} R^{db_1}_{\ \ \ a_1b_1} \cdots R^{a_{N-1}b_{N-1}}_{\ \ \ \ a_{N-1}b_{N-1}]}\Big). \end{aligned} \tag{35}$$

The difference between $\mathbb{R}^{ab}_{cd}$ and the corresponding F is the second term in the bracket, which is the only other tensor that can be written respecting all the relevant symmetries. Both structures are equal in the trivial case of $N = 1$. Note that Lovelock–Ricci tensors arising from F and $\mathbb{R}$ are the same,

$$\mathbb{R}^{a}_{b} = F^{a}_{\ b}, \tag{36}$$

but this does not mean that the extra contribution is traceless. Instead we can write

$$R^{ca_1}_{\ \ \ [ab} R^{bb_1}_{\ \ \ a_1b_1} \ldots R^{a_{N-1}b_{N-1}}_{\ \ \ \ a_{N-1}b_{N-1}]} = -\mathbb{R}^{cb}_{ab} = -\mathbb{R}^{c}_{a}, \tag{37}$$

which clearly leads to (36). The contractions of the other fourth rank tensor $\mathcal{R}_{abcd}$ are different though because of the scalar piece in (32),

$$\mathcal{R}^{a}_{\ b} = \mathbb{R}^{a}_{\ b} - \frac{N-1}{N(d-2)} \mathbb{R}\delta^{a}_{\ b}\,, \tag{38}$$

giving a different normalization to the corresponding scalar,

$$\mathcal{R} = \frac{d-2N}{N(d-2)} \mathbb{R}. \tag{39}$$

This difference pops up also in the corresponding Lovelock–Einstein tensors that have different normalizations for their Ricci and scalar pieces,

$$\mathcal{E}^{a}_{\ b} = 2N \left(\mathcal{R}^{a}_{\ b} - \frac{1}{2}\mathcal{R}\delta^{a}_{\ b}\right) = 2N\mathbb{R}^{a}_{\ b} - \mathbb{R}\delta^{a}_{\ b} \tag{40}$$

and upon contraction we get

$$\mathcal{E}^{a}_{\ a} = -N(d-2)\mathcal{R} = -(d-2N)\mathbb{R}. \tag{41}$$

Kastor's parametrisation makes explicit the fact that in $d = 2N$ the Lovelock action (or $\mathbb{R}$) is a topological density (not necessarily zero) and its variation is therefore zero, $\mathcal{E}^{a}_{\ a} = 0$. In Dadhich's case, this fact gets obscured by the extra $(d-2N)$ factor in the normalization of $\mathcal{R}$, which vanishes as well for $d = 2N$.

4.1 Pure Gauss–Bonnet check

As a check of the above expressions we will analyze the $N = 2$ case of pure Gauss–Bonnet (GB) gravity for which we may compute

$$\mathbb{R}^{cd}_{\ \ ab} = \frac{1}{3}\left(F^{cd}_{\ \ ab} - 2R^{ca_1}_{\ \ \ [ab} R^{db_1}_{\ \ \ a_1b_1]}\right) \tag{42}$$

and

$$F^{cd}_{\ \ ab} = \frac{1}{6}\left(R R^{cd}_{\ \ ab} + 4R^{cd}_{\ \ k[a} R^{k}_{\ b]} + R^{cd}_{\ \ kl} R^{kl}_{\ \ ab}\right). \tag{43}$$

Let us also write the second term explicitly as

$$\begin{aligned} 2R^{ca_1}_{\ \ \ [ab} R^{db_1}_{\ \ \ a_1b_1]} = \frac{2}{3}\Big(& R^{[c|k|}_{\ \ \ \ ab} R^{d]}_{\ \ k} - R^{[c}_{\ \ [a} R^{d]}_{\ \ b]} \\ &+ R^{k[c}_{\ \ \ l[a} R^{d]l}_{\ \ \ b]k}\Big). \end{aligned} \tag{44}$$

As we have seen in previous sections, in $d = 2N + 1$ dimensions ($d = 5$ in this case) the higher rank tensor $\mathbb{R}^{(4N)}$ vanishes for vacuum solutions of pure Lovelock gravity with zero cosmological constant. This tensor is the double Hodge dual of the Lovelock–Einstein tensor (see Eq. (25)). It vanishes whenever $\mathcal{E}^{a}_{\ b} = 0$ and so does any of its contractions, namely Kastor's Lovelock–Riemann tensor. Moreover, we know that both formulations agree at the Ricci tensor level. Thus, whenever the Lovelock–Ricci vanishes in $d = 2N+1$ dimensions, F and the second term in Eq. (42) exactly cancel out each other. For spherically symmetric solutions it turns out that F and the extra term vanish separately in $d = 2N+1$, and that is the basis for Dadhich's conjecture [4,5] for the kinematicity of pure Lovelock gravity in odd critical dimensions. As an example for which both contributions to the Lovelock–Riemann are separately nonzero we can consider a pure GB Kasner vacuum in five dimensions [8]. One such metric is, for instance,

$$ds^2 = -dt^2 + t^{2p_1}dx_1^2 + t^{2p_2}dx_2^2 + t^{2(3-p_1-p_2)}dx_3^2 + dx_4^2, \tag{45}$$

where $p_{1,2}$ are arbitrary constants. In this case the nonzero F is canceled by the extra term to give $\mathbb{R}^{ab}_{cd} = \mathbb{R}^{(4N)} = 0$. In fact, in the context of Kasner type metrics, the different properties displayed by Kastor's and Dadhich's analog Riemann tensors allow for a characterization of this family of solutions in the context of pure Lovelock, splitting them into different classes or isotropy types [8].

5 Discussion

It is interesting that there are two parallel but distinct definitions of a higher order Lovelock–Riemann tensor leading to the same equation of motion. That is, the two constructions describe precisely the same gravitational dynamics, even though there is a nontrivial difference at a kinematic level. This became apparent when a pure GB Kasner vacuum solution was found in five dimensions [8] for which Dadhich's Lovelock–Riemann tensor did not vanish. Besides this was in contradiction with a previous kinematicity conjecture [4,5]. Dadhich's tensor did indeed vanish for spherically symmetric pure GB vacuum solutions [4], and based on that it was proposed that any pure Lovelock vacuum in all odd $d = 2N + 1$ dimensions would be Lovelock flat. A precise realization of this kinematicity property was nonetheless provided by an alternative formulation put forward by Kastor [2]. Kastor's Lovelock–Riemann tensor does indeed vanish for the Kasner vacuum solution in question. In fact this tensor vanishes in all odd critical dimensions for any vacuum solution of pure Lovelock gravity, as we discussed. It became thus pertinent to reconcile both formulations, and this is therefore the main motivation of this investigation. Both Lovelock–Riemann tensors differ in a piece that, remarkably, vanishes in the spherically symmetric case and that is how it was not noticed at first [4].

We have also revisited the kinematicity property and rederived it in a much more direct way by making extensive use of the properties of the Hodge dual map. This is a unique and universal distinguishing feature of pure Lovelock gravity in all odd $d = 2N + 1$ dimensions which is shared by no other theory. It stems from the fact that, for that critical dimensionality, we can write the higher rank tensor $\mathbb{R}^{(4N)}$ as the double Hodge dual of the corresponding Lovelock–Einstein tensor. Thus $\mathbb{R}^{(4N)}$ is completely fixed by the equations of motion. It is important to note that this kinematicity is relative to the Lovelock–Riemann tensor and not to the Riemann curvature. That is, the Lovelock–Riemann tensor vanishes in $d = 2N + 1$ whenever the corresponding Lovelock–Ricci vanishes, but the Riemann curvature may be nonzero. In turn, in $d = 2N + 2$, the Lovelock–Weyl tensor is a priori unrelated to the equation of motion. This is in complete analogy with the behavior of Einstein gravity in three and four dimensions and it can also be generalized in the presence of a nonzero cosmological constant. Pure Lovelock gravity thus unravels a new universal feature of gravity in higher dimensions.

The fact that Dadhich's Riemann analog vanishes, for instance, for spherically symmetric pure Lovelock solutions seems to indicate that this tensor being zero might identify special properties of particular classes of solutions. This intuition has been strengthen by the analysis of pure Lovelock Kasner metrics [8]. These family of vacuum solutions can be divided into several classes and turns out that we can use a set of fourth rank tensors, R_{abcd}, $\mathcal{R}_{abcd}$ and $\mathbb{R}_{abcd}$, to characterize them. Given a particular solution in this family, one may identify which class it belongs to by analyzing which tensors among those in the set vanish.

Pure Lovelock theories posses many interesting properties. Besides the ones already mentioned, thermodynamic parameters of pure Lovelock static black holes bear a universal relation to the horizon radius [9] and bound orbits exist in all even $d = 2N + 2$ dimensions [10] in these spacetimes. It should be pointed out that for Einstein gravity bound orbits around a static black hole exist in 4 dimensions only. All this strongly suggests that pure Lovelock equation is the right equation to describe the gravitational dynamics in higher $d = 2N + 1,\ 2N + 2$ dimensions [5] such that we preserve many interesting properties that Einstein gravity has only for three or four dimensions.

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