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ON m,n-BALANCED PROJECTIVE AND m,n-TOTALLY PROJECTIVE PRIMARY ABELIAN GROUPS

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Dedicated to the memory of Ronald J. Nunke (March 9, 1926-April 3, 2011), whose seminal work on the homological aspects of abelian group theory continues to inspire the authors

Abstract. If m and n are non-negative integers, then three new classes of abelian p-groups are defined and studied: the m,n-*simply presented* groups, the m,n-*balanced projective* groups and the m,n-*totally projective* groups. These notions combine and generalize both the theories of simply presented groups and $p^{\omega+n}$-projective groups. If $m,n=0$, these all agree with the class of totally projective groups, but when $m+n\geq 1$, they also include the $p^{\omega+m+n}$-projective groups. These classes are related to the (strongly) n-simply presented and (strongly) n-balanced projective groups considered in [15] and the n-summable groups considered in [2]. The groups in these classes whose lengths are less than ω^2 are characterized, and if in addition we have $n=0$, they are determined by isometries of their p^m-socles.

0. Introduction, terminology and definitions

By the term "group", we will mean an abelian p-group, where p is a prime fixed for the duration of the paper. In addition, throughout, the letters m and n will denote non-negative integers and we will set $k=m+n$. Our terminology and notation will be based upon [3] and [6]. For example, if α is an ordinal, then a group G will be said to be p^α-*projective* if $p^\alpha \mathrm{Ext}(G,X)=\{0\}$ for all groups X.

The *totally projective* groups have a central position in the study of abelian p-groups (see Chapter XII of [3] or Chapter VI of [6]). One reason for their importance is the number of different ways they can be characterized (see Theorems 81.9, 82.3 and 83.5 of [3]). It is worth pointing out that, unlike the treatment in [3], we do not require a totally projective group to be reduced. A

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totally projective group of length not exceeding ω_1 is a direct sum of countable groups (hereafter abbreviated as a *dsc group*; see [3], Theorem 82.4).

We will assume some familiarity with the theory of *valuated groups* and *valuated vector spaces* (see, for example, [20] and [4]). So if V is a group, then a *valuation* on V is a function $|\ |_V : V \to \mathcal{O}_\infty$ (where $\mathcal{O}_\infty$ is the class of all ordinals plus the symbol ∞) such that for all $x, y \in V$, $|x \pm y|_V \geq \min\{|x|_V, |y|_V\}$ and $|px|_V > |x|_V$. If V is a subgroup of G, then the height function on G, which we also denote by $|\ |_G$, restricts to a valuation on V. Of course, a valuated group is a valuated vector space if it is p-bounded, so that the socle of a group will be a valuated vector space.

A group will be said to be Σ-*cyclic* if and only if it is isomorphic to a direct sum of cyclic groups. The group G is $p^{\omega+n}$-*projective* if and only if there is a subgroup $P \subseteq G[p^n]$ such that G/P is Σ-cyclic ([17]). So a group is p^ω-projective if and only if it is Σ-cyclic. If G_1 and G_2 are $p^{\omega+n}$-projectives, then G_1 and G_2 are isomorphic if and only if $G_1[p^n]$ and $G_2[p^n]$ are isometric as valuated groups (i.e., there exists an isomorphism that preserves the height functions on the two subgroups; see [5]).

This paper is a continuation of a study, initiated in [15], of ways to combine these two branches of knowledge. In that paper a group G was defined to be n-*simply presented* if it has a subgroup $P \subseteq G[p^n]$ such that G/P is simply presented, and *strongly n-simply presented* if this P can be chosen to be a nice subgroup. A short exact sequence

$$0 \to X \to Y \to G \to 0$$

is defined to be n-*balanced exact* if it represents an element of $p^n\mathrm{Bext}(G, X)$. This sequence is *strongly n-balanced exact* if either $n = 0$ and it is just plain balanced, or n is positive and it induces a short exact sequence

$$0 \to X[p^n] \to Y[p^n] \to G[p^n] \to 0$$

which splits in the category of valuated groups; we denote the collection of such sequences by $\mathrm{V}_n\mathrm{ext}(G, X)$. It was shown that there are enough (strongly) n-balanced projectives and that a group satisfies these conditions if and only if it is a summand of a group that is (strongly) n-simply presented. It was also verified that if G has length strictly less than ω^2, then G is (strongly) n-balanced projective if and only if it is (strongly) n-simply presented.

In the first section we unify and generalize these two lines of inquiry. We say a group G is m, n-*simply presented* if there is a subgroup P of $G[p^n]$ such that $H \stackrel{\text{def}}{=} G/P$ is strongly m-simply presented. We call P an m, n-*simply representing* subgroup of G. Observe that "$0, n$-simply presented" = "n-simply presented" and "$n, 0$-simply presented" = "strongly n-simply presented". It is easy to see that if $m > 0$ and G is m, n-simply presented, then it is $m-1, n+1$-simply presented (Proposition 1.1). It follows that if G is strongly $k = m+n$-simply presented, then it is m, n-simply presented, and if it is m, n-simply presented, it is k-simply presented. In other words, being m, n-simply presented is an intermediate condition between being k-simply presented and strongly k-simply presented.

We call a short exact sequence m,n*-balanced exact* if it represents an element of $p^n \mathrm{V}_m \mathrm{ext}(G,X)$. It follows that a group is projective with respect to the m,n-balanced exact sequences if and only if it is a summand of a group that is m,n-simply presented, and that there are enough m,n-balanced projectives (Theorem 1.4).

If λ is an ordinal and G is a group, we will write G_λ for $G/p^\lambda G$; in particular, it is readily checked that if $\lambda = \beta + \gamma$, then $(G_\lambda)_\beta \cong G_\beta$ and $p^\beta(G_\lambda) = (p^\beta G)/(p^\lambda G) = (p^\beta G)_\gamma$. We will say that the groups in some class $\mathcal{C}$ have the λ*-Nunke property* if G is in $\mathcal{C}$ if and only if both $p^\lambda G$ and G_λ are in $\mathcal{C}$. A classical result (due to Nunke, [17]) states that for all λ, the totally projective groups have the λ-Nunke property. Of central importance to the investigations of [15] were two generalizations of this result: For any ordinal λ, the strongly n-simply presented groups have the $\lambda + n$-Nunke property ([15], Theorem 3.4), and the n-simply presented groups have the λ-Nunke property ([15], Theorem 4.4). Parallel results hold for (strongly) n-balanced projective groups. We generalize this to the current context by showing that for any ordinal λ, the m,n-simply presented groups have the $\lambda + k = \lambda + m + n$-Nunke property (Theorem 1.8). Even more satisfactorily, we show that the m,n-balanced projective groups have the $\lambda + m$-Nunke property (Theorem 1.12).

In the second section we generalize Nunke's homological definition of total projectivity. We say the group G is n*-totally projective* if G_λ is $p^{\lambda+n}$-projective for every ordinal λ, and *strongly* n*-totally projective* if $G_{\lambda+n}$ is $p^{\lambda+n}$-projective for every ordinal λ. Note that if $n = 0$, these two definitions reduce to the usual notion of total projectivity.

More generally, we will say the group G is m,n*-totally projective* if for every ordinal λ, $G_{\lambda+m}$ is $p^{\lambda+k}$-projective. A standard argument shows that if this holds for all limit ordinals λ, then it holds for all other ordinals, as well. As before, if $m \geq 1$, then "m,n-totally projective" implies "$m-1, n+1$-totally projective." In particular, this means that "strongly k-totally projective" = "$m+n, 0$-totally projective" implies "m,n-totally projective" implies "$0, m+n$-totally projective" = "k-totally projective"; so again, m,n-total projectivity is an intermediate condition between being strongly k-totally projective and k-totally projective.

It is fairly easy to verify that if G is m,n-balanced projective, then it is m,n-totally projective (Theorem 2.4). In order to discuss the converse, we need consider whether, for an ordinal λ, the m,n-totally projective groups have the $\lambda + m$-Nunke property. It is straightforward to show that if G is m,n-totally projective and λ is any ordinal, then $p^\lambda G$ and G_λ must share this property (Theorem 2.5); so, in particular, this is also true for ordinals of the form $\lambda+m$. The converse is more complicated; we do show that if $G_{\lambda+m}$ is m,n-totally projective and $p^{\lambda+m}G$ is m,n*-balanced* projective, then G is m,n-totally projective (Theorem 2.6). It is also easy to verify that the strongly n-totally projective groups actually do have the $\lambda + n$-Nunke property (Corollary 2.8).

In the third section we apply these notions to the class of groups G whose lengths are strictly less than ω^2. In particular, we show that in this case all these definitions agree, so that G is m, n-simply presented if and only if it is m, n-balanced projective if and only if it is m, n-totally projective (Theorem 3.2); these conditions are also shown to be equivalent to requiring that for all ordinals $\lambda < \omega^2$, $(p^\lambda G)_{\omega+m} = (p^\lambda G)/(p^{\lambda+\omega+m} G)$ is $p^{\omega+k}$-projective. In addition, if G and G' are strongly n-balanced projective groups of length strictly less than ω^2, then G and G' are isomorphic if and only if $G[p^n]$ and $G'[p^n]$ are isometric (Theorem 3.5).

In the fourth section we relate these notions to the following definition from [2]: The group G is said to be *n-summable* if $G[p^n]$ (with the usual valuation) splits into the valuated direct sum of countable valuated groups. Clearly, a dsc group is both strongly n-totally projective and n-summable. This suggests the question of whether the converse holds as well. It is shown that G is a dsc group if and only if it is strongly n-balanced projective and n-summable (Corollary 4.2). In addition, it is established that if G has countable length, then G is a dsc group if and only if it is strongly n-totally projective and n-summable (Theorem 4.6). However, the latter result does not generalize to groups of length ω_1 (Example 4.9); i.e., there are n-summable groups of length ω_1 that are strongly n-totally projective, but not strongly n-balanced projective.

We close the paper with a list of open problems.

1. m, n-simply presented groups

In this section we generalize the results of [15]. Since the proofs will often parallel those found in that paper, we will on occasion simply point out how to make the necessary alterations. We start with the following easy observation.

Proposition 1.1. *If $m > 0$ and G is an m, n-simply presented group, then it is $m-1, n+1$-simply presented.*

Proof. Suppose P is an m, n-representing subgroup of G, so that $p^n P = \{0\}$ and $A \stackrel{\text{def}}{=} G/P$ is strongly m-simply presented. It follows that there is a p^m-bounded nice subgroup N of A such that A/N is simply presented. If P' is the subgroup of G determined by the equation $P'/P = N[p]$, then P' is p^{n+1}-bounded. In addition, $N' \stackrel{\text{def}}{=} N/N[p]$ is a p^{m-1}-bounded nice subgroup of $A' \stackrel{\text{def}}{=} A/N[p] \cong (G/P)/(P'/P) \cong G/P'$, and $A'/N' = (A/N[p])/(N/N[p]) \cong A/N$ is simply presented. Therefore, P' is an $m-1, n+1$-simply representing subgroup of G, as required. $\square$

So if G is strongly $k = m+n$-simply presented, then it is m,n-simply presented; and if G is m,n-simply presented, then it is k-simply presented. Our next result characterizes these classes for $p^{\omega+m}$-bounded groups.

Proposition 1.2. *A $p^{\omega+m}$-bounded group G is m,n-simply presented if and only if it is $p^{\omega+k}$-projective.*

Proof. Assume G is m,n-simply presented and $p^{\omega+m}$-bounded. Let P be a m,n-representing subgroup of G; so $P \subseteq G[p^n]$ and $A \stackrel{\text{def}}{=} G/P$ is strongly m-simply presented. If P' is the subgroup of G defined by the equation $P'/P = p^{\omega+m}A$, then it follows that $P' \subseteq G[p^n]$, and by ([15], Theorem 3.4(a)), $G/P' \cong A_{\omega+m}$ is strongly m-simply presented. Therefore, P' is also an m,n-simply representing subgroup of G. Replacing P by P', we may assume $p^{\omega+m}A = \{0\}$.

By ([15], Proposition 2.5), a $p^{\omega+m}$-bounded group which is strongly m-simply presented, such as A, is $p^{\omega+m}$-projective. This easily implies that G is $p^{\omega+k}$-projective (see, for example, Lemma 2.1(c') below).

Conversely, if G is $p^{\omega+k}$-projective, then again by ([15], Proposition 2.5), G is strongly k-simply presented, and hence m,n-simply presented. □

We continue with another straightforward observation.

Lemma 1.3. *If A' is a subgroup of A such that A/A' is bounded, then A is m,n-simply presented if and only if A' is m,n-simply presented.*

Proof. Suppose first that $n = 0$. By ([15], Theorem 3.4), A (and similarly, A') is strongly m-simply presented if and only if $p^{\omega+m}A$ and $A_{\omega+m}$ are strongly m-simply presented. Since $p^{\omega+m}A = p^{\omega+m}A'$, $p^{\omega+m}A$ is strongly m-simply presented if and only if $p^{\omega+m}A'$ is. And since $A'_{\omega+m}$ embeds as a subgroup of $A_{\omega+m}$ with a bounded cokernel, $A_{\omega+m}$ is strongly m-simply presented if and only if $A_{\omega+m}$ is $p^{\omega+m}$-projective if and only if $A'_{\omega+m}$ is $p^{\omega+m}$-projective if and only if $A'_{\omega+m}$ strongly m-simply presented.

Using the first part of the proof, any m,n-simply representing subgroup $P' \subseteq A'$ can easily seen to be an m,n-simply representing subgroup of A. And conversely, if P is an m,n-simply representing subgroup of A, then $P' \stackrel{\text{def}}{=} P \cap A'$ will be an m,n-simply representing subgroup of A'. □

In particular, the group A is strongly m-simply presented if and only if p^nA has this property. A standard argument then shows that G is m,n-simply presented if and only if there is a strongly m-simply presented group A with a subgroup $Q \subseteq A[p^n]$ such that $G \cong A/Q$. This leads us to a characterization of m,n-balanced projectives. For $m = 0$, it generalizes ([15], Theorem 2.1); and for $n = 0$, it generalizes ([15], Theorem 2.4).

Theorem 1.4. *A group is m,n-balanced projective if and only if it is a summand of a group that is m,n-simply presented. There are enough m,n-balanced projectives.*

Proof. The proof of ([15], Theorem 2.1) was based upon two facts about simply presented = balanced projective groups: (1) there are enough balanced projectives, and (2) if A' is a subgroup of A such that A/A' is bounded, then A is simply presented if and only if A' is simply presented. Since, by Lemma 1.3, both of these statements are equally true when the condition "simply presented" is replaced by "strongly m-simply presented", it follows that the same proof translates over with essentially no changes. $\square$

The following, then, generalizes ([15], Propositions 2.2 and 2.5).

Corollary 1.5. *A $p^{\omega+m}$-bounded group G is m,n-simply presented if and only if it is m,n-balanced projective if and only if it is $p^{\omega+k}$-projective.*

Proof. This follows from Proposition 1.2 and Theorem 1.4, since a summand of a $p^{\omega+m}$-bounded $p^{\omega+k}$-projective group will retain those properties. $\square$

This brings us to another technical observation.

Lemma 1.6. (a) *If A is a strongly m-simply presented group, $p^\lambda A$ is bounded and Z is a subgroup of $p^\lambda A$, then $A' \stackrel{\text{def}}{=} A/Z$ is strongly m-simply presented.*

(b) *If A is a group, $p^{\lambda+m}A$ is bounded, Z is a subgroup of $p^{\lambda+m}A$ and $A' \stackrel{\text{def}}{=} A/Z$ is strongly m-simply presented, then A is also strongly m-simply presented.*

Proof. (a) If Q is a nice p^m-bounded subgroup of A such that A/Q is simply presented, then $Q' = [Q+Z]/Z$ can easily be seen to be a nice p^m-bounded subgroup of A'. In addition, $A'/Q' \cong A/[Q+Z] \cong (A/Q)/([Q+Z]/Q)$, $p^\lambda(A/Q)$ is bounded and $[Q+Z]/Q \subseteq p^\lambda(A/Q)$ implies that A'/Q' is simply presented, as required.

(b) Note that $A_{\lambda+m} \cong A'_{\lambda+m}$ is strongly m-simply presented and $p^{\lambda+m}A$ is bounded, and hence strongly m-simply presented. The result, therefore, follows from ([15], Theorem 3.4(b)). $\square$

The following result and its proof are parallel to ([15], Theorem 3.4(a) and Proposition 3.5(a)); we will therefore pass quickly over a number of details.

Proposition 1.7. *Suppose G is a group and λ is an ordinal. If G is m,n-simply presented or m,n-balanced projective, then $p^\lambda G$ and $G_\lambda = G/p^\lambda G$ share that property.*

Proof. If we can verify this when G is m,n-simply presented, then it immediately follows when it is m,n-balanced projective. So suppose P is an m,n-simply representing subgroup of G and $A \stackrel{\text{def}}{=} G/P$. By ([15], Lemma 3.1(b)), there is an exact sequence

$$0 \to p^{\lambda+n}G/(P \cap p^{\lambda+n}G) \to p^{\lambda+n}A \to B_1 \to 0,$$

where B_1 is bounded. Since A is strongly m-simply presented, so is $p^{\lambda+n}A$. And since B_1 is bounded, by Lemma 1.3, it follows that $p^{\lambda+n}G/(P \cap p^{\lambda+n}G)$ is strongly m-simply presented. Since $p^{\lambda+n}G/(P\cap p^{\lambda+n}G)$ embeds in $p^\lambda G/(P\cap p^\lambda G)$ with a bounded cokernel, it again follows from Lemma 1.3 that $p^\lambda G/(P\cap p^\lambda G)$ is strongly m-simply presented. Since $P\cap p^\lambda G$ is p^n-bounded, we can conclude that $p^\lambda G$ is m,n-simply presented.

We next turn to $G/p^\lambda G$. By ([15], Lemma 3.1(c)), there is a short exact sequence

$$0\to B_2\to A_{\lambda+n}\to G/[p^\lambda G+P]\to 0,$$

where $B_2\subseteq p^\lambda(A_{\lambda+n})$ is bounded. Since A is strongly m-simply presented, so is $A_{\lambda+n}$. And by Lemma 1.6(a), $G/[p^\lambda G+P]$ is strongly m-simply presented. Therefore,

$$G_\lambda/([p^\lambda G+P]/p^\lambda G)\cong G/[p^\lambda G+P]$$

is also strongly m-simply presented. And since $[p^\lambda G+P]/p^\lambda G$ is a p^n-bounded subgroup of G_λ, we can conclude that G_λ is m,n-simply presented, completing the proof. □

We now consider the converse to Proposition 1.7. The following result generalizes ([15], Theorem 3.4(b)), and its proof closely parallels that earlier argument. In fact, it can be thought of as what is obtained if λ is replaced by $\lambda+m$. We therefore again omit a number of details.

Theorem 1.8. *If λ is an ordinal, then the m,n-simply presented groups have the $\lambda+k$-Nunke property.*

Proof. Half the result is a direct consequence of Proposition 1.7. Therefore, suppose that P_1 is a subgroup of G containing $p^{\lambda+k}G$ for which $P_1/p^{\lambda+k}G$ is an m,n-simply representing subgroup of $G_{\lambda+k}$. Let Y be a maximal p^n-bounded summand of $p^{\lambda+m}G$, so that there is a decomposition $p^{\lambda+m}G=X\oplus Y$. Let H be a $p^{\lambda+k}$-high subgroup of G containing Y (i.e., H is maximal with respect to intersecting $p^{\lambda+k}G$ trivially).

It follows as in [15] that $G_{\lambda+k}[p^n]=(X\oplus H[p^n])/p^{\lambda+k}G$, so that $P_1\subseteq X\oplus H[p^n]$. Again as in [15], we let

$$P_2=(X+P_1)\cap H[p^n]\subseteq G[p^n].$$

It follows that

$$X+P_1=X+[(X+P_1)\cap H[p^n]]=X+P_2.$$

We can therefore conclude that $p^{\lambda+m}G+P_1=p^{\lambda+m}G+P_2$.

Next, if P_3 is an m,n-simply representing subgroup of $p^{\lambda+k}G$, then we let $P=P_2+P_3$, so that $P\subseteq G[p^n]$. Let $A=G/P$, which we want to show is strongly m-simply presented. Using ([15], Lemma 3.1(b); with λ replaced by $\lambda+m$), there is a short exact sequence

$$0\to p^{\lambda+k}A\to p^{\lambda+k}G/P_3\to B_1\to 0,$$

where B_1 is bounded. By Lemma 1.3, this implies that $p^{\lambda+k}A$ is strongly m-simply presented.

Since $p^{\lambda+m}(G/P_1)$ is bounded, applying Lemma 1.6(a) to G/P_1, we can deduce that

$$G/[p^{\lambda+m}G + P] = G/[p^{\lambda+m}G + P_1] \cong (G/P_1)/([p^{\lambda+m}G + P_1]/P_1)$$

is strongly m-simply presented. Using ([15], Lemma 3.1(c); again with λ replaced by $\lambda + m$), there is another exact sequence

$$0 \to B_2 \to A_{\lambda+k} \to G/[p^{\lambda+m}G + P] \to 0,$$

where $B_2 \subseteq p^{\lambda+m}A_{\lambda+k}$. Therefore, by Lemma 1.6(b), $A_{\lambda+k}$ will also be strongly m-simply presented.

Finally, since $\lambda + k = (\lambda + n) + m$, by ([15], Theorem 3.4(b)), $A = G/P$ is strongly m-simply presented, as desired. □

The last result has the following consequence, which is proven exactly as in ([15], Proposition 3.5).

Corollary 1.9. *If λ is an ordinal, then the m, n-balanced projective groups have the $\lambda + k$-Nunke property.*

Note that G_λ is totally projective if and only if $G_{\lambda+k}$ is totally projective, and $p^\lambda G$ is m, n-simply presented or m, n-balanced projective if and only if $p^{\lambda+k}G$ has the corresponding property. The following, then, is a direct consequence of Theorem 1.8 and Corollary 1.9.

Corollary 1.10. *If λ is an ordinal, G is a group and G_λ is totally projective, then G is m, n-simply presented or m, n-balanced projective if and only if $p^\lambda G$ shares that property.*

We want to improve on Corollary 1.9 by showing that the m, n-balanced projective groups have the $\lambda + m$-Nunke property; i.e., we want to reduce from $k = m + n$ to m. The next result is the key step in this reduction. If G is a group, α is an ordinal and $j < \omega$, let $G[p^j_\alpha] = \{x \in G : p^j x \in p^\alpha G\}$; note that $G[p^j_\alpha]/p^\alpha G = (G_\alpha)[p^j]$.

Lemma 1.11. *If λ is an ordinal, G is a group, $p^{\lambda+m}G$ is bounded and $G_{\lambda+m}$ is m, n-balanced projective, then G is m, n-balanced projective.*

Proof. The result is easily checked when λ is finite, so assume $\lambda \geq \omega$. Let V be the valuated group $G[p^m_{\lambda+m}] = G[p^m] + p^\lambda G$, with the height valuation from G, and T be a group containing V as a nice subgroups such that the valuation on V also agrees with the height function on T and T/V is simply presented of length λ (see [20] for this standard construction).

Claim 1: T is strongly m-simply presented.

Let M be a $p^{\lambda+m}$-high subgroup of G. There is a decomposition $p^\lambda G = (p^\lambda M) \oplus X$ which leads to a valuated decomposition $V = M[p^m] \oplus X$. Because X is bounded, $X[p^m]$ is nice in X (as a valuated group), so that $V[p^m] = M[p^m] \oplus X[p^m]$ is nice in $V = M[p^m] \oplus X$. Since V is nice in T, this implies that $V[p^m]$ is nice in T (because niceness is transitive for valuated groups even though it is not transitive for non-valuated groups). In addition, $V/V[p^m] \cong X/X[p^m]$ is a bounded subgroup of $p^\lambda(T/V[p^m])$. Since $(T/V[p^m])/(V/V[p^m]) \cong T/V$ is simply presented of length λ, it follows that $V/V[p^m] = p^\lambda(T/V[p^m])$ and $T/V[p^m]$ is simply presented. Therefore, T is strongly m-simply presented, establishing the claim.

The identity map $V \to G[p^m_{\lambda+m}]$ extends to a homomorphism $\phi : T \to G$; denote the kernel of ϕ by K. If $m > 0$, it is easy to check that ϕ must be surjective. If $m = 0$, then $V = p^\lambda G$, and if necessary, we can replace T by a direct sum, $T \oplus X$, where X is a totally projective group of length λ, and extend ϕ to this larger group so that

$$0 \to K \to T \to G \to 0$$

is balanced exact.

There is a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K[p^n] & = & K[p^n] & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & K & \longrightarrow & T & \longrightarrow & G & \to & 0 \\
 & & \downarrow{\scriptstyle p^n} & & \downarrow{\scriptstyle \gamma} & & \| & & \\
0 & \to & K & \longrightarrow & H & \longrightarrow & G & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & K/p^nK & = & K/p^nK & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Let E denote the upper short exact row, so that the lower one is p^nE. We will now break the argument into two parts.

Claim 2: H is m,n-simply presented.

First, T is strongly m-simply presented. This implies that $H' \stackrel{\text{def}}{=} \gamma(T)$ is m,n-simply presented. Since H/H' is bounded, by Lemma 1.3, we can conclude that H is m,n-simply presented.

The proof therefore reduces to the next statement.

Claim 3: p^nE is splitting exact.

Note that $V(\lambda+m)$ maps isometrically onto $p^{\lambda+m}G$. This induces the top row of another commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & K & \longrightarrow & T/V(\lambda+m) & \longrightarrow & G_{\lambda+m} & \to & 0 \\
 & & \downarrow{\scriptstyle p^n} & & \downarrow & & \| & & \\
0 & \to & K & \longrightarrow & H' & \longrightarrow & G_{\lambda+m} & \to & 0
\end{array}$$

where the bottom row is just a push-out. Let E' be the upper row of this, so that p^nE' is its lower row. If $m > 0$, then since there is an isometry of $V/V(\lambda+m)$ with $G_{\lambda+m}[p^m] = G[p^m_{\lambda+m}]/p^{\lambda+m}G$, we can conclude that E' is strongly m-balanced; in other words, $E' \in \mathrm{V}_m\mathrm{ext}(G_{\lambda+m}, K)$. If $m = 0$, the fact that E is balanced also implies that $E' \in \mathrm{Bext}(G_\lambda, K) = \mathrm{V}_m\mathrm{ext}(G_\lambda, K)$. Therefore, in either case, $p^n E' \in p^n \mathrm{V}_m\mathrm{ext}(G_{\lambda+m}, K)$ is m, n-balanced. Since $G_{\lambda+m}$ is assumed to be m, n-balanced projective, we can infer that $p^n E'$ splits.

If $\pi : G \to G_{\lambda+m}$ is the canonical surjection, then $E = \pi^*(E')$, where $\pi^* : \mathrm{V}_m\mathrm{ext}(G_{\lambda+m}, K) \to \mathrm{V}_m\mathrm{ext}(G, K)$ is the usual functorial homomorphism. Therefore, $p^n E = p^n \pi^*(E') = \pi^*(p^n E') = \pi^*(0) = 0$, so that $p^n E$ splits, as required. □

This leads to the following very satisfactory result.

Theorem 1.12. *If λ is an ordinal, then the m, n-balanced projectives have the $\lambda + m$-Nunke property.*

Proof. One implication is a consequence of Proposition 1.7, so assume $G_{\lambda+m}$ and $p^{\lambda+m}G$ are m, n-balanced projective. Clearly, if $p^{\lambda+m}G$ is m, n-balanced projective, then the same holds for $p^{\lambda+k}G$. Next, $(G_{\lambda+k})_{\lambda+m} \cong G_{\lambda+m}$ is m, n-balanced projective; and therefore by Lemma 1.11, $G_{\lambda+k}$ is m, n-balanced projective. So by Corollary 1.9, G is m, n-balanced projective. □

For an arbitrary ordinal λ, in ([15], Theorem 4.4) it was shown that n-simply presented groups have the λ-Nunke property, but the argument was long and difficult. By way of comparison, using a much simpler argument, Theorem 1.12 with $m = 0$ states that the n-balanced projective groups have the λ-Nunke property. In other words, in verifying the λ-Nunke property, the n-simply presented groups are much harder to handle than their *summands*, the n-balanced projective groups. Of course, in addition, Theorem 1.12 also applies when $m > 0$.

2. m, n-totally projective groups

We will find it convenient to denote the torsion product of the groups A and B by $A \bigtriangledown B = \mathrm{Tor}(A, B)$ (this notation - originally suggested by Claudia Metelli - is not only more compact, but it also better reflects the fact that $\bigtriangledown$ is the derived functor of the tensor product, $\otimes$, as well as better reflecting that this is actually a *product* in the category of primary abelian groups).

A short exact sequence

$$0 \to X \to Y \to G \to 0$$

is p^α*-pure* if it represents an element of $p^\alpha\mathrm{Ext}(G, X)$, and G is p^α-projective if all such sequences split. For the aid of the reader, we state and give quick proofs of some of the main properties of p^α-projective groups, beyond what can be found in [3] or [6]. Most of these facts are due to Nunke ([17], [18], [19]).

Denote the *generalized Prüfer group* by H_α (there will be no danger of confusion with the notation $G_\alpha = G/p^\alpha G$ employed elsewhere). For every group G there is a natural homomorphism $\partial_G^\alpha : H_\alpha \bigtriangledown G \to G$. If α is finite, then ∂_G^α can be identified with the inclusion $G[p^\alpha] \subseteq G$, and G is p^α-projective if and only if it is p^α-bounded. If $\alpha = \lambda + \xi$, then $p^\lambda H_\alpha$ can be identified with H_ξ, and $\partial_{p^\lambda G}^\xi : H_\xi \bigtriangledown p^\lambda G \to p^\lambda G$ can be identified with $\partial_G^\alpha|_{p^\lambda(H_\alpha \bigtriangledown G)} : p^\lambda H_\alpha \bigtriangledown p^\lambda G \to p^\lambda G$ (where we are using [3], Theorem 64.2).

Lemma 2.1. *Let G be a group and $\alpha = \lambda + \xi$ be an ordinal.*

(a) *G is p^α-projective if and only if ∂_G^α has a right inverse $\nu : G \to H_\alpha \bigtriangledown G$ (i.e., $\partial_G^\alpha \circ \nu = 1_G$)).*

(b) *If $p^\alpha G = \{0\}$, then $p^\alpha \mathrm{Ext}(G,X) \subseteq \mathrm{Bext}(G,X)$.*

(c) *If G is p^α-projective and $P \subseteq G[p^n]$, then G/P is $p^{\alpha+n}$-projective.*

(c') *If $P \subseteq G[p^n]$ and G/P is p^α-projective, then G is $p^{\alpha+n}$-projective.*

(d) *If G is p^α-projective, then $p^\alpha G = \{0\}$.*

(e) *If X is a subgroup of Y, and Y/X is isomorphic to a subgroup of a p^α-projective group G, Z is a group and $\epsilon : X \to p^\alpha Z$ is a homomorphism, then ϵ extends to a homomorphism $\mu : Y \to Z$ (see [9]).*

(f) *If G is p^α-projective, then $p^\lambda G$ is p^ξ-projective.*

(f') *If G is p^α-projective, then G_λ is p^α-projective.*

(g) *If G is p^α-pure projective, then any $p^{\alpha+1}$-pure exact sequence $E : 0 \to X \to G \to Y \to 0$ necessarily splits.*

Proof. To begin, (a) and (b) are restatements of ([6], Lemma 85(a) and Theorem 91), and (c) is an obvious extension of ([3], Lemma 82.1), and (c') follows from virtually identical reasoning. Turning to (d), if G is p^α-projective, then (a) implies that G is isomorphic to a summand of $H_\alpha \bigtriangledown G$. The result then follows from the isomorphism $p^\alpha(H_\alpha \bigtriangledown G) \cong p^\alpha H_\alpha \bigtriangledown p^\alpha G = \{0\} \bigtriangledown p^\alpha G = \{0\}$.

Considering (e), if $D \cong \oplus_I \mathbb{Z}_{p^\infty}$ is a divisible hull for G, then G is isomorphic to a summand of $H_\alpha \bigtriangledown G \subseteq H_\alpha \bigtriangledown D \cong \oplus_I (H_\alpha \bigtriangledown \mathbb{Z}_{p^\infty}) \cong \oplus_I H_\alpha$. We may therefore assume that G is a totally projective group of length α.

The surjectivity of the map $\mathrm{Ext}(G,X) \to \mathrm{Ext}(Y/X,X)$ implies that we can find a group Y_1 containing Y such that $Y_1/X \cong G$. We can easily construct another group Y_2 containing X such that $p^\alpha Y_2 = X$ and $Y_2/X = G'$ is also totally projective of length α. We let Y_3 be the sum of Y_1 and Y_2 along X (i.e., $Y_3 = (Y_1 \oplus Y_2)/\{(x,-x) : x \in X\}$). Note that $Y \subseteq Y_1 \subseteq Y_3$, $p^\alpha Y_3 = X$ and $Y_3/X \cong G \oplus G'$ is totally projective of length α. Therefore, by ([3], Corollary 84.1), $\epsilon : X \to p^\alpha Z$ must extend to a homomorphism $Y_3 \to Z$, which restricts to the desired homomorphism $\mu : Y \to Z$.

For (f) and (f'), if $\pi : G \to G_\lambda$ is the canonical surjection, then since $H_\xi \bigtriangledown p^\lambda G$ can be identified with $p^\lambda(H_\alpha \bigtriangledown G)$, there is a commutative diagram:

$$\begin{array}{ccc}
H_\xi \bigtriangledown p^\lambda G & \xrightarrow{\partial^\xi_{p^\lambda G}} & p^\lambda G \\
\downarrow \subseteq & & \downarrow \subseteq \\
H_\alpha \bigtriangledown G & \xrightarrow{\partial^\alpha_G} & G \\
\downarrow 1\bigtriangledown\pi & & \downarrow \pi \\
H_\alpha \bigtriangledown G_\lambda & \xrightarrow{\partial^\alpha_{G_\lambda}} & G_\lambda
\end{array}$$

Since G is p^α-projective, there is a right inverse, ν to ∂^α_G.

Considering (f), since $\nu|_{p^\lambda G}$ will be a right inverse of $\partial^\xi_{p^\lambda G}$, it follows that $p^\lambda G$ is p^ξ-projective.

Regarding (f'), since $p^\lambda(H_\alpha \bigtriangledown G_\lambda) = \{0\}$, it follows that $(1 \bigtriangledown \pi)(\nu(p^\lambda G)) = \{0\}$, so that $(1 \bigtriangledown \pi) \circ \nu$ determines a homomorphism $\nu' : G_\lambda \to H_\alpha \bigtriangledown G_\lambda$ which will be a right inverse to $\partial^\alpha_{G_\lambda}$.

Finally, as to (g), for any group Z we have part of a long exact sequence

$$\to \mathrm{Hom}(X, Z) \xrightarrow{\delta} p^\alpha \mathrm{Ext}(Y, Z) \to 0 \quad (= p^\alpha \mathrm{Ext}(G, Z)).$$

Since E is $p^{\alpha+1}$-pure, the image of δ is contained in $p^{\alpha+1}\mathrm{Ext}(Y, Z)$. This implies that $p^\alpha \mathrm{Ext}(Y, Z) = p^{\alpha+1}\mathrm{Ext}(Y, Z)$. Since Y is torsion, by ([3], Lemma 55.3), $\mathrm{Ext}(Y, Z)$ is reduced. It follows that $p^{\alpha+1}\mathrm{Ext}(Y, Z) = \{0\}$, so that Y is $p^{\alpha+1}$-projective. However, this show that E splits, as stated. □

We introduce some new, somewhat ad hoc, terminology. If G is a group and γ is an ordinal, we will say G is *$p^{\gamma+(n)}$-projective* if there is a group K and a subgroup $P \subseteq (G \oplus K)[p^n]$ such that $(G \oplus K)/P$ is p^γ-projective. It follows from Lemma 2.1(c') that if G is $p^{\gamma+(n)}$-projective, then it is $p^{\gamma+n}$-projective. Note that if $n = 0$ or $\gamma \le \omega$, then G is $p^{\gamma+(n)}$-projective if and only if it is $p^{\gamma+n}$-projective.

Lemma 2.2. *Suppose λ and ξ are ordinals, $\xi \ge \omega$ and G is a group.*

(a) *If $p^{\lambda+m}G = \{0\}$ and G is strongly m-simply presented, then it is $p^{\lambda+m}$-projective.*

(b) *If G is m, n-balanced projective, then $G_{\lambda+m}$ is $p^{\lambda+m+(n)}$-projective.*

(c) *If $G_{\lambda+m}$ is $p^{\lambda+k}$-projective and $p^{\lambda+m}G$ is $p^{\xi+m+(n)}$-projective, then G will be $p^{\lambda+\xi+k}$-projective.*

Proof. Denote $\lambda + m$ by γ. Starting with (a), suppose $N \subseteq G[p^m]$ is nice and G/N is totally projective. So $N' = N + p^\lambda G \subseteq G[p^m]$ and $G/N' \cong (G/N)/([N + p^\lambda G]/N) = (G/N)/p^\lambda(G/N)$ is also totally projective. So G/N' is p^λ-projective, so that G is p^γ-projective.

For (b), suppose $G \oplus K \overset{\text{def}}{=} G'$ is m, n-simply presented. Consequently, by Proposition 1.7, $G_\gamma \oplus K_\gamma \cong G'_\gamma$ is m, n-simply presented, so that we may assume $p^\gamma G = p^\gamma K = \{0\}$. Let P be an m, n-simply representing subgroup of G'. Now, $p^\gamma(G'/P) \subseteq G'[p^n]/P$, so let $P' \subseteq G'[p^n]$ be the subgroup containing P such that $P'/P = p^\gamma(G'/P)$. It follows that $G'/P' \cong (G'/P)/p^\gamma(G'/P)$ is strongly m-simply presented and $p^\gamma = p^{\lambda+m}$-bounded. Therefore by (a), G'/P' is p^γ-projective. It follows that G', and hence G, will be $p^{\gamma+(n)}$-projective.

As to (c), denote $\xi + m$ by μ. Since $p^\gamma G$ is $p^{\mu+(n)}$-projective, there is a group K and a subgroup $P \subseteq (p^\gamma G \oplus K)[p^n]$ such that $(p^\gamma G \oplus K)/P$ is p^μ-projective. Let L be a group such that $p^\gamma L = K$ and $L/p^\gamma L$ is totally projective. If we replace G by $G \oplus L$, then we may assume $P \subseteq (p^\gamma G)[p^n]$ and $p^\gamma G/P$ is p^μ-projective.

As we have observed before, since μ is infinite, $p^{\gamma+n}H_{\gamma+\mu}$ can be identified with H_μ. Again, this means we can identify $p^{\gamma+n}(H_{\gamma+\mu}\bigtriangledown G)$ with $H_\mu \bigtriangledown p^{\gamma+n}G$; let $\partial = \partial_G^{\gamma+\mu+n} : H_{\gamma+\mu+n} \bigtriangledown G \to G$ and $\partial' = \partial_G^{\gamma+\mu} : H_{\gamma+\mu} \bigtriangledown G \to G$ which restricts to $\partial' = \partial^{\mu}_{p^{\gamma+n}G} : H_\mu \bigtriangledown p^{\gamma+n}G \to p^{\gamma+n}G$. There is a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & K & \to & H_{\gamma+\mu+n} \bigtriangledown G & \xrightarrow{\partial} & G & \to & 0 \\
 & & \| & & \downarrow & & \downarrow p^n & & \\
0 & \to & K & \to & H_{\gamma+\mu} \bigtriangledown G & \xrightarrow{\partial'} & G & \to & 0 \\
 & & \uparrow \subseteq & & \uparrow \subseteq & & \uparrow \subseteq & & \\
0 & \to & p^{\gamma+n}K & \to & H_\mu \bigtriangledown p^{\gamma+n}G & \xrightarrow{\partial'} & p^{\gamma+n}G & \to & 0
\end{array}$$

Note that $p^n : p^\gamma G \to p^{\gamma+n}G$ induces a homomorphism $p^\gamma G/P \to p^{\gamma+n}G$. Since $p^\gamma G/P$ is p^μ-projective and the lower row of our diagram is p^μ-pure, there is a homomorphism $\phi_0 : p^\gamma G \to H_\mu \bigtriangledown p^{\gamma+n}G$ such that $\partial' \circ \phi_0 = p^n|_{p^\gamma G}$. Since G_γ is $p^{\gamma+n}$-projective and $\phi_0(p^\gamma G) \subseteq p^{\gamma+n}(H_{\gamma+\mu} \bigtriangledown G)$, it follows from Lemma 2.1(e) that ϕ_0 extends to a homomorphism $\phi : G \to H_{\gamma+\mu}\bigtriangledown G$. Because $(\partial' \circ \phi - p^n)(p^\gamma G) = \{0\}$, $\partial' \circ \phi - p^n$ induces a homomorphism $G_\gamma \to G$. Since G_γ is $p^{\gamma+n}$-projective and the middle row of our diagram is $p^{\gamma+\mu}$-pure, and so $p^{\gamma+n}$-pure, it follows that there is a homomorphism $\rho : G \to H_{\gamma+\mu} \bigtriangledown G$ such that $\partial' \circ \phi - p^n = \partial' \circ \rho$. Since $\partial' \circ (\phi - \rho) = p^n$, it follows that the upper row of our diagram splits, so that G is $p^{\gamma+\mu+n}$-projective, as required. □

By letting $m = n = 0$ in Lemma 2.2(c) we get another useful theorem of Nunke.

Corollary 2.3. *Suppose λ and ξ are ordinals and G is a group. If G_λ is p^λ-projective and $p^\lambda G$ is p^ξ-projective, then G is $p^{\lambda+\xi}$-projective.*

Proof. If ξ is infinite, this follows from Lemma 2.2(c), and if ξ is finite, it follows from Lemma 2.1(c'). □

Returning to our main investigation, we have the following consequence of Lemma 2.2(b).

Theorem 2.4. *If a group G is m,n-balanced projective, then it is m,n-totally projective.*

Proof. If λ is any ordinal, then $G_{\lambda+m}$ is $p^{\lambda+m+(n)}$-projective, and hence $p^{\lambda+k}$-projective, as required. □

As was the case for m,n-simply presented groups and m,n-balanced projectives, half of the λ-Nunke property for m,n-totally projective groups is easy.

Theorem 2.5. *If λ is an ordinal and G is an m,n-totally projective group, then $p^\lambda G$ and $G_\lambda = G/p^\lambda G$ are m,n-totally projective.*

Proof. Let μ be any ordinal, so that $G_{\mu+m}$ is $p^{\mu+k}$-projective. If $\mu+m \leq \lambda$, then $(G_\lambda)_{\mu+m} \cong G_{\mu+m}$ is $p^{\mu+k}$-projective. On the other hand, if $\mu + m > \lambda$, then by Lemma 2.1(f'), $(G_\lambda)_{\mu+m} \cong G_\lambda \cong (G_{\mu+m})_\lambda$ is $p^{\mu+k}$-projective. Therefore, G_λ is m, n-totally projective.

In addition, since $G_{\lambda+\mu+m}$ is $p^{\lambda+\mu+k}$-projective, by Lemma 2.1(f) we can conclude $(p^\lambda G)_{\mu+m} = p^\lambda(G_{\lambda+\mu+m})$ is $p^{\mu+k}$-projective, so that $p^\lambda G$ is m, n-totally projective. □

We now consider the converse of Theorem 2.5. The following, which parallels Theorem 1.12, is slightly unsatisfactory in the sense that it requires that we strengthen our assumptions regarding $p^{\lambda+m}G$.

Theorem 2.6. *Suppose G is a group and λ is an ordinal. If $p^{\lambda+m}G$ is m, n-balanced projective and $G_{\lambda+m} = G/p^{\lambda+m}G$ is m, n-totally projective, then G is m, n-totally projective.*

Proof. If μ is a limit ordinal, then we need to show $G_{\mu+m}$ is $p^{\mu+k}$-projective. If $\mu \le \lambda$, then $G_{\mu+m} \cong (G_{\lambda+m})_{\mu+m}$ is $p^{\mu+k}$-projective.

On the other hand, if $\mu > \lambda$, then let ξ be defined by the equation $\mu = \lambda + \xi$, so that ξ is infinite. By Theorem 1.12, $p^{\lambda+m}G_{\mu+m} \cong (G_{\lambda+m})_{\xi+m}$ is m, n-balanced projective; it follows from Lemma 2.2(b) that $p^{\lambda+m}G_{\mu+m}$ is $p^{\xi+m+(n)}$-projective. Since $(G_{\mu+m})_{\lambda+m} \cong G_{\lambda+m}$ is m, n-totally projective, it must be $p^{\lambda+k}$-projective. So, by Lemma 2.2(c), we can conclude that $G_{\mu+m}$ is $p^{\lambda+\xi+k} = p^{\mu+k}$-projective, as required. □

We now briefly discuss one special case in which a λ-Nunke-type result occurs.

Proposition 2.7. *Suppose λ is an ordinal and G is a group such that G_λ is p^λ-projective (e.g., G_λ could be totally projective). Then G is m, n-totally projective if and only if both $p^\lambda G$ and G_λ are m, n-totally projective.*

In particular, G is m, n-totally projective if and only if $p^\lambda G$ is m, n-totally projective, provided G_λ is totally projective.

Proof. One implication is a direct consequence of Theorem 2.5, so suppose $p^\lambda G$ and G_λ are m, n-totally projective. Let μ be any ordinal. If $\mu + m \le \lambda$, then since G_λ is m, n-totally projective, $G_{\mu+m} \cong (G_\lambda)_{\mu+m}$ is $p^{\mu+k}$-projective.

Next, suppose $\lambda < \mu + m < \lambda + \omega$ and $\mu + m = \lambda + j$. Since $p^\lambda(G_{\mu+m})$ is p^j-bounded, and $(G_{\mu+m})_\lambda \cong G_\lambda$ is p^λ-projective, it follows that $G_{\mu+m}$ is $p^{\lambda+j} = p^{\mu+m}$-projective; and hence $p^{\mu+k}$-projective.

Finally, if $\lambda+\omega \le \mu+m$, then let ξ be defined by $\mu+m = \lambda+\xi+m$. We have $p^\lambda(G_{\mu+m}) = (p^\lambda G)_{\xi+m}$ is $p^{\xi+k}$-projective. In addition, since $(G_{\mu+m})_\lambda \cong G_\lambda$ is p^λ-projective, by Corollary 2.3, $G_{\mu+m}$ is $p^{\lambda+\xi+k} = p^{\mu+k}$-projective. So G is m, n-totally projective, as stated.

The final part is immediate. □

The next result, which parallels Theorem 1.12 and ([15], Theorem 3.4(b)), shows that in one extreme case we get the desired result.

Corollary 2.8. *If λ is an ordinal, then the strongly n-totally projective groups have the $\lambda + n$-Nunke property.*

Proof. Suppose $p^{\lambda+n}G$ and $G_{\lambda+n}$ are strongly n-totally projective. Since $G_{\lambda+n}$ will be $p^{\lambda+n}$-projective, by Proposition 2.7 (with $m = n$, $n = 0$, $\lambda = \lambda + n$), G is strongly n-totally projective. The converse follows from Theorem 2.5. $\square$

3. Groups of length less than ω^2

The following is a key step in discussing groups of length less than ω^2. Its proof is a version of the argument used in ([15], Theorem 4.5); however, since it only deals with the ordinal ω and Σ-cyclic groups, as opposed to a general limit ordinal and all totally projective groups, it is substantially simpler.

Lemma 3.1. *If G is an m,n-balanced projective group and $p^{\omega+m}G$ is bounded, then G is m,n-simply presented.*

Proof. If $m = 0$, this follows from ([15], Corollary 4.7), so we may assume $m > 0$. We now induct on n. If $n = 0$, the result is an immediate consequence of ([15], Corollary 3.6). So assume $n > 0$ and the result holds for $n - 1$.

Since a bounded group, such as $p^{\omega+m}G$, is clearly m,n-balanced projective, by Theorem 1.12 and Corollary 1.5, G is m,n-balanced projective if and only if $G_{\omega+m}$ is $p^{\omega+k}$-projective. Since a group A such that $p^{\omega+m}A$ is bounded is strongly m-simply presented if and only if $A_{\omega+m}$ is $p^{\omega+m}$-projective, the result will follow by induction from the following statement:

Claim 1: There is a subgroup $X \subseteq G[p]$ such that $G' \stackrel{\text{def}}{=} G/X$ is $m, n-1$-balanced projective (i.e., $G'_{\omega+m}$ is $p^{\omega+k-1}$-projective).

After separating off a bounded summand, we may assume that G has rank and final rank equaling some cardinal κ. If κ is countable, then G will be a dsc group and the result clearly follows; without loss of generality, then, assume κ is uncountable. Note that G is $p^{\omega+\ell}$-projective for some $\ell < \omega$, so by ([11], Corollary 25) it is *far from thick*. This means that there is a Σ-cyclic group S and a surjective homomorphism $\pi : G \to S$ such that for all $j < \omega$ we have $r(\pi((p^jG)[p])) = \kappa$; let P' be the kernel of π. There is a subgroup $P \subseteq G[p^k_{\omega+m}]$ containing $p^{\omega+m}G$ such that $G/P \cong G_{\omega+m}/(P/p^{\omega+m}G)$ is Σ-cyclic. Replacing P by $P \cap P'$, if necessary, we may assume that $p^kP \subseteq p^{\omega+m}G$ and the above cardinality condition holds for $\pi : G \to G/P = S$. Since G/P is separable, we have $p^\omega G \subseteq P$.

Fix a decomposition $S = \bigoplus_{i\in I} S_i$, where each S_i is a cyclic group, and let π_i be the composition $G \to G/P \to S_i$. If $x \in G$, let $\mathrm{supp}(x)$ be the support of $\pi(x)$ in this decomposition. If $J \subseteq I$, let $\Sigma_J = \bigoplus_{i\in J} S_i$.

Let L be the set of limit ordinals in κ and x'_γ for $\gamma \in L$ be a listing of $P(\omega + m - 1) = P \cap p^{\omega+m-1}G$, where we simply repeat terms if $|P(\omega + m - 1)| < \kappa$. For each $x'_\gamma \in P(\omega+m-1)$, choose $x_\gamma \in P(\omega) = p^\omega G$ such that $p^{m-1}x_\gamma = x'_\gamma$. We inductively pick $y_\alpha \in G[p]$ and $z_\alpha \in G[p^1_\omega]$ such that

(a) $y_\alpha \in (p^jG)[p] - P$, where $\gamma \in L$, $j < \omega$ and $\alpha = \gamma + j$;

(b) $\mathrm{supp}(y_\alpha) \cap K_\alpha = \emptyset$, where $K_\alpha \stackrel{\text{def}}{=} \bigcup_{\beta<\alpha}(\mathrm{supp}(y_\beta) \cup \mathrm{supp}(z_\beta))$;

(c) $pz_\alpha = x_\gamma$ (where $\gamma \in L$ is as in (a));
(d) $|\pi(y_\alpha)|_S < |z_\alpha|_G$;
(e) $\mathrm{supp}(z_\alpha) \cap \mathrm{supp}(y_\alpha) = \emptyset$.

Basically, one chooses $y_\alpha \in G[p]$ such that (a) and (b) hold; this can be done since the projection $\Sigma_I \to \Sigma_{K_\alpha}$ restricted to $\pi((p^j G)[p]) \to \Sigma_{K_\alpha}$ must have a non-zero kernel. Then choose z_α such that $pz_\alpha = x_\gamma$ and $|\pi_i(y_\alpha)|_S < |z_\alpha|_G$ for all $i \in \mathrm{supp}(y_\alpha)$.

For all $\alpha < \kappa$ we now let $r_\alpha = (y_\alpha + z_\alpha) - (y_{\alpha+1} + z_{\alpha+1})$. Clearly $pr_\alpha = 0$, so let

$$X = P[p] + \langle r_\alpha : \alpha < \kappa \rangle \subseteq G[p].$$

If $G' = G/X$, then we need to show that $G'_{\omega+m}$ is $p^{\omega+k-1}$-projective. To that end, let

$$Q = P + \langle y_\alpha + z_\alpha : \alpha < \kappa \rangle,$$

so that $X \subseteq Q$. We divide our argument into two statements.

Claim 2: $p^{k-1}(Q/X) \subseteq p^{\omega+m}G'$.

Claim 3: $G'/(Q/X) \cong G/Q \cong S/(Q/P)$ is Σ-cyclic.

Regarding Claim 2, we begin with the following:

Subclaim 2': $[P(\omega + m - 1) + X]/X \subseteq p^{\omega+m}G'$.

If $\gamma \in L$ and $x'_\gamma \in P(\omega + m - 1)$, then $x'_\gamma = p^m(y_\gamma + z_\gamma)$. For every $j < \omega$,

$$y_\gamma + z_\gamma = (y_{\gamma+j+1} + z_{\gamma+j+1}) + (r_{\gamma+j} + \cdots + r_{\gamma+1} + r_\gamma).$$

Note that by (a) and (d), $|y_{\gamma+j+1} + z_{\gamma+j+1}|_G > j$, and $r_{\gamma+j} + \cdots + r_{\gamma+1} + r_\gamma \in X$; it follows that $y_\gamma + z_\gamma + X \in p^\omega G'$. Therefore, $x'_\gamma + X \in p^{\omega+m}G'$, as stated.

Since $p^{k-1}P \subseteq P(\omega + m - 1) + P[p] \subseteq P(\omega + m - 1) + X$, it follows from Subclaim 2' that $p^{k-1}([P + X]/X) \subseteq p^{\omega+m}G'$.

When $\gamma \in L$, $j < \omega$ and $\alpha = \gamma + j$, then $(y_\alpha + z_\alpha) + X = (y_\gamma + z_\gamma) + X \in p^\omega G'$, which gives $p^m(y_\alpha + z_\alpha) + X \in p^{\omega+m}G'$. Note also that $k - 1 \geq m$, so that $p^{k-1}(y_\alpha + z_\alpha) + X \in p^{\omega+m}G'$. This concludes the proof of Claim 2.

Turning to Claim 3, let $\pi(y_\alpha + z_\alpha) = s_\alpha + t_\alpha$, where $s_\alpha \in \Sigma_{(K_{\alpha+1} - K_\alpha)}$ and $t_\alpha \in \Sigma_{K_\alpha}$ (In fact, we will have $\mathrm{supp}(s_\alpha) = K_{\alpha+1} - K_\alpha$). It follows from (b), (d) and (e) in the construction of y_α and z_α that $|s_\alpha|_S \leq |t_\alpha|_S$.

For $\alpha \leq \kappa$, let $S_\alpha = \Sigma_{K_\alpha}$ and $Q_\alpha = \langle s_\nu + t_\nu : \nu < \alpha \rangle$. Clearly $S/(Q/P) \cong (\Sigma_{I - K_\kappa}) \oplus (S_\kappa/Q_\kappa)$, where the first term is certainly Σ-cyclic. To show the second term is also Σ-cyclic, note that S_κ/Q_κ is the direct limit of $\{S_\alpha/Q_\alpha\}_{\alpha<\kappa}$. Claim 3, therefore, follows from the next statement.

Subclaim 3': For every $\alpha < \kappa$ we have a split-exact sequence

$$0 \to S_\alpha/Q_\alpha \to S_{\alpha+1}/Q_{\alpha+1} \to \Sigma_{(K_{\alpha+1} - K_\alpha)}/\langle s_\alpha \rangle \to 0,$$

where the right-hand term is finite, and hence Σ-cyclic.

Since $|s_\alpha|_S \leq |t_\alpha|_S$, the map $s_\alpha \mapsto t_\alpha$ extends to a homomorphism

$$\phi : \Sigma_{(K_{\alpha+1} - K_\alpha)} \to \Sigma_{K_\alpha} = S_\alpha.$$

Therefore, $(u, v) \mapsto (u + \phi(v), v)$ is an automorphism of $S_{\alpha+1} = \Sigma_{K_{\alpha+1}}$ which fixes S_α and takes $Q_\alpha \oplus \langle s_\alpha \rangle$ to $Q_{\alpha+1}$. In particular, we have

$$S_{\alpha+1}/Q_{\alpha+1} \cong (S_\alpha/Q_\alpha) \oplus (\Sigma_{(K_{\alpha+1}-K_\alpha)}/\langle s_\alpha \rangle).$$

This establishes Claim 3'; and hence Claim 3; and hence Claim 1; and hence the lemma. □

This brings us to an extension of ([15], Corollaries 3.6 and 4.7). The new result applies not only when both m and n are positive, but also includes the condition of m,n-totally projectivity.

Theorem 3.2. *Suppose G is a reduced group of length strictly less than ω^2. The following are equivalent*:

(a) *G is m,n-simply presented*;

(b) *G is m,n-balanced projective*;

(c) *G is m,n-totally projective*;

(d) *for every $\lambda < \omega^2$, $(p^\lambda G)_{\omega+m}$ is $p^{\omega+k}$-projective.*

Proof. Clearly (a) implies (b). By Theorem 2.4, (b) implies (c). Assuming (c), then to verify (d), let $\lambda < \omega^2$. It follows from Theorem 2.5 that $(p^\lambda G)_{\omega+m}$ is m,n-totally projective. However, since this factor is $p^{\omega+m}$-bounded, it must be $p^{\omega+k}$-projective.

Finally, we assume (d) is true and verify (a). We induct on ℓ, which we define to be the smallest non-negative integer such that $p^{\omega\cdot\ell}G = 0$. Observe that $p^{\omega+k}G$ also satisfies (d), and has a smaller corresponding value of ℓ. It follows by induction that $p^{\omega+k}G$ is m,n-simply presented. Next, observe that $p^{\omega+m}(G_{\omega+k})$ is bounded (by p^n) and $(G_{\omega+k})_{\omega+m} \cong G_{\omega+m}$ is $p^{\omega+k}$-projective. It follows from Lemma 3.1 that $G_{\omega+k}$ is m,n-simply presented. Therefore, (a) follows from Theorem 1.8. □

The following is a slight extension of the last result but is its direct consequence.

Corollary 3.3. *Suppose G is a group, $\gamma < \omega^2$ and $p^\gamma G$ is m,n-simply presented. Then* (a) *through* (d) *of Theorem 3.2 are still equivalent.*

Proof. Note that the first paragraph of the last proof applies without change. Suppose then that $p^\gamma G$ is m,n-simply presented and G satisfies (d); we need to verify that G is m,n-simply presented. Clearly, $p^{\gamma+k}G = p^k(p^\gamma G)$ is m,n-simply presented. A now standard argument shows that $G_{\gamma+k}$ also satisfies (d). However, since $\gamma + k < \omega^2$, by Theorem 3.2, $G_{\gamma+k}$ is m,n-simply presented. Therefore, by Theorem 1.8, G is m,n-simply presented, as required. □

We have the following containments:

"$k,0$-balanced projectives" $\subseteq$ "m,n-balanced projectives" $\subseteq$ "$m-1,n+1$-balanced projectives" $\subseteq$ "$0,k$-balanced projectives"

Example 3.4. If $m > 0$, then there is a group G that is $m-1, n+1$-balanced projective, but not m, n-balanced projective.

Proof. Consider any group G of length $\omega + m$ which is not $p^{\omega+k}$-projective, but $G/p^{\omega+m-1}G$ is $p^{\omega+k}$-projective. It follows from Theorem 3.2 that such a group has the specified properties. To be a bit more specific, let B be an unbounded Σ-cyclic group with torsion completion $\overline{B}$ and V be the valuated group $\overline{B}[p^{k+1}]/B[p^m]$, where $|x|_V = |x|_{\overline{B}/B[p^m]}$ for all $x \in V$. Next, let G be a group containing V such that G/V is Σ-cyclic and $|x|_V = |x|_G$ for all $x \in V$. We leave it to the reader to verify that this G has the indicated properties. □

This example also shows that the m, n-balanced projective groups do not have the $\omega + m - 1$-Nunke property, so that Theorem 1.12 is the best possible result. It also shows that the m, n-simply presented and the m, n-totally projective groups do not have the $\omega + m - 1$-Nunke property (cf. Theorems 1.8 and 2.6).

Let $\mathcal{S}_n$ be the collection of groups G such that for some $\gamma < \omega^2$, G_γ is strongly n-simply presented and $p^\gamma G$ is totally projective. Clearly $\mathcal{S}_0$ is just the totally projective groups. For $n > 0$, the following shows that the groups in $\mathcal{S}_n$ are determined by their p^n-socles.

Theorem 3.5. *Suppose $n > 0$, and G_1 and G_2 are in $\mathcal{S}_n$. Then G_1 and G_2 are isomorphic if and only if $G_1[p^n]$ and $G_2[p^n]$ are isometric.*

Proof. Certainly, if G_1 and G_2 are isomorphic, then they have isometric p^n-socles. For the converse, if $\ell < \omega$, let $\mathcal{S}_n^\ell$ be the collection of $G \in \mathcal{S}_n$ such that $p^{(\omega\cdot\ell)+n}G$ is totally projective. Clearly, $\mathcal{S}_n$ is the ascending union of the $\mathcal{S}_n^\ell$. We induct on ℓ to show that the groups in $\mathcal{S}_n^\ell$ are determined by the isometry classes of their p^n-socles. Since $\mathcal{S}_n^0$ is just the simply presented groups, this is true for $\ell = 0$. Suppose now that this holds for ℓ. Since $G \in \mathcal{S}_n^{\ell+1}$ if and only if $p^{\omega+n}G \in \mathcal{S}_n^\ell$ and $G_{\omega+n}$ is $p^{\omega+n}$-projective, it follows from ([14], Theorem 3.16) that the groups in $\mathcal{S}_n^{\ell+1}$ are also determined by the isometry classes of their p^n-socles. Therefore, by induction, the result follows for $\mathcal{S}_n = \cup_\ell \mathcal{S}_n^\ell$. □

One important and useful property of $p^{\omega+1}$-projective groups G is that they always split into $G = S \oplus T$, where S is separable and T is totally projective. The following shows that a variation on this property generalizes to the groups in $\mathcal{S}_1$. If $\lambda \le \omega_1$ is an ordinal, then G is a C_λ *group* if for every $\alpha < \lambda$ one (and hence all) p^α-high subgroups of G are dsc groups. If λ is a limit ordinal, this is equivalent to requiring that G_α is a dsc group for every $\alpha < \lambda$ (see, for example, [12], Theorem 8). All groups are C_ω groups.

Proposition 3.6. *A group G is in $\mathcal{S}_1$ if and only if*

$$G \cong H \oplus (\bigoplus_{1\le\ell\le j} A_\ell),$$

where

(a) *j is a non-negative integer*;
(b) *H is totally projective*;
(c) *A_ℓ is a $p^{(\omega\cdot\ell)+1}$-projective $C_{\omega\cdot\ell}$ group with $p^{\omega\cdot\ell}A_\ell = \{0\}$.*

Proof. It is easy to check that any group of the indicated form is in $\mathcal{S}_1$. For the converse, let j be the smallest non-negative integer such that $p^{\omega\cdot j}G$ is simply presented. If $j = 0$, the result is obvious, so assume it holds for all groups in $\mathcal{S}_1$ with a smaller corresponding value of j. Note that $p^{\omega+1}G$ satisfies the hypothesis with $j-1$, so there is a corresponding decomposition $p^{\omega+1}G \cong H' \oplus (\bigoplus_{1\le\ell\le j-1} A'_\ell)$.

Let Y be a $p^{\omega+1}$-high subgroup of G. Since Y embeds in $G_{\omega+1}$, it is $p^{\omega+1}$-projective. In particular, Y must be C-decomposable, so that $Y \cong A_1 \oplus C \oplus T$, where A_1 is a separable $p^{\omega+1}$-projective, T is simply presented of length $\omega+1$ and C is a Σ-cyclic group whose final rank is at least as large as $r(p^\omega G)$.

Note that $G[p]$ is isometric to $Y[p]\oplus(p^{\omega+1}G)[p]$. A simple (but rather tedious) computation in valuated vector spaces, which we omit, then shows that $G[p]$ is isometric to the socle of a group of the form $H \oplus (\bigoplus_{1\le\ell\le j} A_\ell)$, where H is simply presented with $p^{\omega+1}H \cong H'$ and for $2 \le \ell \le j$, $p^{\omega+1}A_\ell \cong A'_{\ell-1}$ and $A_\ell/p^{\omega+1}A_\ell$ is a dsc group.

By Theorem 3.5, then, G is isomorphic to this direct sum. $\square$

4. n-summable groups

Throughout this section we will assume n is positive. We now consider groups of length not exceeding ω_1. The following definition appeared in [2]: A group G is *n-summable* if the valuated group $G[p^n]$ is isometric to the valuated direct sum of a collection of countable valuated groups. In particular, a group is 1-summable if and only if it is summable in the usual sense of the term. For more detailed information about summable and n-summable groups, see [2], [3], [9] and [10].

We now relate this to our current discussions. Recall from [15] that for a group G, a group $H(G)$ is defined such that $H(G)$ has a nice subgroup V which is isometric to $G[p^n]$ and such that $H(G)/V$ is simply presented. We identify V with $G[p^n]$. This group was used to construct a strongly n-balanced projective resolution, $0 \to K(G) \to H(G) \to G \to 0$, of G.

Theorem 4.1. *A group G is n-summable if and only if $H(G)$ is a dsc group.*

Proof. Suppose first that G is n-summable. By ([2], Theorem 2.1), $G[p^n]$, as a valuated group, has a nice composition series, $\{N_i\}_{i<\alpha}$. It is readily checked that each N_i is also nice in $H(G)$. Since $H(G)/G[p^n]$ is totally projective, it has a nice composition series $\{M_j\}_{j<\beta}$. If M'_j is the subgroup of $H(G)$ containing $G[p^n]$ such that $M'_j/G[p^n] = M_j$, then the Ns together with the M's form a nice composition series for G.

Conversely, if $H(G)$ is a dsc group, where $V = G[p^n]$, then the proof of ([2], Theorem 2.1) it is clearly n-summable. Since $G[p^n]$ is a valuated summand of

$H(G)[p^n]$, it will also be a direct sum of countable valuated groups. Therefore, G is n-summable. □

Corollary 4.2. *A group G is a dsc group if and only if it is strongly n-balanced projective and n-summable.*

Proof. Certainly, we know that a dsc group is strongly n-balanced projective and n-summable. Conversely, if G is n-summable, it follows from Theorem 4.1 that $H(G)$ is a dsc group. And if G is also strongly n-balanced projective, then it is isomorphic to a summand of $H(G)$, so that it, too, is a dsc group. □

Theorem 4.1 allows us to derive properties of n-summable groups from the corresponding classical results for dsc groups. We present a couple of examples.

Corollary 4.3. *If $G = \cup_{i<\omega} G_i$, where $G_i \subseteq G_{i+1}$ are n-summable isotype subgroups of G, then G is n-summable.*

Proof. Note that $H(G)$ will be the ascending union of the isotype subgroups $H(G_i)$. If the latter are all dsc groups, then by a result of Hill ([7]), so is $H(G)$, which implies that G is n-summable, as required. □

Proposition 4.4. *Let G be an isotype subgroup of the n-summable group A. If G is summable, then it is n-summable.*

Proof. The result being trivial if $n = 1$, we assume that $n > 1$. Again, $H(G)$ will be an isotype subgroup of $H(A)$, and since A is n-summable, $H(A)$ is a dsc group. We next show that $H(G)$ is summable: There is a valuated decomposition $H(G)[p^n] \cong (K(G)[p^n]) \oplus (G[p^n])$. Therefore, if $H' \overset{\text{def}}{=} H(G)/G[p^{n-1}]$, then $(p^{n-1}G)[p] \cong G[p^n]/G[p^{n-1}] \overset{\text{def}}{=} V'$ is a nice subgroup of H' such that $H'/V' \cong H(G)/G[p^n]$ is a dsc group. It follows that $0 \to K(G) \to H' \to p^{n-1}G \to 0$ is a strongly 1-balanced projective resolution of $p^{n-1}G$. However, since G is summable, so is $p^{n-1}G$; and this implies that H' is actually a dsc group. Since $H'[p]$ is isometric to $(K(G)[p]) \oplus V'$, $K(G)$ is summable. And since $H(G)[p]$ is isometric to $(K(G)[p]) \oplus (G[p])$, we have that $H(G)$ is also summable.

Therefore, again by a result of Hill ([8]), $H(G)$ (as an isotype and summable subgroup of a dsc group) is also a dsc group. However, in view of Theorem 4.1, this gives that G is n-summable, as stated. □

We want to consider what happens in Corollary 4.2 when the condition "strongly n-balanced projective" is replaced with the possibly weaker condition "strongly n-totally projective." To that end, we have the following intermediate step.

Lemma 4.5. *If G is a strongly n-totally projective group, α is a countable ordinal and X is $p^{\alpha+n-1}$-high in G, then X is also strongly n-totally projective.*

Proof. By ([6], Theorem 92), X is $p^{\alpha+n}$-pure in G. Let λ be an ordinal. If $\lambda + n \leq \alpha + n$, then by ([6], Proposition 87), we can infer that $X_{\lambda+n}$ embeds as a $p^{\lambda+n}$-pure subgroup of $G_{\lambda+n}$. Since $G_{\lambda+n}$ is $p^{\lambda+n}$-projective and $\lambda+n$ is countable, it follows that $X_{\lambda+n}$ is also $p^{\lambda+n}$-projective (see, for example, [17]). If $\lambda+n > \alpha+n$, then we already know that $X \cong X_{\lambda+n}$ will be $p^{\alpha+n}$-projective, and hence $p^{\lambda+n}$-projective. This shows, therefore, that X is strongly n-totally projective. □

This brings us to one of the main results of this section.

Theorem 4.6. *Suppose G is a group of countable length. Then G is a dsc group if and only if it is strongly n-totally projective and n-summable.*

Proof. Certainly if G is a dsc group, then it satisfies these two conditions. For the converse, we induct on the length of G, which we denote by μ; so suppose that the result holds for all groups of shorter length. If $\mu < \omega$, the result is trivial, so we may assume μ is infinite.

Case 1: $\mu = \alpha + n$ for some $\alpha < \mu$. Let X be $p^{\alpha+n-1}$-high in G. By ([2], Corollary 3.1(c)), X is n-summable and by Lemma 4.5, it is strongly n-totally projective; so by induction on lengths, X must be a dsc. It follows that G is a p^{μ}-projective C_μ group. By ([13], Proposition 2), this implies that G is a dsc group.

Case 2: $\lambda \le \mu \le \lambda + n - 1$, where λ is a limit ordinal. If $\alpha < \lambda$ and X is a $p^{\alpha+n-1}$-high subgroup of G, it follows as above that X is a dsc group. Therefore, G is a C_λ group.

Since G is n-summable and of countable length, it follows from ([2], Theorem 2.2) that $G[p^n]$ is the ascending union of a sequence of subgroup $\{S_\ell\}_{\ell<\omega}$, such that each $|S_\ell|_G = \{|x|_G : x \in S_\ell\}$ is finite. Since $p^{n-1}(p^\lambda G) = 0$, it follows that $G_\lambda[p] \subseteq (G[p^n])/p^\lambda G = \cup_{\ell<\omega}(S_\ell + p^\lambda G)$. Since $|S_\ell + p^\lambda G|_{G/p^\lambda G} \subseteq |S_\ell|_G$ is finite, we can conclude from ([3], Theorem 84.1) that G_λ is summable. So by Megibben's result on summable C_λ groups (see [16]), G_λ is a dsc group. However, since $p^\lambda G$ is bounded (and hence Σ-cyclic), it follows that G is also a dsc group. □

Corollary 4.7. *If G is an n-summable strongly n-totally projective group, then G is a C_{ω_1} group.*

Proof. If $\alpha < \omega_1$, it follows from [2] that being an isotype subgroup any $p^{\alpha+n-1}$-high subgroup X of G is n-summable, and by Lemma 4.5, strongly n-totally projective. So by Theorem 4.6, X must be a dsc group. Since this is valid for all countable α, G must be a C_{ω_1} group (see [12]), as claimed. □

We finish with a couple of examples. The first shows that in Corollary 4.2 and Theorem 4.6, we cannot drop the word "strongly".

Example 4.8. There is a group G of length $\omega + 1$ which is 1-simply presented (and so 1-balanced projective and 1-totally projective) and 1-summable, but is not a dsc group.

Proof. Let H be any separable group which is $p^{\omega+1}$-projective, but not Σ-cyclic. If B is a basic subgroup of H, we can let $G = H/B[p]$. It is readily checked that $G[p]$ is isometric to $p^\omega G \oplus (pB)[p]$, so that G is 1-summable. Since $G_\omega \cong pH$ is $p^{\omega+1}$-projective, G is 1-simply presented. Since G_ω is not Σ-cyclic, G cannot be a dsc group. □

The last example also shows that, in contrast to Theorem 3.5, the 1-balanced projective groups are not determined by isometries of their (p^1-)socles.

Our final example demonstrates that Theorem 4.6 and the strong case of Theorem 3.2 do not immediately generalize to groups of uncountable length. In other words, though every m, n-balanced projective group is m, n-totally projective, the converse does not hold for strongly n-totally projective groups of uncountable length, even in the case of groups that are n-summable.

Example 4.9. There is an n-summable group G that is strongly n-totally projective, but not strongly n-balanced-projective - which, by Corollary 4.2, is equivalent to it failing to be a dsc group.

Proof. We assume that $n = 1$, though a similar construction would be possible for larger values. Again, let H_{ω_1+1} be the generalized Prüfer group of length $\omega_1 + 1$. In [1] a (1-)summable C_{ω_1} group X of length ω_1 was constructed which is not a dsc group.

Let $G = X \bigtriangledown H_{\omega_1+1}$. Since X and H_{ω_1+1} are C_{ω_1} groups, by ([13], Proposition 4), so is G. This implies that G_α is p^α-projective for all $\alpha < \omega_1$. Next, since H_{ω_1+1} is p^{ω_1+1}-projective, by ([6], Theorem 82), $G_{\omega_1+1} \cong G$ is also p^{ω_1+1}-projective. Together, this means that G is strongly 1-totally projective.

On the other hand, it is clear that the summability of X implies that there is a direct sum of copies of X such that $\oplus_I X[p]$ is isometric to $Y[p]$, where Y is a dsc group. Observe that $Y \bigtriangledown H_{\omega_1+1}$ is a dsc group (see, for instance, [13], Theorem 1), and since the torsion product behaves well with respect to socles and heights, we can conclude that $(\oplus_I X \bigtriangledown H_{\omega_1+1})[p] \cong \oplus_I G$ is isometric to $(Y \bigtriangledown H_{\omega_1+1})[p]$. Since the later is a free valuated vector space, it follows that $\oplus_I G$, and hence G itself, is summable.

There is a p^{ω_1+1}-pure exact sequence

$$0 \to X \bigtriangledown M_{\omega_1+1} \to X \bigtriangledown H_{\omega_1+1}(= G) \to X \to 0.$$

If G were a dsc group, it would follow that it is p^{ω_1}-projective. By Lemma 2.1(g), we could conclude that $G \cong X \oplus (X \bigtriangledown M_{\omega_1+1})$, which would imply that X is a dsc group. Since X is not a dsc group, G must not be a dsc group, either. □

5. Some open problems

In what follows, G is a group and λ is an ordinal. The following is clearly important.

Problem 5.1. Do the m,n-simply presented groups have the $\lambda+m$-Nunke property?

Using Theorem 1.8, as in the proof of Theorem 1.12, it suffices to consider the case where $p^{\lambda+m}G$ is bounded.

Problem 5.2. Do the m,n-totally projective groups have the $\lambda+m$-Nunke property?

Problem 5.2 would be a consequence of the following, which is of independent interest.

Problem 5.3. If G is $p^{\lambda+n}$-projective, can we conclude that it is also $p^{\lambda+(n)}$-projective?

The next five questions have affirmative answers for groups of length less than ω^2.

Problem 5.4. If G is m,n-balanced projective, does it follow that it is m,n-simply presented?

In other words, is a summand of an m,n-simply presented group also m,n-simply presented?

Problem 5.5. If G is an m,n-totally projective group of countable length, does it follow that it is m,n-balanced projective?

By Example 4.9, this does not hold if $m>0$, $n=0$ and G has length ω_1.

Problem 5.6. If $n>0$, and G_1 and G_2 are strongly n-balanced projective groups such that $G_1[p^n]$ is isometric to $G_2[p^n]$, can we conclude that G_1 is isomorphic to G_2?

Note that if G_1 is n-summable, then G_2 is, as well. Hence both will be dsc groups, and therefore isomorphic.

The following generalizes a classical result about isotype subgroups of totally projective groups due to Hill.

Problem 5.7. Suppose G is an m,n-totally projective (or m,n-balanced projective or m,n-simply presented) group of countable length and A is an isotype subgroup of G. Can we conclude that A is also m,n-totally projective (or m,n-balanced projective or m,n-simply presented)?

Our next question is a weakened version of Problem 5.4.

Problem 5.8. If G is m,n-balanced projective, can we conclude that there is a subgroup $P\subseteq G[p^n]$ such that G/P is strongly m-balanced projective?

The following primarily concerns groups of uncountable lengths.

Problem 5.9. If G is an IT-group that is strongly n-totally projective, can we conclude that G is an A-group?

We close with a generalization of Proposition 3.6.

Problem 5.10. Suppose G is strongly 1-simply presented. Can we write $G \cong H \oplus (\oplus_\lambda A_\lambda)$, where λ ranges over the limit ordinals such that each A_λ is a C_λ group of length λ, and H is totally projective?

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