On magic graphs

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Abstract

A (p, q)-graph $G = (V, E)$ is said to be magic if there exists a bijection $f : V \cup E \to \{1, 2, 3, \ldots, p + q\}$ such that for all edges uv of G, $f(u) + f(v) + f(uv)$ is a constant. The minimum of all constants say, $m(G)$, where the minimum is taken over all such bijections of a magic graph G, is called the magic strength of G. In this paper we define the maximum of all constants say, $M(G)$, analogous to $m(G)$, and introduce strong magic, ideal magic, weak magic labelings, and prove that some known classes of graphs admit such labelings.

1 Introduction

For all standard notation and terminology in graph theory we follow [4]. *Graph labelings* where the vertices are assigned real values subject to certain conditions, have often been motivated by practical problems, but they are also of logico-mathematical interest in their own right. An enormous body of literature has grown around the subject, especially in the last thirty years or so, and is presented in a survey [3].

A (p, q)-graph $G = (V, E)$ is said to be *magic* if there exists a bijection $f : V \cup E \to \{1, 2, 3, \ldots, p + q\}$ such that for all edges uv of G, $f(u) + f(v) + f(uv)$ is a constant (see [6]). Such a bijection is called a *magic labeling* of G.

For any *magic labeling* f of G, there is a constant $c(f)$ such that for all edges uv of G, $f(u) + f(v) + f(uv) = c(f)$. The *magic strength* $m(G)$, is defined as the minimum of $c(f)$ where the minimum is taken over all magic labelings of G.

The *magic strength*, $m(G)$ of the following graphs has already been established.

1. $m(P_{2n}) = 5n + 1$ and $m(P_{2n+1}) = 5n + 3$ where P_{2n} is a path with $2n$ vertices (see [9, 10]).

2. $m(B_{n,n}) = 5n + 6$ where $B_{n,n}$ is the graph obtained from two copies of $K_{1,n}$ by joining the vertices of maximum degree by an edge which is called a *bistar* (see [9]).

3. $m(C_{2n}) = 5n + 4$, $m(C_{2n+1}) = 5n + 2$, where C_n is a cycle of length n (see [9, 10]).

4. $m(K_{1,n}) = 2n + 4$ (see [2, 9, 10]).

5. $m((2n+1)P_2) = 9n+6$, where $(2n+1)P_2$ is the disjoint union of $2n+1$ copies of P_2. (Note that $2nP_2$ is not magic) (see [6, 9]).

Also some new constructions of magic graphs have been established [1, 7, 8].

We call $m(G)$ the *minimum magic strength* of a magic graph G, and analogously we define the *maximum magic strength*, $M(G)$, as the maximum of all $c(f)$. That is, $M(G) = \max\{c(f) : f \text{ is a magic labeling of } G\}$. Clearly for any magic labeling f of a (p, q)-graph G, we get

$$p + q + 3 \leq m(G) \leq c(f) \leq M(G) \leq 2(p + q). \tag{1}$$

In this paper we introduce *strong magic*, *ideal magic*, and *weak magic* labelings of graphs and study these parameters for some well-known graphs.

The above three notions decompose the set $\mathbf{M}$, of all magic graphs, into three mutually disjoint subsets whose union is $\mathbf{M}$.

A magic graph G is said to be

1. *strong magic* if $m(G) = M(G)$,

2. *ideal magic* if $1 \leq M(G) - m(G) \leq p$, and

3. *weak magic* if $M(G) - m(G) > p$.

2 Strong, Ideal and Weak magic graphs

In this section we study some trees, cycles, cycle related graphs $W_\circ(t, 3)$, etc. for their strong magic, ideal magic and weak magic nature. Also we construct some weak magic graphs.

The *crown* $C_n \odot K_1$, is the graph obtained from a cycle C_n by attaching a pendant edge at each vertex of the cycle. The *web graph* without center $W_\circ(2, n)$ is the graph

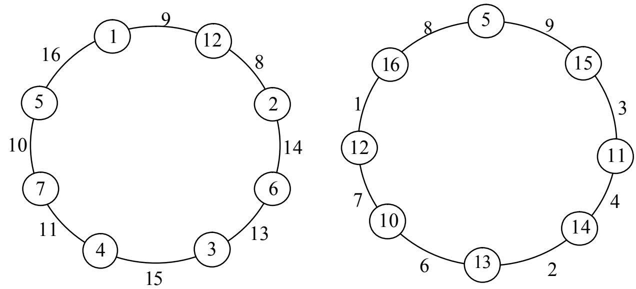

Figure 1: A primal magic labeling and dual magic labeling of C_8

obtained from $C_n \odot K_1$ by joining the pendant vertices to form the cycle and then adding a single pendant edge at each vertex of the outer cycle. The *generalized web graph without center* $W_\circ(t,n)$ is the graph obtained by iterating the process of constructing $W_\circ(2,n)$ from $C_n \odot K_1$ so that the web has exactly t cycles. We prove that the graph $W_\circ(t,3)$, is weak magic for all $t \geq 1$.

The following theorem gives a relation between the magic strengths $m(G)$ and $M(G)$ of any magic graph G.

Theorem 2.1 ([10]): *A (p,q)-graph G is magic with minimum magic strength $m(G)$ if and only if it is magic with maximum magic strength $M(G) = 3(p+q+1) - m(G)$.*

Remark 2.2: Let G be a magic graph. Then for every magic labeling f of G, we obtain another magic labeling g of G. We call f, a *primal magic labeling* of G and g, the *dual magic labeling* of G with respect to f. One can see that the dual of the dual of any primal magic labeling of G is itself. For example, a primal magic labeling and its dual magic labeling of C_8 using Theorem 2.1, are illustrated in Figure 1. (Also $m(C_8) = 22$ and $M(C_8) = 29$.)

Therefore for every magic labeling f of a magic graph G there exists a positive integer k such that $c(f) = p + q + 2 + k$ and $c(\text{dual of } f) = 2p + 2q + 1 - k$. From Theorem 2.1 and (1) at least one k is such that $k \leq (p + q - 1)/2$. Hence we have the following corollary.

Corollary 2.3: *If the (p,q)-graph G is magic then there exists at least one magic labeling f of G such that $c(f) \leq (3p + 3q + 5)/2$.*

Theorem 2.4: *A path P_n of n vertices $(n > 1)$, is strong magic if and only if $n = 2$. Further for all $n \neq 2$, P_n is ideal magic.*

Proof. Let P_n be a strong magic path of n vertices. When n is odd, we have from [9] that $m(P_n) = m(P_{2k+1}) = 5k + 3$. Then by Theorem 2.1, $M(P_n) = M(P_{2k+1}) = 7k + 3$. For P_n to be strong magic, $m(P_n) = M(P_n)$, that is $5k + 3 = 7k + 3$ which implies $k = 0$. Therefore for odd n, P_n is not strong magic.

When n is even, $m(P_n) = m(P_{2k}) = 5k+1$, and $M(P_n) = M(P_{2k}) = 7k-1$. Now $5k+1 = 7k-1$ holds only when $k = 1$. The converse is obvious as P_2 is magic with $m(P_2) = M(P_2) = 6$.

Further, for odd n, $M(P_n) - m(P_n) = 2k < 2k+1$, and for even n, $M(P_n) - m(P_n) = 2k-2 < 2k$. Hence all P_n except P_2 are ideal-magic. ■

Theorem 2.5: *The star $K_{1,n}$ is*

1. *strong magic for $n = 1$,*
2. *ideal magic for $n = 2, 3$, and*
3. *weak magic for $n > 3$.*

Proof. Note that $m(K_{1,n}) = 2n+4$ and by Theorem 2.1, $M(K_{1,n}) = 4n+2$. Hence the theorem follows because $M(K_{1,n}) - m(K_{1,n}) = 2n-2$. ■

Theorem 2.6: *The graph n-bistar $B_{n,n}$ is ideal-magic for all $n \geq 1$.*

Proof. Since $m(B_{n,n}) = 5n+6$ and $M(B_{n,n}) = 7n+6$, the proof is trivial. ■

Analogously one can prove the following two theorems.

Theorem 2.7: *The graph ${}_{(2n+1)}P_2$, is strong magic for all $n \geq 0$.*

(Note that $m({}_{(2n+1)}P_2) = 9n+6 = M({}_{(2n+1)}P_2))$.

Theorem 2.8: *All cycles are ideal-magic.*

(Note that $m(C_{2n}) = 5n+4$, $m(C_{2n+1}) = 5n+2$, $M(C_{2n}) = 7n+1$, and $M(C_{2n+1}) = 7n+5$.)

Theorem 2.9: *Let $(u_i, w_i, v_{i,1}, v_{i,2}, \ldots, v_{i,n})$, $1 \leq i \leq t$ be a collection of t disjoint graphs $P_2 + \overline{K_n}$ such that $\deg(u_i) = \deg(w_i) = n+1$ and $\deg(v_{i,j}) = 2$, $1 \leq j \leq n$. Then the graph $G = (V, E)$ obtained by joining $v_{i,n}$ to u_{i+1}, $v_{i+1,1}$ and $v_{i+1,2}$, for $1 \leq i \leq t-1$ is weak magic for all integers $t > 1$.*

Proof. Define a labeling $f : V \cup E \to \{1, 2, 3, \ldots, (3n+6)t - 3\}$ by

$$\begin{cases} f(u_i) = ni - n + 2i - 1, & 1 \leq i \leq t \\ f(w_i) = ni + 2i, & 1 \leq i \leq t \\ f(v_{i,j}) = f(u_i) + j, & 1 \leq j \leq n. \\ f(uv) = (3n+6)t - (f(u) + f(v)), & \text{for all } uv, \text{ where } uv \text{ is an edge of } G. \end{cases}$$

Then one easily checks that f so defined is a magic labeling of the graph G. Now $c(f) = (3n+6)t$ and therefore $m(G) \leq c(f) = (3n+6)t$. But for any magic (p,q)-graph G, $p+q+3 \leq m(G)$ which implies $(3n+6)t \leq m(G)$.

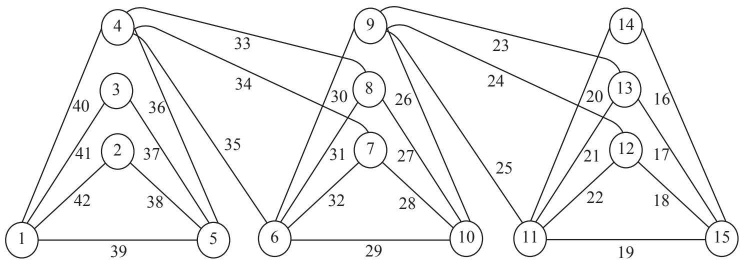

Figure 2: A magic labeling of the graph G when $t = 3$ and $n = 3$, using Theorem 2.9.

Therefore we get, $m(G) = (3n+6)t$ and hence by Theorem 2.1, we have $M(G) = (6n+12)t-6$. Clearly $M(G) - m(G) = (3n+6)t - 6 > (n+2)t$ for all $t > 1$. (Note that $(n+2)t$ is the number of vertices of G.) Hence G is weak magic for all $t > 1$. ■

For example, a magic labeling of G when $t = 3$ and $n = 3$, using Theorem 2.9, is illustrated in Figure 2. (Minimum magic strength $m(G) = 45$.)

Theorem 2.10: *The graph* $W_\circ(t,3)$*, is weak-magic for all* $t \geq 1$.

Proof. Name the vertices of the innermost cycle of $W_\circ(t,3)$ successively as $v_{1,1}, v_{1,2}, v_{1,3}$. Label the vertices adjacent to $v_{1,1}$, $v_{1,2}$, $v_{1,3}$ on the second cycle as $v_{2,3}, v_{2,1}, v_{2,2}$ respectively and the vertices adjacent to $v_{2,3}, v_{2,1}, v_{2,2}$ on the third cycle as $v_{3,2}$, $v_{3,3}$, $v_{3,1}$ and so on, the vertices adjacent to $v_{i,x}, v_{i,y}, v_{i,z}$ on the $(i+1)^{\text{th}}$ cycle as $v_{i+1,z}$,$v_{i+1,x}$, $v_{i+1,y}$.

Define a labeling $f : V(W_\circ(t,3)) \cup E(W_\circ(t,3)) \to \{1, 2, 3, \ldots, 9t+3\}$ by

$$\begin{cases} f(v_{i,j}) = 3(i-1) + j, & 1 \leq i \leq t+1, j = 1, 2, 3 \\ f(v_{i,j}v_{m,n}) = 9t + 6 - (f(v_{i,j}) + f(v_{m,n})), & \text{where } v_{i,j}v_{m,n} \text{ is an edge.} \end{cases}$$

That is, $v_{1,1}$, $v_{1,2}$, $v_{1,3}$,$v_{2,1}$, $v_{2,2}$, $v_{2,3}$, ... are labeled respectively $1, 2, 3, 4, 5, 6, \ldots$ and every edge is labeled by subtracting the sum of the labels of its end vertices from $9t+6$. Clearly one can verify that f defined above is a magic labeling of $W_\circ(t,3)$.

Since $c(f) = 9t+6$, $m(W_\circ(t,3)) \leq c(f) = 9t+6$. We have, for any magic (p,q)-graph G, $p+q+3 \leq m(G)$.

Therefore we get $9t+6 \leq m(W_\circ(t,3))$ and hence $m(W_\circ(t,3)) = 9t+6$. By Theorem 2.1, we have $M(W_\circ(t,3)) = 18t+6$. Clearly $M(W_\circ(t,3)) - m(W_\circ(t,3)) = 9t > 3t+3$ for all $t \geq 1$. Hence $W_\circ(t,3)$ is weak magic for all $t \geq 1$. ■

For example, a magic labeling of $W_\circ(4,3)$, with minimum magic strength $m(W_\circ(4,3)) = 42$, using Theorem 2.10, is illustrated in Figure 3.

Theorem 2.11: *Let* $(u_i, v_i, w_i, x_i, y_i, z_i)$*,* $1 \leq i \leq t$ *be a collection of* t *disjoint* 3*-regular graphs with six vertices such that* u_i, v_i, w_i*, are adjacent respectively to*

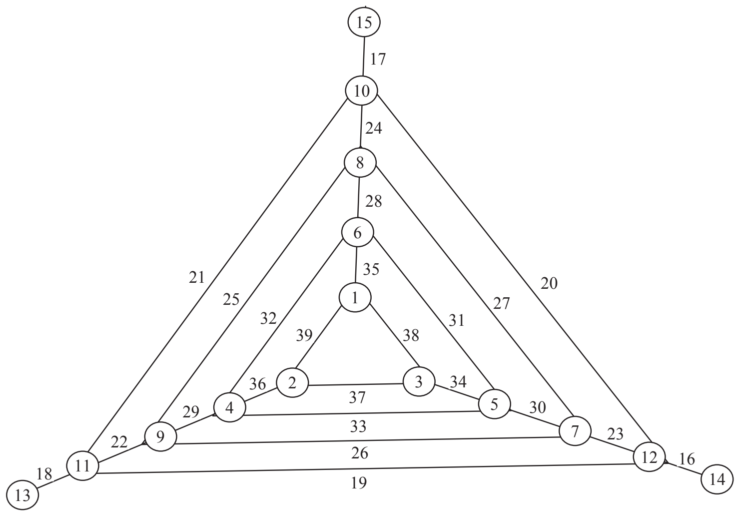

Figure 3: A magic labeling of $W_{\circ}(4,3)$ using Theorem 2.10.

x_i, y_i, z_i, $1 \leq i \leq t-1$. *Then the graph* $G = (V, E)$ *obtained by joining* z_i *to* u_{i+1}, v_{i+1} *and* w_{i+1}, $1 \leq i \leq t-1$ *is weak magic for all integers* $t \geq 1$.

Proof. Define a labeling $f : V \cup E \to \{1, 2, 3, \ldots, 18t-3\}$ by

$$\begin{cases} f(u_i) = 6i-4, & \\ f(v_i) = 6i-3, & \\ f(w_i) = 6i-5, & \\ f(x_i) = 6i, & \\ f(y_i) = 6i-2, & \\ f(z_i) = 6i-1, & 1 \leq i \leq t, \\ f(uv) = 18t - (f(u) + f(v)), & \text{where } uv \text{ is an edge of the graph } G. \end{cases}$$

Then one easily checks that f so defined is a magic labeling of the graph G and $c(f) = 18t$. Therefore $m(G) \leq c(f) = 18t$ and then $m(G) = 18t$.

Now by Theorem 2.1, $M(G) = 36t - 6$. Clearly $M(G) - m(G) = 18t - 6 > 6t$ for all $t \geq 1$.

Hence G is weak magic for all $t \geq 1$. ■

For example, a magic labeling of the graph G defined in Theorem 2.11 when $t = 3$, is illustrated in Figure 4. (Minimum magic strength $= m(G) = 54$).

3 An observation

In this section we propose a conjecture, about caterpillars. A *caterpillar* is a tree, the deletion of whose pendant vertices results in a path.

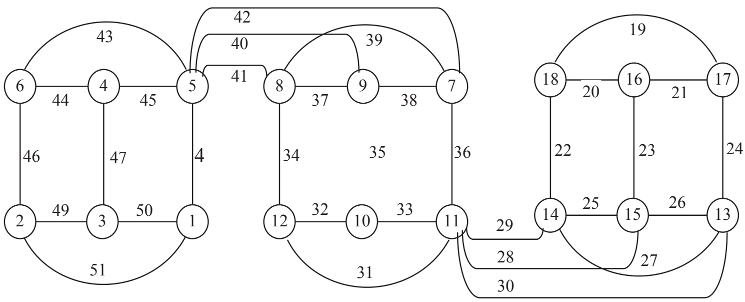

Figure 4: A magic labeling of the graph G when $t = 3$, using Theorem 2.11.

Let $C_{(a,b)}$ be a caterpillar with bipartition $\{A, B\}$ of its vertex set $V(C_{(a,b)})$, where $A = \{u_1, u_2, \ldots, u_a\}$ and $B = \{v_1, v_2, \ldots, v_b\}$. It is well known that every caterpillar $C_{(a,b)}$ has a plane representation, such as ones that are shown in Figure 5, with the a-vertices u_i appearing from the top to bottom on the left hand side column and the b-vertices v_i appearing from the top to bottom on the right hand side column. Note that a, $b > 1$ and also without loss of generality one can assume $a \leq b$.

We now define four different magic labelings of $C_{(a,b)}$ as follows.

1. $f : V(C_{(a,b)}) \cup E(C_{(a,b)}) \to \{1, 2, 3, \ldots, 2(a+b)-1\}$ by

$$\begin{cases} f(u_i) = i, & i \in \{1, 2, \ldots, a\}, \\ f(v_i) = a + i, & i \in \{1, 2, \ldots, b\}, \\ f(u_i v_j) = 2(a+b) - (i+j-1), & \text{for all } i, j \text{ where } u_i v_j \text{ is an edge.} \end{cases}$$

2. Dual of f.

3. $g : V(C_{(a,b)}) \cup E(C_{(a,b)}) \to \{1, 2, 3, \ldots, 2(a+b)-1\}$ by

$$\begin{cases} g(v_i) = i, & i \in \{1, 2, \ldots, b\}, \\ g(u_i) = b + i, & i \in \{1, 2, \ldots, a\}, \\ g(u_i v_j) = 2(a+b) - (i+j-1), & \text{for all } i, j \text{ where } u_i v_j \text{ is an edge.} \end{cases}$$

4. Dual of g.

Also, one can observe that $c(f) = 3a + 2b + 1$, $c(\text{dual of } f) = 3a + 4b - 1$, $c(g) = 2a + 3b + 1$, and $c(\text{dual of } g) = 4a + 3b - 1$.

On the basis of the above observation we propose the following conjecture.

Conjecture 3.1: *The caterpillar $C_{(a,b)}$, $a \leq b$ is*

1. *ideal magic if $b = a, a+1, a+2$.*

2. *weak magic if $b > a + 2$.*

Magic labelings f and g as defined above, for a caterpillar $C_{(4,6)}$, are illustrated in Figure 5.

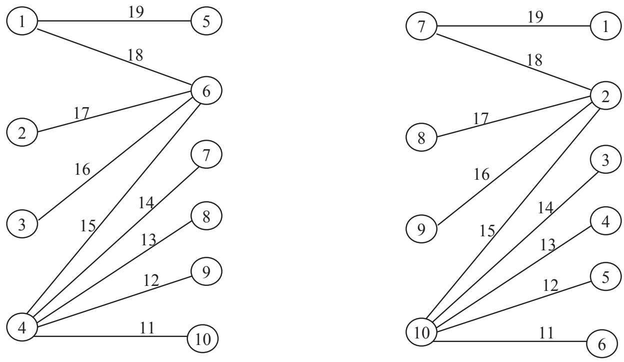

Figure 5: Magic labelings of a caterpillar $C_{(4,6)}$

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