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On magnitude, asymptotics and duration of drawdowns for Lévy models

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This paper considers magnitude, asymptotics and duration of drawdowns for some Lévy processes. First, we revisit some existing results on the magnitude of drawdowns for spectrally negative Lévy processes using an approximation approach. For any spectrally negative Lévy process whose scale functions are well-behaved at $0+$, we then study the asymptotics of drawdown quantities when the threshold of drawdown magnitude approaches zero. We also show that such asymptotics is robust to perturbations of additional positive compound Poisson jumps. Finally, thanks to the asymptotic results and some recent works on the running maximum of Lévy processes, we derive the law of duration of drawdowns for a large class of Lévy processes (with a general spectrally negative part plus a positive compound Poisson structure). The duration of drawdowns is also known as the "Time to Recover" (TTR) the historical maximum, which is a widely used performance measure in the fund management industry. We find that the law of duration of drawdowns qualitatively depends on the path type of the spectrally negative component of the underlying Lévy process.

Keywords: asymptotics; drawdown; duration; Lévy process; magnitude; parisian stopping time

1. Introduction

Drawdowns relate to an investor's sustained loss from a market peak. It is one of the most frequently quoted indices for downside risks in the fund management industry. Drawdown quantities appear in performance measures such as the Calmar ratio, the Sterling ratio, the Burke ratio, and others; see, for example, Schuhmacher and Eling [34] for a collection of such drawdown-based performance measures. Furthermore, drawdown problems have drawn considerable theoretical and practical interest in various research areas including probability, finance, risk management, and statistics; see Section 1.1 for a brief literature review.

In this paper, we consider a one-dimensional Lévy process $X = \{X_t, t \geq 0\}$ defined on $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$, a filtered probability space satisfying the usual conditions. The drawdown process of X is defined as

$$Y_t = M_t - X_t, \qquad t \geq 0,$$

where $M_t = \sup_{0\leq u\leq t} X_u$ is the running maximum (historical peak) of X at time t. Let

$$\tau_a = \inf\{t \geq 0 : Y_t > a\},$$

be the first time the magnitude of drawdowns exceeds a pre-specified threshold $a > 0$. Given that $(\max_{0\le s\le t} Y_s > a) = (\tau_a < t)$ $\mathbb{P}$-a.s., the distributional study of the maximum drawdown in magnitude is equivalent to the study of the stopping time τ_a.

However, from a risk management standpoint, the magnitude itself is not sufficient to provide a comprehensive risk evaluation of extreme drawdown risks. For instance, for extreme risks such as tornado and flooding, it is natural to also investigate the frequency and the duration of drawdowns. Landriault *et al.* [22] recently studied the frequency of drawdowns for a Brownian motion process by defining two types of drawdown time sequences depending on whether a historical running maximum is reset or not. In this paper, we will consider the duration of drawdowns, also known as "Time to Recover" (TTR) the historic running maximum in the fund management industry.

Mathematically, the duration of drawdowns of a stochastic process can be considered as the length of excursions from its running maximum. For $t \ge 0$, let $G_t := \sup\{0 \le s \le t : Y_s = 0\}$ be the last time the process Y is at level 0 (or equivalently $X = M$) before or at time t. The drawdown duration at time t is therefore $t - G_t$. We then define a stopping time

$$\eta_b = \inf\{t \ge b : t - G_t \ge b\}, \tag{1.1}$$

that is the first time the duration of drawdowns exceeds a pre-specified time threshold $b > 0$. Equivalently, the event $(\eta_b > t)$ implies that the maximum duration of drawdowns before time t is shorter than b.

The stopping time η_b is related to the so-called Parisian time, which is the first time the length of excursions from a fixed spatial level (rather than its running maximum) exceeds a pre-specified time threshold; see, for example, Chesney *et al.* [9] and Czarna and Palmowski [11]. Further, Loeffen *et al.* [26] provided an unified proof to derive the probability that the Parisian time occurs in an infinite time horizon (known as the Parisian ruin probability in actuarial science) for spectrally negative Lévy processes. Notice that, in contrast to the Parisian time, the stopping time η_b is almost surely finite (e.g., page 105 of Bertoin [4]), which motivates us to study the Laplace transform (LT) of η_b in this paper. Another related concept is the so-called red period of the insurance surplus process; see Kyprianou and Palmowski [21]. The red period corresponds to the length of time an insurance surplus process shall take to recover its deficit at ruin. But it is different than the distributional study of η_b, especially when X has no negative jumps (e.g., Brownian motion).

1.1. Literature review on drawdowns

Taylor [36] first derived the joint Laplace transform of τ_a and M_{τ_a} for Brownian motion processes. Later on, it was generalized by Lehoczky [24] to time-homogeneous diffusion processes. Douady *et al.* [13] and Magdon *et al.* [28] derived infinite series expansions for the distribution of τ_a for a standard and drifted Brownian motion, respectively. For spectrally negative Lévy processes, Mijatović and Pistorius [29] obtained a general sextuple formula for the joint Laplace transform of τ_a and the last passage time at level M_{τ_a} prior to τ_a, together with the joint distribution of the running maximum, the running minimum, and the overshoot of Y at τ_a. Also, an extensive body of literature exists on the dual of drawdowns, drawups, which measure the increase in value of an underlying process from its running minimum; see, for instance, Pistorius [30], Hadjiliadis and Večeř [16], Pospisil *et al.* [32], and Zhang and Hadjiliadis [38,39].

In finance and risk management, researchers have devoted considerable effort in assessing, managing, and reducing drawdown risks. For instance, Grossman and Zhou [15] examined a portfolio selection problem subject to drawdown constraints. Cvitanic and Karatzas [10] extended the discussion to multiple assets. Chekhlov *et al.* [8] proposed a new family of risk measures and studied problems of parameter selection and portfolio optimization under the new measures. Pospisil and Večeř [31] invented a new class of Greeks to examine the sensitivity of investment portfolios to drawdowns. Carr *et al.* [5] designed some European-style digital drawdown insurance contracts and proposed semi-static hedging strategies using barrier options and vanilla options. Other recent works on drawdown insurance are Zhang *et al.* [37,40], among others.

In addition, a few a priori unrelated problems in finance and insurance are also closely connected to the drawdown problematic. For instance, the pricing of Russian options (e.g., Shepp and Shiryaev [35], Asmussen *et al.* [1] and Avram *et al.* [2]), and the optimal dividend models with "reflecting barriers" (e.g., Avram et al. [3], Kyprianou and Palmowski [21], and Loeffen [27]) are two common examples.

1.2. Objective and structure

In this article, we begin by developing an approximation technique in the spirit of Lehoczky [24] to revisit several known LT results on the magnitude of drawdowns of spectrally negative Lévy processes via basic fluctuation identities.

Second, as the threshold of drawdown magnitude $a \downarrow 0$, we examine the asymptotic behavior of those LTs for any spectrally negative Lévy process whose scale functions are well-behaved at $0+$ (see Assumption 4.1 below). We also show that such asymptotics are robust with respect to the perturbation of arbitrary positive compound Poisson jumps, and hence obtain the asymptotics of drawdown estimates for a class of Lévy models with two-sided jumps.

Finally, we study the duration of drawdowns via the LT of η_b. First, an approximate scheme for the LT of η_b is developed. To obtain a well-defined limit, we turn our problem to the behavior of the density of the running maximum process M and the convergence of some potential measure of the drawdown process Y. Thanks to the asymptotic results obtained and some recent works on the distribution of the running maxima of Lévy processes (e.g., Chaumont [6], Chaumont and Małecki [7], and Kwaśnicki *et al.* [19]), we obtain the law of η_b in terms of the right tail of the ascending ladder time process for a class of Lévy process with two-sided jumps (a general spectrally negative part plus a positive compound Poisson structure).

The rest of the paper is organized as follows. In Section 2, we review the scale function of spectrally negative Lévy processes and the ascending ladder process of a general Lévy process. In Section 3, we revisit some known LT results on the magnitude of drawdowns of spectrally negative Lévy processes based on an approximation approach. The asymptotic behavior of these LTs for small threshold is studied in Section 4, where we also examine the asymptotic behavior in the presence of positive compound Poisson jumps. In Section 5, the LT of η_b is derived for a large class of Lévy processes with two-sided jumps. Some explicit examples are presented in Section 6. For completeness, some results on the extended continuity theorem are presented in the Appendix.

2. Preliminaries

In this section, we briefly introduce some preliminary results for Lévy processes. Readers are referred to Bertoin [4] and Kyprianou [20] for a more detailed background.

For ease of notation, throughout the paper, we let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{H}^+ = \{s \in \mathbb{C} : \mathrm{Re}(s) \geq 0\}$. We denote by $\mathbb{P}_x$ the law of a Lévy process with $X_0 = x \in \mathbb{R}$. For brevity, we write $\mathbb{P} = \mathbb{P}_0$. The minimum of real numbers u, v is denoted by $u \wedge v = \min\{u, v\}$. For a function $f(\cdot)$ on $(0, \infty)$ and $x_0 \in [0, \infty]$, we write $f(x) = o(g(x))$ as $x \to x_0$ for a positive function $g(\cdot)$ if $\lim_{x\to x_0} f(x)/g(x) = 0$.

2.1. Spectrally negative Lévy processes and scale functions

Consider a spectrally negative Lévy process $X = \{X_t, t \geq 0\}$. Throughout the paper, we assume that $|X|$ is not a subordinator and hence 0 is regular for $(0, \infty)$ (see Definition 6.4 and Theorem 6.5 of Kyprianou [20] for the definition and equivalent characterizations of the regularity). The Laplace exponent of X is given by

$$\psi(s) := \frac{1}{t} \log \mathbb{E}\big[e^{sX_t}\big] = -\mu s + \frac{1}{2}\sigma^2 s^2 + \int_{(-\infty,0)} \big(e^{sx} - 1 - sx1_{\{x>-1\}}\big)\Pi(\mathrm{d}x), \tag{2.1}$$

for every $s \in \mathbb{H}^+$. Here, $\sigma \geq 0$ and the Lévy measure $\Pi(\mathrm{d}x)$ is supported on $(-\infty, 0)$ with

$$\int_{(-\infty,0)} \big(1 \wedge x^2\big)\Pi(\mathrm{d}x) < \infty.$$

It is known that X has paths of bounded variation if and only if $\int_{(-1,0)} |x|\Pi(\mathrm{d}x) < \infty$ and $\sigma = 0$. In this case, we can rewrite (2.1) as

$$\psi(s) = sd + \int_{(-\infty,0)} \big(e^{sx} - 1\big)\Pi(\mathrm{d}x), \qquad s \geq 0, \tag{2.2}$$

where the drift $d := -\mu + \int_{(-1,0)} |x|\Pi(\mathrm{d}x) > 0$ as $|X|$ is not a subordinator. For any given $q \geq 0$, the equation $\psi(s) = q$ has at least one positive solution, and we denote the largest one by $\Phi(q)$.

It is well known that $\{e^{cX_t - \psi(c)t}, t \geq 0\}$ is a martingale for any $c \geq 0$. This gives rise to the change of measure

$$\frac{\mathrm{d}\mathbb{P}^c}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_t} = e^{cX_t - \psi(c)t}, \qquad t \geq 0. \tag{2.3}$$

Under the new measure $\mathbb{P}^c$, X is still a spectrally negative Lévy process, and its Laplace exponent is given by $\psi_c(s) = \psi(s + c) - \psi(c)$ for all $s \in \mathbb{C}$ such that $s + c \in \mathbb{H}^+$.

For any $q \geq 0$, the q-scale function $W^{(q)} : \mathbb{R} \mapsto [0, \infty)$ is the unique function supported on $(0, \infty)$ with Laplace transform

$$\int_{(0,\infty)} e^{-sx} W^{(q)}(x)\,\mathrm{d}x = \frac{1}{\psi(s) - q}, \qquad s > \Phi(q).$$

It is known that $W^{(q)}$ is continuous and increasing on $(0,\infty)$. Henceforth, we assume that the jump measure $\Pi(\mathrm{d}x)$ has no atom, then it follows that $W^{(q)}\in C^1(0,\infty)$ (e.g., Lemma 2.4 of Kuznetsov *et al.* [18]). Moreover, if the Gaussian coefficient $\sigma>0$ then $W^{(q)}\in C^2(0,\infty)$ for all $q\geq 0$ (e.g., Theorem 3.10 of Kuznetsov *et al.* [18]). The q-scale function $W^{(q)}$ is closely related to exit problems of the spectrally negative Lévy process X with respect to first passage times of the form

$$T_x^{+(-)}=\inf\{t\geq 0: X_t\geq(\leq)x\},\qquad x\in\mathbb{R}.$$

Two well-known fluctuation identities of spectrally negative Lévy processes are given below (e.g., Kyprianou [20], Theorem 8.1). For $q\geq 0$ and $0\leq x\leq a$, we have

$$\mathbb{E}_x\big[e^{-qT_a^+}1_{\{T_a^+<T_0^-\}}\big]=\frac{W^{(q)}(x)}{W^{(q)}(a)} \tag{2.4}$$

and

$$\mathbb{E}_x\big[e^{-qT_0^-}1_{\{T_0^-<T_a^+\}}\big]=Z^{(q)}(x)-Z^{(q)}(a)\frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{2.5}$$

where $Z^{(q)}(x)=1+q\int_0^x W^{(q)}(y)\,\mathrm{d}y$.

The following lemma gives the behavior of scale functions at $0+$ and ∞; see, for example, Lemmas 3.1, 3.2 of Kuznetsov *et al.* [18]. Relation (2.6) is from (3.13) of Egami *et al.* [14].

Lemma 2.1. *For any* $q\geq 0$,

$$W^{(q)}(0+)=\begin{cases}0, & \textit{if } \sigma>0 \textit{ or } \int_{(-1,0)}|x|\Pi(\mathrm{d}x)=\infty \ (\textit{unbounded variation}),\\ \frac{1}{d}, & \textit{otherwise } (\textit{bounded variation}),\end{cases}$$

$$W^{(q)\prime}(0+)=\begin{cases}\frac{2}{\sigma^2}, & \textit{if } \sigma>0,\\ \infty, & \textit{if } \sigma=0 \textit{ and } \Pi(-\infty,0)=\infty,\\ \frac{q+\Pi(-\infty,0)}{d^2}, & \textit{if } \sigma=0 \textit{ and } \Pi(-\infty,0)<\infty,\end{cases}$$

and

$$\lim_{x\to\infty}\frac{W^{(q)\prime}(x)}{W^{(q)}(x)}=\Phi(q). \tag{2.6}$$

2.2. The ascending ladder process of general Lévy processes

In this subsection, we consider a general Lévy process $X=\{X_t,t\geq 0\}$ characterized by its characteristic exponent

$$\Psi(s):=-\frac{1}{t}\log\mathbb{E}\big[e^{isX_t}\big]=i\mu s+\frac{1}{2}\sigma^2s^2+\int_{\mathbb{R}\setminus\{0\}}\big(1-e^{isx}+isx1_{\{|x|<1\}}\big)\Pi(\mathrm{d}x), \tag{2.7}$$

for all $s \in \mathbb{R}$. If X has bounded variation, we can rewrite (2.7) as

$$\Psi(s) = -isd + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{isx}\right)\Pi(\mathrm{d}x), \tag{2.8}$$

where the drift $d := -\mu - \int_{0<|x|<1} x\Pi(\mathrm{d}x)$.

The local time of X at its running maximum, denoted by $L = \{L_t, t \geq 0\}$, is a continuous, non-decreasing, $\mathbb{R}_+$-valued process. The inverse local time process, also known as the ascending ladder time process, is defined as $L^{-1} = \{L_t^{-1}, t \geq 0\}$ where

$$L_t^{-1} := \begin{cases} \inf\{s > 0 : L_s > t\}, & \text{if } t < L_\infty, \\ \infty, & \text{otherwise.} \end{cases}$$

The ladder height process $H = \{H_t, t \geq 0\}$ is defined as

$$H_t := \begin{cases} X_{L_t^{-1}}, & \text{if } t < L_\infty, \\ \infty, & \text{otherwise.} \end{cases}$$

The inverse local time L^{-1} corresponds to the real times at which new maxima are reached, and the ascending ladder height process H corresponds to the set of new maxima.

The bivariate process $(L^{-1}, H) = \{(L_t^{-1}, H_t), t \geq 0\}$, called the ascending ladder process of X, is a two-dimensional (possibly killed) subordinator with joint Laplace transform

$$\mathbb{E}\left[e^{-\alpha L_t^{-1} - \beta H_t} 1_{\{t<L_\infty\}}\right] = e^{-\kappa(\alpha,\beta)t}, \qquad \alpha, \beta \geq 0.$$

The joint Laplace exponent is given by

$$\kappa(\alpha, \beta) = \kappa(0, 0) + \alpha d_L + \beta d_H + \int_{(0,\infty)^2} \left(1 - e^{-\alpha x - \beta y}\right)\Lambda(\mathrm{d}x, \mathrm{d}y), \qquad \alpha, \beta \geq 0, \tag{2.9}$$

where $(d_L, d_H) \in \mathbb{R}_+^2$ and Λ is a bivariate intensity measure on $(0, \infty)^2$ satisfying

$$\int_{(0,\infty)^2} \left(1 \wedge \sqrt{x^2 + y^2}\right)\Lambda(\mathrm{d}x, \mathrm{d}y) < \infty.$$

When L^{-1} and H are independent, Λ takes the form $\Lambda(\mathrm{d}x, \mathrm{d}y) = \Lambda_L(\mathrm{d}x)\delta_0(\mathrm{d}y) + \Lambda_H(\mathrm{d}y)\delta_0(\mathrm{d}x)$ for $x, y \geq 0$. In particular, if X is a spectrally negative Lévy process, one can choose $L_t = M_t$, which implies that $L_t^{-1} = T_t^+$, $H_t = X_{T_t^+} = t$ on $\{t < L_\infty\}$, and further $\kappa(\alpha, \beta) = \Phi(\alpha) + \beta$.

By letting $\beta = 0$ in (2.9), we obtain the Laplace exponent of the ascending ladder time process,

$$-\frac{1}{t}\log \mathbb{E}\left[e^{-\alpha L_t^{-1}} 1_{\{t<L_\infty\}}\right] = \kappa(\alpha, 0) = \kappa(0, 0) + \alpha d_L + \int_{(0,\infty)} \left(1 - e^{-\alpha x}\right)\nu_L(\mathrm{d}x), \qquad \alpha \geq 0,$$

where $\nu_L(\mathrm{d}x) = \Lambda(\mathrm{d}x, (0, \infty))$ is the jump measure of L^{-1}. It follows from integration by parts that

$$\kappa(\alpha, 0) - \kappa(0, 0) = \alpha\left(d_L + \int_{(0,\infty)} e^{-\alpha x}\bar{\nu}_L(x)\,\mathrm{d}x\right), \qquad \alpha \geq 0, \tag{2.10}$$

where $\bar{\nu}_L(x) := \nu_L(x, \infty)$.

The renewal function h associated with the ladder height process H is defined as

$$h(x) := \int_{(0,\infty)} \mathbb{P}\{H_t \le x\}\,\mathrm{d}t, \qquad x \ge 0. \tag{2.11}$$

When X is a spectrally negative Lévy process, it is easily seen that $h(x) = \int_{(0,x)} e^{-\Phi(0)t}\,\mathrm{d}t$ for $x \ge 0$. We recall the follow results (see Theorem 5 in Chapter III and Theorem 19 in Chapter VI of Bertoin [4]) on the connection between the renewal function and the creeping property. Here we say X creeps across x if it enters (x, ∞) continuously.

Lemma 2.2. *The following assertions are equivalent.*

(i) $\mathbb{P}\{X$ *creeps across* $x\} > 0$ *for some* $x > 0$.
(ii) *The drift coefficient* $d_H > 0$.
(iii) *The renewal function h is absolute continuous and* h' *is bounded.*

Moreover, when these assertions hold, there is a version h' *that is continuous and positive on* $(0, \infty)$. *Finally,* $\lim_{x\downarrow 0} h'(x) = \frac{1}{d_H} > 0$ *and* $\mathbb{P}\{X$ *creeps across* $x\} = d_H h'(x)$ *for all* $x > 0$.

3. Magnitude of drawdowns revisited

In this section, we revisit some known results of the magnitude of drawdowns of spectrally negative Lévy processes via fluctuation identities and an approximation approach introduced by Lehoczky [24]. Such approach is in the spirit of the general Itô's excursion theory.

Lemma 3.1. *For* $q \ge 0$ *and* $x > 0$, *we have*

$$\mathbb{E}\big[e^{-qT_x^+} 1_{\{M_{\tau_a} \ge x\}}\big] = \exp\left\{-\frac{W^{(q)\prime}(a)}{W^{(q)}(a)}x\right\}. \tag{3.1}$$

Proof. For fixed $x > 0$ and $n \in \mathbb{N}$, let $\{s_{n,i}, i = 0, \dots, n\}$ be a sequence of increasing partitions of the interval $[0, x]$ with $0 = s_{n,0} < s_{n,1} < \cdots < s_{n,n} = x$ and such that $\Delta_n = \max_{1\le i\le n}(s_{n,i} - s_{n,i-1})$ decreases to 0 as $n \to \infty$. Using the strong Markov property of X, we propose to approximate the event $(M_{\tau_a} \ge x)$ by $\bigcap_{m=1}^n (T^+_{s_{n,i}} < T^-_{s_{n,i-1}-a} | X_0 = s_{n,i-1})$, and thus use

$$E_n := \prod_{i=1}^n \mathbb{E}\big[e^{-qT^+_{s_{n,i}}} 1_{\{T^+_{s_{n,i}} < T^-_{s_{n,i-1}-a}\}} | X_0 = s_{n,i-1}\big],$$

as an approximation of $\mathbb{E}[e^{-qT_x^+} 1_{\{M_{\tau_a} \ge x\}}]$. By (2.4), we have

$$E_n = \prod_{i=1}^n \frac{W^{(q)}(a)}{W^{(q)}(a + s_{n,i} - s_{n,i-1})} = \exp\left\{\sum_{i=1}^n \ln\left\{1 - \frac{W^{(q)}(a + s_{n,i} - s_{n,i-1}) - W^{(q)}(a)}{W^{(q)}(a + s_{n,i} - s_{n,i-1})}\right\}\right\}.$$

Since $W^{(q)} \in C^1(0,\infty)$ and is increasing on $(0,\infty)$, we have

$$\left(\frac{W^{(q)}(a+s_{n,i}-s_{n,i-1})-W^{(q)}(a)}{W^{(q)}(a+s_{n,i}-s_{n,i-1})}\right)^2 \le \left(\frac{W^{(q)}(a+\Delta_n)-W^{(q)}(a)}{W^{(q)}(a)}\right)^2 \le K(\Delta_n)^2,$$

for all $1 \le i \le n$ and some constant $K > 0$. By the fact that $-\ln(1-\varepsilon) = \varepsilon + o(\varepsilon)$ for small $\varepsilon > 0$, it follows that

$$\begin{aligned}
&\mathbb{E}\big[e^{-qT_x^+} 1_{\{M_{\tau_a} \ge x\}}\big] \\
&\quad = \lim_{n\to\infty} \exp\left\{\sum_{i=1}^n \ln\left\{1 - \frac{W^{(q)}(a+s_{n,i}-s_{n,i-1})-W^{(q)}(a)}{W^{(q)}(a+s_{n,i}-s_{n,i-1})}\right\}\right\} \\
&\quad = \lim_{n\to\infty} \exp\left\{-\sum_{i=1}^n \frac{W^{(q)}(a+s_{n,i}-s_{n,i-1})-W^{(q)}(a)}{W^{(q)}(a+s_{n,i}-s_{n,i-1})}\right\} \\
&\quad = \exp\left\{-\frac{W^{(q)\prime}(a)}{W^{(q)}(a)} x\right\},
\end{aligned}$$

which completes the proof. □

By letting $q = 0$ in (3.1), it is easy to see that M_{τ_a} follows an exponential distribution with mean $W(a)/W'(a)$. Then it follows from (3.1) that, for $q \ge 0$ and $x \ge 0$,

$$\mathbb{E}\big[e^{-qT_x^+} | M_{\tau_a} = x\big] = \exp\left\{-\left(\frac{W^{(q)\prime}(a)}{W^{(q)}(a)} - \frac{W'(a)}{W(a)}\right)x\right\}. \tag{3.2}$$

Next, we consider the following lemma which relates to downward exiting.

Lemma 3.2. *For $q, s \ge 0$, we have*

$$\mathbb{E}_a\big[e^{-qT_0^- - s(a - X_{T_0^-})} | T_0^- < T_a^+\big] = \frac{W(a)}{W'(a)} \frac{Z_s^{(p)}(a) W_s^{(p)\prime}(a) - p W_s^{(p)}(a)^2}{W_s^{(p)}(a)}, \tag{3.3}$$

where $p = q - \psi(s)$, $W_s^{(p)}$ and $Z_s^{(p)}$ are p-scale functions under $\mathbb{P}^s$.

Proof. We first consider that $s \le \Phi(q)$, or equivalently, $q \ge \psi(s)$. For $0 \le x \le y$, since $T_0^- \wedge T_y^+$ is a.s. finite, by change of measure (2.3) and (2.5),

$$\begin{aligned}
\mathbb{E}_x\big[e^{-qT_0^- - s(x - X_{T_0^-})} 1_{\{T_0^- < T_y^+\}}\big] &= \mathbb{E}_x^s\big[e^{-pT_0^-} 1_{\{T_0^- < T_y^+\}}\big] \\
&= Z_s^{(p)}(x) - Z_s^{(p)}(y) \frac{W_s^{(p)}(x)}{W_s^{(p)}(y)}.
\end{aligned} \tag{3.4}$$

It follows from (2.4) and (3.4) that

$$\begin{aligned}
&\mathbb{E}_a\big[e^{-qT_0^- - s(a - X_{T_0^-})} | T_0^- < T_a^+\big] \\
&= \lim_{\varepsilon \downarrow 0} \mathbb{E}_a\big[e^{-qT_0^- - s(a - X_{T_0^-})} | T_0^- < T_{a+\varepsilon}^+\big] \\
&= \lim_{\varepsilon \downarrow 0} \left(Z_s^{(p)}(a) - Z_s^{(p)}(a+\varepsilon) \frac{W_s^{(p)}(a)}{W_s^{(p)}(a+\varepsilon)} \right) \frac{W(a+\varepsilon)}{W(a+\varepsilon) - W(a)} \\
&= \frac{W(a)}{W'(a)} \frac{Z_s^{(p)}(a) W_s^{(p)\prime}(a) - p W_s^{(p)}(a)^2}{W_s^{(p)}(a)}.
\end{aligned} \tag{3.5}$$

The other side of the approximation $\lim_{\varepsilon \downarrow 0} \mathbb{E}_{a-\varepsilon}[e^{-qT_0^- - s(a - X_{T_0^-})} | T_0^- < T_a^+]$ also results in (3.5). The proof is then completed through an analytical extension of (3.3) to $s \geq 0$. □

To obtain the main result of this section, we notice that a sample path of X until τ_a can be splitted into two parts: the rising part and the subsequent crashing part. Because of the regularity of 0 for $(0, \infty)$, we know that the last passage time $(G_{\tau_a} | M_{\tau_a} = x) = (T_x^+ | M_{\tau_a} = x)$, $\mathbb{P}$-a.s. (see also discussions on page 158 of Kyprianou [20]). Our analysis essentially follows this idea: relations (3.2) and (3.3) correspond to the rising and the crashing part, respectively. The following quadruple LT is obtained by pasting these two parts at the turning point G_{τ_a}.

Theorem 3.1. *For $q, r, s, \delta \geq 0$, we have*

$$\mathbb{E}\big[e^{-q\tau_a - rG_{\tau_a} - sY_{\tau_a} - \delta M_{\tau_a}}\big] = \frac{W^{(q+r)}(a)}{\delta W^{(q+r)}(a) + W^{(q+r)\prime}(a)} \frac{Z_s^{(p)}(a) W_s^{(p)\prime}(a) - p W_s^{(p)}(a)^2}{W_s^{(p)}(a)}, \tag{3.6}$$

where $p = q - \psi(s)$.

Proof. By conditioning on the event $(M_{\tau_a} = x)$ for $x > 0$, we have $\tau_a = G_{\tau_a} + T_{x-a}^- \circ \theta_{G_{\tau_a}}$ and $T_{x-a}^- \circ \theta_{G_{\tau_a}} < T_x^+ \circ \theta_{G_{\tau_a}}$, $\mathbb{P}$-a.s. where θ is the Markov shift operator defined as $X_t \circ \theta_s = X_{t+s}$. Therefore, by (3.2) and (3.3), we obtain

$$\begin{aligned}
&\mathbb{E}\big[e^{-q\tau_a - rG_{\tau_a} - sY_{\tau_a}} | M_{\tau_a} = x\big] \\
&= \mathbb{E}\big[e^{-(q+r)G_{\tau_a} - q(\tau_a - G_{\tau_a}) - sY_{\tau_a}} | M_{\tau_a} = x\big] \\
&= \mathbb{E}\big[e^{-(q+r)G_{\tau_a}} \mathbb{E}\big[e^{-qT_{x-a}^- \circ \theta_{G_{\tau_a}} - s(x - X_{T_{x-a}^-})} | T_{x-a}^- \circ \theta_{G_{\tau_a}} < T_x^+ \circ \theta_{G_{\tau_a}}\big] | M_{\tau_a} = x\big] \\
&= \mathbb{E}\big[e^{-(q+r)G_{\tau_a}} | M_{\tau_a} = x\big] \mathbb{E}_x\big[e^{-qT_{x-a}^- - s(x - X_{T_{x-a}^-})} | T_{x-a}^- < T_x^+\big] \\
&= \exp\left\{ -\left(\frac{W^{(q+r)\prime}(a)}{W^{(q+r)}(a)} - \frac{W'(a)}{W(a)} \right) x \right\} \frac{W(a)}{W'(a)} \frac{Z_s^{(p)}(a) W_s^{(p)\prime}(a) - p W_s^{(p)}(a)^2}{W_s^{(p)}(a)}.
\end{aligned} \tag{3.7}$$

Multiplying (3.7) by the density of M_{τ_a} and then integrating with respect to x, we obtain (3.6). □

Relation (3.6) generalizes Theorem 1 of Avram *et al.* [1] by incorporating the joint LT of G_{τ_a} and M_{τ_a}. Moreover, by a similar approximation argument, one can solve for the joint distribution of equation (3.6) but with the law of Y_{τ_a}, which then recovers the sextuple law in Theorem 1 of Mijatović and Pistorius [29] (the running minimum at τ_a can also be easily incorporated).

4. Asymptotics of magnitude of drawdowns

In this section, we investigate the asymptotics of the LT (3.6) of the magnitude of drawdowns as $a \downarrow 0$ for spectrally negative Lévy processes. Furthermore, we show that such asymptotics are robust with respect to the perturbation by positive compound Poisson jumps.

4.1. Spectrally negative Lévy processes

Henceforth, we make the following assumption on the behavior of the scale function at 0+.

Assumption 4.1.

$$\lim_{x \downarrow 0} x W'(x) = 0.$$

In fact, since $x W'(x) \geq 0$ for all $x > 0$, as long as W' is well-behaved at 0+ in the sense that

$$\lim_{x \downarrow 0} x W'(x) = c \qquad \text{for some } c \in [0, \infty],$$

one deduces from the integrability of W' at 0+ that $c = 0$.

Remark 4.1. From Lemma 2.1, it is clear that Assumption 4.1 holds if the Gaussian component $\sigma > 0$ or $\Pi(-\infty, 0) < \infty$. Moreover, the spectrally negative α-stable process with index $\alpha \in (1, 2)$, whose Laplace exponent $\psi(s) = s^\alpha$ and scale function

$$W(x) = 1_{\{x \geq 0\}} \frac{x^{\alpha - 1}}{\Gamma(\alpha)},$$

also satisfies Assumption 4.1.

Since scale functions are only known in a few cases, we examine sufficient conditions on the Laplace exponent to identify cases when Assumption 4.1 holds.

Remark 4.2. For a general spectrally negative Lévy process with Laplace exponent ψ, by Lemma 2.1, one can choose an arbitrary $s_0 > \Phi(0)$ and define a function $g(x) := 1_{\{x>0\}} e^{-s_0 x} x W'(x)$, which is non-negative and continuous on $\mathbb{R} \setminus \{0\}$. By Lemma 3.3 of

Kuznetsov [18] and (2.6), we further know that $g(x) \in L^1(\mathbb{R})$. By integration by parts and analytical continuation, one obtains that

$$\int_{\mathbb{R}} e^{isx} g(x)\,\mathrm{d}x = \varphi(s_0 - is), \qquad s \in \mathbb{R},$$

where $\varphi(s) := \frac{s\psi'(s)-\psi(s)}{\psi(s)^2}$ for $\mathrm{Re}(s) \geq 0$. By the Fourier inversion and the dominated convergence theorem, we know that a sufficient condition for Assumption 4.1 to hold is that $\varphi(s_0 - i\cdot) \in L^1(\mathbb{R})$ as it implies that $g(\cdot)$ is continuous over $\mathbb{R}$.

Lemma 4.1. *Under Assumption 4.1, we have* $\lim_{x\downarrow 0} xW^{(q)\prime}(x) = 0$ *for every* $q \geq 0$.

Proof. Since the scale function W is supported on $(0,\infty)$, for any $k \geq 1$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} W^{*(k+1)}(x) &= \int_{(0,x)} W'(x-y)W^{*k}(y)\,\mathrm{d}y + W(0+)W^{*k}(x) \\
&\leq \frac{x^{k-1}}{(k-1)!} W^k(x)\left(\int_{(0,x)} W'(x-y)\,\mathrm{d}y + W(0+)\right) \\
&= \frac{x^{k-1}}{(k-1)!} W^{k+1}(x),
\end{aligned} \tag{4.1}$$

where the inequality above is due to equation (8.23) of Kyprianou [20] and the monotonicity of W. By (4.1) and taking derivatives term by term to the well-known identity $W^{(q)}(x) = \sum_{k=0}^{\infty} q^k W^{*(k+1)}(x)$, where W^{*k} is the kth convolution of W with itself, we obtain

$$\begin{aligned}
xW^{(q)\prime}(x) &= xW'(x) + x\sum_{k=1}^{\infty} q^k \frac{\mathrm{d}}{\mathrm{d}x} W^{*(k+1)}(x) \\
&\leq xW'(x) + qxW^2(x)\sum_{k=1}^{\infty} \frac{(qxW(x))^{k-1}}{(k-1)!} \\
&= xW'(x) + qxW^2(x)e^{qxW(x)}.
\end{aligned}$$

This ends the proof as the right-hand side of the last equation approaches 0 as $x \downarrow 0$ by Assumption 4.1. $\square$

Lemma 4.1 is paramount to derive the following asymptotic results.

Theorem 4.1. *Consider a spectrally negative Lévy process X satisfying Assumption 4.1. For any $q, s \geq 0$, we have*

$$\lim_{\varepsilon\downarrow 0} \frac{W^{(q)\prime}(\varepsilon)}{W^{(q)}(\varepsilon)}\left(1 - \mathbb{E}\left[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}}\right]\right) = \begin{cases} s, & \text{if } X \text{ has unbounded variation,} \\ s + \dfrac{q-\psi(s)}{d}, & \text{if } X \text{ has bounded variation.} \end{cases}$$

Proof. Using (3.6), one deduces that

$$\begin{aligned}
&\frac{W^{(q)\prime}(\varepsilon)}{W^{(q)}(\varepsilon)}\big(1-\mathbb{E}\big[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}}\big]\big) \\
&\quad = s - \big(q-\psi(s)\big)\frac{W^{(q)\prime}(\varepsilon)}{W^{(q)}(\varepsilon)}\int_{(0,\varepsilon)} e^{-sx}W^{(q)}(x)\,\mathrm{d}x \\
&\qquad + s\big(q-\psi(s)\big)\int_{(0,\varepsilon)} e^{-sx}W^{(q)}(x)\,\mathrm{d}x + \big(q-\psi(s)\big)e^{-s\varepsilon}W^{(q)}(\varepsilon).
\end{aligned} \tag{4.2}$$

From the monotonicity of $W^{(q)}(\cdot)$, we have

$$0 \le \frac{W^{(q)\prime}(\varepsilon)}{W^{(q)}(\varepsilon)}\int_{(0,\varepsilon)} e^{-sx}W^{(q)}(x)\,\mathrm{d}x \le \frac{W^{(q)\prime}(\varepsilon)}{W^{(q)}(\varepsilon)}\varepsilon W^{(q)}(\varepsilon) = \varepsilon W^{(q)\prime}(\varepsilon).$$

It follows from (4.2) and Lemma 4.1 that

$$\lim_{\varepsilon\downarrow 0}\frac{W^{(q)\prime}(\varepsilon)}{W^{(q)}(\varepsilon)}\big(1-\mathbb{E}\big[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}}\big]\big) = s + \big(q-\psi(s)\big)W^{(q)}(0+),$$

which ends the proof by Lemma 2.1. □

4.2. A class of Lévy models with two-sided jumps

Next, we consider a class of Lévy process with two-sided jumps of the form

$$X_t = \tilde{X}_t + S_t^+, \tag{4.3}$$

where $\tilde{X}$ a spectrally negative Lévy process satisfying Assumption 4.1, and S^+ is a compound Poisson process with arrival rate $\lambda^+ = \Pi(0,\infty) \in (0,\infty)$ and i.i.d. positive jump size with distribution function F^+. The two processes $\tilde{X}$ and S^+ are assumed to be independent. Since we assume that $|\tilde{X}|$ is not a subordinator and is regular for $(0,\infty)$, it is clear that the same holds for X.

The characteristic exponent of X is given by

$$\Psi(s) = \tilde{\Psi}(s) + \lambda^+\int_0^\infty \big(1-e^{isx}\big)F^+(\mathrm{d}x), \qquad s\in\mathbb{R}, \tag{4.4}$$

where $\tilde{\Psi}(\cdot)$ is the characteristic exponent of $\tilde{X}$. Henceforth, we add the symbol $\tilde{}$ to all quantities when they relate to the spectrally negative Lévy component $\tilde{X}$ only.

By conditioning on the first positive jump arrival time and the jump size, we have the following representation of the joint Laplace transform of $(\tau_\varepsilon, Y_{\tau_\varepsilon})$.

Lemma 4.2. *For $q, s \geq 0$ and $\varepsilon > 0$, we have*

$$\mathbb{E}\big[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}}\big] = \frac{\mathbb{E}[e^{-(q+\lambda^+)\tilde{\tau}_\varepsilon - s\tilde{Y}_{\tilde{\tau}_\varepsilon}}] + \mathbb{E}[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}} 1_{\{\tau_\varepsilon > \xi_1^+, J_1^+ < Y_{\xi_1^+ -}\}}]}{1 - (\lambda^+/(q+\lambda^+))(1 - \mathbb{E}[e^{-(q+\lambda^+)\tilde{\tau}_\varepsilon}]) + \mathbb{E}[e^{-q\xi_1^+} 1_{\{\tilde{\tau}_\varepsilon > \xi_1^+, J_1^+ < \tilde{Y}_{\xi_1^+ -}\}}]}, \quad (4.5)$$

where ξ_1^+ and J_1^+ are the time and size of the first upward jump of X, respectively.

Proof. Recall that ξ_1^+ is exponentially distributed with mean $1/\lambda^+$. By the strong Markov property of X and the fact that $(\tau_\varepsilon < \xi_1^+) = (\tilde{\tau}_\varepsilon < \xi_1^+)$ a.s.,

$$\begin{aligned}
&\mathbb{E}\big[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}}\big] \\
&\quad = \mathbb{E}\big[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}} 1_{\{\tau_\varepsilon < \xi_1^+\}}\big] + \mathbb{E}\big[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}} 1_{\{\tau_\varepsilon > \xi_1^+\}}\big] \\
&\quad = \mathbb{E}\big[e^{-q\tilde{\tau}_\varepsilon - s\tilde{Y}_{\tilde{\tau}_\varepsilon}} 1_{\{\tilde{\tau}_\varepsilon < \xi_1^+\}}\big] + \mathbb{E}\big[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}} 1_{\{\tau_\varepsilon > \xi_1^+, J_1^+ \geq Y_{\xi_1^+ -}\}}\big] \\
&\qquad + \mathbb{E}\big[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}} 1_{\{\tau_\varepsilon > \xi_1^+, J_1^+ < Y_{\xi_1^+ -}\}}\big] \\
&\quad = \mathbb{E}\big[e^{-(q+\lambda^+)\tilde{\tau}_\varepsilon - s\tilde{Y}_{\tilde{\tau}_\varepsilon}}\big] + \mathbb{E}\big[e^{-q\xi_1^+} 1_{\{\tilde{\tau}_\varepsilon > \xi_1^+, J_1^+ \geq \tilde{Y}_{\xi_1^+ -}\}}\big]\mathbb{E}\big[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}}\big] \\
&\qquad + \mathbb{E}\big[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}} 1_{\{\tau_\varepsilon > \xi_1^+, J_1^+ < Y_{\xi_1^+ -}\}}\big].
\end{aligned}$$

Solving for $\mathbb{E}[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}}]$, one obtains

$$\mathbb{E}\big[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}}\big] = \frac{\mathbb{E}[e^{-(q+\lambda^+)\tilde{\tau}_\varepsilon - s\tilde{Y}_{\tilde{\tau}_\varepsilon}}] + \mathbb{E}[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}} 1_{\{\tau_\varepsilon > \xi_1^+, J_1^+ < Y_{\xi_1^+ -}\}}]}{1 - \mathbb{E}[e^{-q\xi_1^+} 1_{\{\tilde{\tau}_\varepsilon > \xi_1^+, J_1^+ \geq \tilde{Y}_{\xi_1^+ -}\}}]}. \quad (4.6)$$

For the denominator on the right-hand side of (4.6), we notice that

$$\begin{aligned}
&\mathbb{E}\big[e^{-q\xi_1^+} 1_{\{\tilde{\tau}_\varepsilon > \xi_1^+, J_1^+ \geq \tilde{Y}_{\xi_1^+ -}\}}\big] \\
&\quad = \mathbb{E}\big[e^{-q\xi_1^+}\big] - \mathbb{E}\big[e^{-q\xi_1^+} 1_{\{\tilde{\tau}_\varepsilon < \xi_1^+\}}\big] - \mathbb{E}\big[e^{-q\xi_1^+} 1_{\{\tilde{\tau}_\varepsilon > \xi_1^+, J_1^+ < \tilde{Y}_{\xi_1^+ -}\}}\big] \qquad (4.7) \\
&\quad = \frac{\lambda^+}{q+\lambda^+}\big(1 - \mathbb{E}\big[e^{-(q+\lambda^+)\tilde{\tau}_\varepsilon}\big]\big) - \mathbb{E}\big[e^{-q\xi_1^+} 1_{\{\tilde{\tau}_\varepsilon > \xi_1^+, J_1^+ < \tilde{Y}_{\xi_1^+ -}\}}\big].
\end{aligned}$$

The proof of (4.5) is completed by substituting (4.7) to (4.6). □

We present an analogue of Theorem 4.1 for the Lévy process (4.3) with two-sided jumps. Note that by (4.4), the drift of the characteristic exponent d of $\tilde{X}$ and X are the same when $\tilde{X}$ has bounded variation.

Theorem 4.2. *Consider the Lévy model* (4.3). *For* $q, s \geq 0$, *we have*

$$\lim_{\varepsilon\downarrow 0} \frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)}\big(1-\mathbb{E}\big[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}}\big]\big) = \begin{cases} s, & \text{if } \tilde{X} \text{ has unbounded variation,} \\ s + \dfrac{q-\tilde{\psi}(s)}{d}, & \text{if } \tilde{X} \text{ has bounded variation.} \end{cases}$$

Proof. Since $\tilde{X}$ and S^+ are independent and $(\tau_\varepsilon < \xi_1^+) = (\tilde{\tau}_\varepsilon < \xi_1^+)$ a.s., we have

$$\begin{aligned} \mathbb{P}\big\{\tau_\varepsilon > \xi_1^+, J_1^+ < Y_{\xi_1^+-}\big\} &= \mathbb{P}\big\{\tilde{\tau}_\varepsilon > \xi_1^+, J_1^+ < \tilde{Y}_{\xi_1^+-}\big\} \\ &\leq \mathbb{P}\big\{\tilde{\tau}_\varepsilon > \xi_1^+, J_1^+ < \varepsilon\big\} \\ &= \big(1-\mathbb{E}\big[e^{-\lambda^+\tilde{\tau}_\varepsilon}\big]\big)\mathbb{P}\big\{J_1^+ < \varepsilon\big\} \\ &\leq \big(1-\mathbb{E}\big[e^{-(q+\lambda^+)\tilde{\tau}_\varepsilon}\big]\big)F^+(\varepsilon). \end{aligned} \tag{4.8}$$

It follows from Theorem 4.1 that

$$\begin{aligned} &\frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)}\mathbb{P}\big\{\tau_\varepsilon > \xi_1^+, J_1^+ < Y_{\xi_1^+-}\big\} \\ &\quad \leq \frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)}\big(1-\mathbb{E}\big[e^{-(q+\lambda^+)\tilde{\tau}_\varepsilon}\big]\big)F^+(\varepsilon) = o(1), \end{aligned} \tag{4.9}$$

for small $\varepsilon > 0$. By (4.5), (4.8) and (4.9), one obtains that

$$\begin{aligned} &\frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)}\big(1-\mathbb{E}\big[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}}\big]\big) \\ &= \frac{\frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)}(1-\mathbb{E}[e^{-(q+\lambda^+)\tilde{\tau}_\varepsilon - s\tilde{Y}_{\tilde{\tau}_\varepsilon}}] - \frac{\lambda^+}{q+\lambda^+}(1-\mathbb{E}[e^{-(q+\lambda^+)\tilde{\tau}_\varepsilon}])) + o(1)}{1-\frac{\lambda^+}{q+\lambda^+}(1-\mathbb{E}[e^{-(q+\lambda^+)\tilde{\tau}_\varepsilon}]) + o(1)}. \end{aligned} \tag{4.10}$$

We first consider $\tilde{X}$ has unbounded variation. From Lemma 2.1, we deduce that $\frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)} \to \infty$ as $\varepsilon \downarrow 0$. By Theorem 4.1, this further implies that

$$1-\mathbb{E}\big[e^{-(q+\lambda^+)\tilde{\tau}_\varepsilon}\big] = o(1). \tag{4.11}$$

One concludes from (4.8) and (4.11) that the denominator on the right-hand side of (4.10) approaches 1 as $\varepsilon \downarrow 0$. Moreover, by Theorem 4.1,

$$\lim_{\varepsilon\downarrow 0}\frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)}\big(1-\mathbb{E}\big[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}}\big]\big) = \lim_{\varepsilon\downarrow 0}\frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)}\big(1-\mathbb{E}\big[e^{-(q+\lambda^+)\tilde{\tau}_\varepsilon - s\tilde{Y}_{\tilde{\tau}_\varepsilon}}\big]\big) = s.$$

When $\tilde{X}$ has bounded variation but the Lévy measure $\Pi(-\infty, 0) = \infty$, note that (4.11) still holds by Lemma 2.1. Hence, it follows from (4.8) that the denominator on the right-hand side of

(4.10) also approaches 1 as $\varepsilon \downarrow 0$. Furthermore, by Theorem 4.1, we obtain

$$\lim_{\varepsilon\downarrow 0} \frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)}\big(1-\mathbb{E}\big[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}}\big]\big) = s + \frac{q+\lambda^+ - \tilde{\psi}(s)}{d} - \frac{\lambda^+}{q+\lambda^+}\frac{q+\lambda^+}{d} = s + \frac{q-\tilde{\psi}(s)}{d}.$$

Finally, when $\tilde{X}$ has bounded variation and $\Pi(-\infty,0)<\infty$, by Lemma 2.1 and Theorem 4.1,

$$\lim_{\varepsilon\downarrow 0}\big(1-\mathbb{E}\big[e^{-q\tilde{\tau}_\varepsilon - s\tilde{Y}_{\tilde{\tau}_\varepsilon}}\big]\big) = \frac{q+sd-\tilde{\psi}(s)}{q+\Pi(-\infty,0)}. \tag{4.12}$$

Then, by (4.10), (4.8) and Theorem 4.1, it is straightforward to verify that

$$\lim_{\varepsilon\downarrow 0} \frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)}\big(1-\mathbb{E}\big[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}}\big]\big) = s + \frac{q-\tilde{\psi}(s)}{d},$$

which completes the proof. □

5. Duration of drawdowns

In this section, we examine the duration of drawdowns via the LT of the stopping time η_b defined in (1.1) for the Lévy model with two-sided jumps (4.3).

To do so, we use the perturbation approach which has been developed by many researchers in similar contexts (e.g., Dassios and Wu [12], Landriault *et al.* [23], Li and Zhou [25], Loeffen *et al.* [26], and Zhang [37]). To present the main idea, let $\varepsilon > 0$ and define the following sequence of stopping times:

$$\tau_\varepsilon^1 = \tau_\varepsilon, \qquad \vartheta_0^1 = \tau_\varepsilon^1 + T^+_{M_{\tau_\varepsilon^1}} \circ \theta_{\tau_\varepsilon^1}, \ldots, \tau_\varepsilon^i = \vartheta_0^i + \tau_\varepsilon \circ \theta_{\vartheta_0^i}, \qquad \vartheta_0^i = \tau_\varepsilon^i + T^+_{M_{\tau_\varepsilon^i}} \circ \theta_{\tau_\varepsilon^i},$$

for $i \in \mathbb{N}$ where we recall θ stands for the Markov shift operator. An approximation of η_b is given by

$$\eta_b^\varepsilon = \inf\big\{t \in (\tau_\varepsilon^i, \vartheta_0^i] : t - \tau_\varepsilon^i \ge b \text{ for some } i \in \mathbb{N}\big\},$$

for which only excursions of Y with height over ε are considered. By construction, it is clear that η_b^ε is monotonically decreasing as $\varepsilon \downarrow 0$, and $\eta_b = \lim_{\varepsilon\downarrow 0}\eta_b^\varepsilon$, $\mathbb{P}$-a.s.

For fixed $q > 0$, we consider an independent exponential rv $\mathbf{e}_q$ with mean $1/q$. By the strong Markov property of X,

$$\begin{aligned}\mathbb{P}\big\{\mathbf{e}_q > \eta_b^\varepsilon\big\} &= \mathbb{P}\big\{\mathbf{e}_q > \eta_b^\varepsilon, \vartheta_0^1 > \tau_\varepsilon^1 + b\big\} + \mathbb{P}\big\{\mathbf{e}_q > \eta_b^\varepsilon, \vartheta_0^1 < \tau_\varepsilon^1 + b\big\} \\ &= \mathbb{P}\big\{\mathbf{e}_q \wedge \vartheta_0^1 > \tau_\varepsilon^1 + b\big\} + \mathbb{P}\big\{\vartheta_0^1 < \mathbf{e}_q \wedge \big(\tau_\varepsilon^1 + b\big)\big\}\mathbb{P}\big\{\mathbf{e}_q > \eta_b^\varepsilon\big\},\end{aligned}$$

which yields

$$\mathbb{P}\big\{\mathbf{e}_q > \eta_b^\varepsilon\big\} = \frac{\mathbb{P}\{\mathbf{e}_q \wedge \vartheta_0^1 > \tau_\varepsilon^1 + b\}}{1-\mathbb{P}\{\vartheta_0^1 < \mathbf{e}_q \wedge (\tau_\varepsilon^1 + b)\}}. \tag{5.1}$$

By conditioning on $Y_{\tau_\varepsilon^1}$ and then using the strong Markov property of X at time τ_ε^1, we find

$$\begin{aligned}\mathbb{P}\{\vartheta_0^1 < \mathbf{e}_q \wedge (\tau_\varepsilon^1 + b)\} &= \int_{[\varepsilon,\infty)} \mathbb{E}\big[e^{-q\tau_\varepsilon} 1_{\{Y_{\tau_\varepsilon} \in dy\}}\big] \mathbb{P}\{T_y^+ < \mathbf{e}_q \wedge b\} \\ &= \mathbb{E}\big[e^{-q\tau_\varepsilon}\big] - \int_{[\varepsilon,\infty)} \mathbb{E}\big[e^{-q\tau_\varepsilon} 1_{\{Y_{\tau_\varepsilon} \in dy\}}\big] \mathbb{P}\{T_y^+ > \mathbf{e}_q \wedge b\}\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}\mathbb{P}\{\mathbf{e}_q \wedge \vartheta_0^1 > \tau_\varepsilon^1 + b\} &= \int_{[\varepsilon,\infty)} \mathbb{E}\big[e^{-q\tau_\varepsilon} 1_{\{Y_{\tau_\varepsilon} \in dy\}}\big] \mathbb{P}\{\mathbf{e}_q \wedge T_y^+ > b\} \\ &= e^{-qb} \int_{[\varepsilon,\infty)} \mathbb{E}\big[e^{-q\tau_\varepsilon} 1_{\{Y_{\tau_\varepsilon} \in dy\}}\big] \mathbb{P}\{T_y^+ > b\}.\end{aligned} \tag{5.3}$$

Substituting (5.2) and (5.3) into (5.1), we obtain

$$\mathbb{E}\big[e^{-q\eta_b^\varepsilon}\big] = \mathbb{P}\{\mathbf{e}_q > \eta_b^\varepsilon\} = \frac{e^{-qb} \int_{[\varepsilon,\infty)} \mathbb{E}[e^{-q\tau_\varepsilon} 1_{\{Y_{\tau_\varepsilon} \in dy\}}] \mathbb{P}\{T_y^+ > b\}}{1 - \mathbb{E}[e^{-q\tau_\varepsilon}] + \int_{[\varepsilon,\infty)} \mathbb{E}[e^{-q\tau_\varepsilon} 1_{\{Y_{\tau_\varepsilon} \in dy\}}] \mathbb{P}\{T_y^+ > \mathbf{e}_q \wedge b\}}. \tag{5.4}$$

From the representation (5.4), it seems relevant to define, for $x > 0$ and $p \geq 0$, a bounded auxiliary function

$$\begin{aligned} f_\varepsilon^{(p)}(t) &:= \int_{[\varepsilon,\infty)} \mathbb{E}\big[e^{-q\tau_\varepsilon} 1_{\{Y_{\tau_\varepsilon} \in dy\}}\big] \mathbb{P}\{T_y^+ > \mathbf{e}_p \wedge t\} \\ &= \int_{[\varepsilon,\infty)} \mathbb{E}\big[e^{-q\tau_\varepsilon} 1_{\{Y_{\tau_\varepsilon} \in dy\}}\big] \mathbb{P}\{M_{\mathbf{e}_p \wedge t} \leq y\},\end{aligned} \tag{5.5}$$

where the dependence of (5.5) on q is silently assumed. Hence, we rewrite (5.4) as

$$\mathbb{E}\big[e^{-q\eta_b^\varepsilon}\big] = \frac{e^{-qb} f_\varepsilon^{(0)}(b)}{1 - \mathbb{E}[e^{-q\tau_\varepsilon}] + f_\varepsilon^{(q)}(b)}. \tag{5.6}$$

To obtain a well-defined asymptotics for $f_\varepsilon^{(p)}(b)$ as $\varepsilon \downarrow 0$, the key is to investigate the convergence of the measure $\mathbb{E}[e^{-q\tau_\varepsilon} 1_{\{Y_{\tau_\varepsilon} \in dy\}}]$ as $\varepsilon \downarrow 0$, which is closely related to the asymptotic results of Section 4. As we will see below, the convergence of the measure differs according to whether the Lévy process has bounded or unbounded variation.

5.1. Bounded variation case

We first show that the distribution function of the running maximum of X is well-behaved.

Proposition 5.1. *Let X be a Lévy process of bounded variation with a drift $d > 0$ in its characteristic exponent representation* (2.8)*. Then, for any fixed $p \geq 0$ and $t > 0$, the function*

$\mathbb{P}\{M_{\mathbf{e}_p \wedge t} \le y\}/y$ is bounded for $y \in (0,\infty)$. Moreover, if we further assume that $\Pi(-\infty,0) = \infty$ and Π has no atoms on $(-\infty,0)$, the function $\mathbb{P}\{M_{\mathbf{e}_p \wedge t} \le y\}/y$ is also continuous for every $y \in (0,\infty)$.

Proof. For any fixed $p \ge 0$ and $t > 0$, we denote by

$$F_t^{(p)}(y) := \mathbb{P}\{M_{\mathbf{e}_p \wedge t} \le y\}/y.$$

We first consider the case $p = 0$. Using the upper bound in equation (4.16) of Chaumont and Małecki [7] (which holds for a general Lévy process), we know that

$$F_t^{(0)}(y) \le \frac{e}{e-1}\kappa\left(\frac{1}{t},0\right)\frac{h(y)}{y}, \tag{5.7}$$

where we recall $h(\cdot)$ is the renewal function defined in (2.11). Since X has bounded variation and $d > 0$, we deduce that X creeps upwards by Theorem 7.11 of Kyprianou [20]. From Lemma 2.2 we know that $h(y)/y$ converges to a finite limit as $y \downarrow 0$. Therefore, we conclude from (5.7) that $F_t^{(0)}(y)$ is bounded for $y \in (0,\infty)$.

Next, we consider the case $p > 0$. By Wiener–Hopf factorization, it is well known that $\tilde{M}_{\mathbf{e}_p}$ follows an exponential distribution with mean $1/\tilde{\Phi}(p) > 0$. Moreover, since $M_t \ge \tilde{M}_t$ a.s. for any $t \ge 0$, one obtains that

$$\begin{aligned}
F_t^{(p)}(y) &= \int_{(0,t)} pe^{-ps}\frac{\mathbb{P}\{M_s \le y\}}{y}\,\mathrm{d}s + e^{-pt}\frac{\mathbb{P}\{M_t \le y\}}{y} \\
&\le \int_{(0,\infty)} pe^{-ps}\frac{\mathbb{P}\{\tilde{M}_s \le y\}}{y}\,\mathrm{d}s + e^{-pt}F_t^{(0)}(y) \\
&= \frac{1 - e^{-\tilde{\Phi}(p)y}}{y} + e^{-pt}F_t^{(0)}(y).
\end{aligned}$$

By the boundedness of $F_t^{(0)}(\cdot)$, we deduce that $F_t^{(p)}(y)$ is also bounded for $y \in (0,\infty)$.

Finally, suppose that we also have $\Pi(-\infty,0) = \infty$ and Π has no atoms on $(-\infty,0)$. For any fixed $t > 0$, by Theorem 27.7 of Sato [33], we know that the law of $\tilde{X}_t$ is absolute continuous with respect to the Lebesgue measure, so is the law of X_t from the property of convolutions. In addition, by Theorem 6.5 of Kyprianou [20], we know X is regular for $(0,\infty)$ as X has bounded variation and $d > 0$. Therefore, from Theorem 1 of Chaumont [6], we conclude the law of M_t is absolute continuous with respect to the Lebesgue measure. As a consequence, $\mathbb{P}\{M_{\mathbf{e}_p \wedge t} \le y\}/y$ is continuous for every $y \in (0,\infty)$. □

Remark 5.1. For the Lévy model (4.3) with $\tilde{X}$ has bounded variation and $\Pi(-\infty,0) = \infty$, it follows that $\mathbb{P}\{M_{\mathbf{e}_p \wedge t} \le y\}/y$ is bounded and continuous for $y \in (0,\infty)$ due to our assumptions that $|\tilde{X}|$ is not a subordinator, $\tilde{X}$ is regular for $(0,\infty)$, and Π has no atom on $(-\infty,0)$.

We are now ready to present the main result of this subsection.

Theorem 5.1. *Consider the Lévy model* (4.3). *If $\tilde{X}$ has bounded variation and satisfies Assumption* 4.1, *for any $q > 0$, we have*

$$\mathbb{E}\big[e^{-q\eta_b}\big] = e^{-qb} \frac{\int_{(0,\infty)} \mathbb{P}\{M_b \le y\}\Pi(-\mathrm{d}y)}{q + \int_{(0,\infty)} \mathbb{P}\{M_{\mathbf{e}_q \wedge b} \le y\}\Pi(-\mathrm{d}y)}.$$

Proof. We first consider the case $\Pi(-\infty, 0) = \infty$. From (5.5) with $p \ge 0$, we have

$$\begin{aligned}
&\frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)} f_\varepsilon^{(p)}(b) \\
&\quad = \frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)} \int_{[\varepsilon,\infty)} \mathbb{E}\big[e^{-q\tau_\varepsilon} 1_{\{Y_{\tau_\varepsilon} \in \mathrm{d}y\}}\big] \mathbb{P}\{M_{\mathbf{e}_p \wedge b} \le y\} \\
&\quad = \int_{(0,\infty)} \frac{\mathbb{P}\{M_{\mathbf{e}_p \wedge b} \le y\}}{1 - e^{-y}} \cdot \frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)} 1_{\{y \ge \varepsilon\}} \big(1 - e^{-y}\big) \mathbb{E}\big[e^{-q\tau_\varepsilon} 1_{\{Y_{\tau_\varepsilon} \in \mathrm{d}y\}}\big] \\
&\quad = \int_{(0,\infty)} \frac{\mathbb{P}\{M_{\mathbf{e}_p \wedge b} \le y\}}{1 - e^{-y}} \mu_\varepsilon(\mathrm{d}y),
\end{aligned} \tag{5.8}$$

where $\mu_\varepsilon(\mathrm{d}y)$ is a finite measure on $(0, \infty)$ defined as

$$\mu_\varepsilon(\mathrm{d}y) = \frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)} 1_{\{y \ge \varepsilon\}} \big(1 - e^{-y}\big) \mathbb{E}\big[e^{-q\tau_\varepsilon} 1_{\{Y_{\tau_\varepsilon} \in \mathrm{d}y\}}\big]. \tag{5.9}$$

By Theorem 4.2, we have

$$\begin{aligned}
\lim_{\varepsilon \downarrow 0} \int_{(0,\infty)} e^{-sy} \mu_\varepsilon(\mathrm{d}y) &= \lim_{\varepsilon \downarrow 0} \frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)} \big(\mathbb{E}\big[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}}\big] - \mathbb{E}\big[e^{-q\tau_\varepsilon - (s+1)Y_{\tau_\varepsilon}}\big]\big) \\
&= 1 + \frac{\tilde{\psi}(s) - \tilde{\psi}(s+1)}{d},
\end{aligned}$$

for all $s \ge 0$. On the other hand, we notice from (2.2) that

$$\begin{aligned}
&\int_{(0,\infty)} e^{-sy} \frac{1 - e^{-y}}{d} \Pi(-\mathrm{d}y) \\
&\quad = \frac{1}{d} \int_{(-\infty,0)} \big(e^{sy} - 1\big) \Pi(\mathrm{d}y) - \frac{1}{d} \int_{(-\infty,0)} \big(e^{(s+1)y} - 1\big) \Pi(\mathrm{d}y) \\
&\quad = 1 + \frac{\tilde{\psi}(s) - \tilde{\psi}(s+1)}{d}.
\end{aligned}$$

Hence, by Proposition A.1, one concludes that, as $\varepsilon \downarrow 0$, $\mu_\varepsilon(\mathrm{d}y)$ weakly converges to the measure $d^{-1}(1 - e^{-y})\Pi(-\mathrm{d}y)$, which is a finite measure on $(0, \infty)$ because $\tilde{X}$ has bounded variation.

From Proposition 5.1 and Remark 5.1, we know the function $\mathbb{P}\{M_{\mathbf{e}_p\wedge b}\le y\}/(1-e^{-y})$ is bounded and continuous for $y\in(0,\infty)$. By the definition of weak convergence, it follows from (5.8) that

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)}f_\varepsilon^{(p)}(b)&=\lim_{\varepsilon\downarrow 0}\int_{(0,\infty)}\frac{\mathbb{P}\{M_{\mathbf{e}_p\wedge b}\le y\}}{1-e^{-y}}\mu_\varepsilon(\mathrm{d}y)\\&=\int_{(0,\infty)}\frac{\mathbb{P}\{M_{\mathbf{e}_p\wedge b}\le y\}}{1-e^{-y}}\frac{1}{d}\big(1-e^{-y}\big)\Pi(-\mathrm{d}y)\\&=\frac{1}{d}\int_{(0,\infty)}\mathbb{P}\{M_{\mathbf{e}_p\wedge b}\le y\}\Pi(-\mathrm{d}y).\end{aligned}\tag{5.10}$$

Therefore, by (5.6), (5.10) and Theorem 4.2, we have

$$\begin{aligned}\mathbb{E}\big[e^{-q\eta_b}\big]&=\frac{e^{-qb}\lim_{\varepsilon\downarrow 0}\frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)}f_\varepsilon^{(0)}(b)}{\lim_{\varepsilon\downarrow 0}\frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)}(1-\mathbb{E}[e^{-q\tau_\varepsilon}])+\lim_{\varepsilon\downarrow 0}\frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)}f_\varepsilon^{(q)}(b)}\\&=\frac{e^{-qb}\int_{(0,\infty)}\mathbb{P}\{M_b\le y\}\Pi(-\mathrm{d}y)}{q+\int_{(0,\infty)}\mathbb{P}\{M_{\mathbf{e}_q\wedge b}\le y\}\Pi(-\mathrm{d}y)}.\end{aligned}$$

Finally, we consider the case that $\Pi(-\infty,0)<\infty$. By (4.12), for any $s\ge 0$, we have

$$\lim_{\varepsilon\downarrow 0}\int_{(0,\infty)}e^{-sy}\mathbb{E}\big[e^{-q\tau_\varepsilon}1_{\{Y_{\tau_\varepsilon}\in\mathrm{d}y\}}\big]=\frac{\tilde{\psi}(s)-sd+\Pi(-\infty,0)}{q+\Pi(-\infty,0)}=\int_{(0,\infty)}e^{-sy}\frac{\Pi(-\mathrm{d}y)}{q+\Pi(-\infty,0)}.$$

By Proposition A.1, we see that the measure $\mathbb{E}[e^{-q\tau_\varepsilon}1_{\{Y_{\tau_\varepsilon}\in\mathrm{d}y\}}]$ weakly converges to the measure $\Pi(-\mathrm{d}y)/(q+\Pi(-\infty,0))$ as $\varepsilon\downarrow 0$. Since $\mathbb{P}\{M_{\mathbf{e}_p\wedge b}\le y\}$ is bounded and upper semi-continuous in $y\in(0,\infty)$, it follows from Portemanteau theorem of weak convergence that

$$\begin{aligned}\limsup_{\varepsilon\downarrow 0}f_\varepsilon^{(p)}(b)&=\limsup_{\varepsilon\downarrow 0}\int_{(0,\infty)}\mathbb{P}\{M_{\mathbf{e}_p\wedge b}\le y\}\mathbb{E}\big[e^{-q\tau_\varepsilon}1_{\{Y_{\tau_\varepsilon}\in\mathrm{d}y\}}\big]\\&\le\frac{1}{q+\Pi(-\infty,0)}\int_{(0,\infty)}\mathbb{P}\{M_{\mathbf{e}_p\wedge b}\le y\}\Pi(-\mathrm{d}y).\end{aligned}\tag{5.11}$$

On the other hand, since $\mathbb{P}\{M_{\mathbf{e}_p\wedge b}<y\}$ is lower semi-continuous in $y\in(0,\infty)$. By Portemanteau theorem again, we have

$$\begin{aligned}\liminf_{\varepsilon\downarrow 0}f_\varepsilon^{(p)}(b)&\ge\liminf_{\varepsilon\downarrow 0}\int_{(0,\infty)}\mathbb{P}\{M_{\mathbf{e}_p\wedge b}<y\}\mathbb{E}\big[e^{-q\tau_\varepsilon}1_{\{Y_{\tau_\varepsilon}\in\mathrm{d}y\}}\big]\\&\ge\frac{1}{q+\Pi(-\infty,0)}\int_{(0,\infty)}\mathbb{P}\{M_{\mathbf{e}_p\wedge b}<y\}\Pi(-\mathrm{d}y)\\&=\frac{1}{q+\Pi(-\infty,0)}\int_{(0,\infty)}\mathbb{P}\{M_{\mathbf{e}_p\wedge b}\le y\}\Pi(-\mathrm{d}y),\end{aligned}\tag{5.12}$$

where the last equality holds because Π has no atom on $(-\infty, 0)$ and $\mathbb{P}\{M_{\mathbf{e}_p \wedge b} < y\} = \mathbb{P}\{M_{\mathbf{e}_p \wedge b} \leq y\}$ for almost all $y > 0$. By letting $\varepsilon \downarrow 0$ in each term of (5.6) and using (5.11), (5.12) and (4.12), this completes the proof of Theorem 5.1. □

5.2. Unbounded variation case

We now consider the unbounded variation case for which the following assumption on the density of X_t is made.

Assumption 5.1. *If X has unbounded variation, we assume that the density of X_t, namely $p_t^X(x)$, is bounded for all $t > 0$.*

Remark 5.2. We point out that Assumption 5.1 is identical to assumption (H1) in Chaumont and Małecki [7], which is equivalent to the assumption that the characteristic function $e^{-t\Psi(\cdot)} \in L^2(\mathbb{R})$, for all $t > 0$. It is also clear that, if X is a spectrally negative Lévy process with unbounded variation and Y is an arbitrary Lévy process independent of X, then the sum $X + Y$ satisfies Assumption 5.1 as long as X does. Hence, examples of Levy processes satisfying Assumption 5.1 include processes with $\sigma > 0$, or $\sigma = 0$ and with a spectrally negative α-stable jump distribution with $\alpha \in (1, 2)$.

The following proposition shows that, for a Lévy process with unbounded variation satisfying Assumption 5.1, the density of the running maximum at $0+$ is well-behaved.

Proposition 5.2. *Let X be a Lévy process with unbounded variation that creeps upwards and satisfies Assumption 5.1. Then the running maximum M_t has a continuous density $p_t^M(\cdot)$ for every $t > 0$ and further,*

$$\lim_{x \downarrow 0} p_t^M(x) = \frac{\bar{\nu}_L(t) + \kappa(0,0)}{d_H} > 0,$$

where $\bar{\nu}_L(\cdot)$ is the tail of the jump measure of the ascending ladder time process (see (2.10)).

Proof. From Lemma 2.2, we know that the renewal density h' can be chosen to be a continuous function with well-defined limit $h'(0) = \frac{1}{d_H} > 0$. Since X has unbounded variation, assumption (H2) of Chaumont and Małecki [7] also holds. By Proposition 2 and Theorem 1 of Chaumont and Małecki [7], we know that M_t has a continuous density $p_t^M(x)$ for every $t > 0$, and also,

$$\lim_{x \downarrow 0} \frac{p_t^M(x)}{h'(x)} = d_H \lim_{x \downarrow 0} p_t^M(x) = n(\zeta > t),$$

where $n(\zeta > t)$ is the excursion measure of excursions with length over $t > 0$. From (6.11) and (6.14) of Kyprianou [20] (see also Section IV.4 of Bertoin [4]), we know that

$$n(\zeta > t) = \bar{\nu}_L(t) + \kappa(0,0) > 0,$$

which completes the proof. □

Corollary 5.1. *Under the conditions of Proposition 5.2, for any fixed $p \geq 0$ and $t > 0$, the function $\mathbb{P}\{M_{\mathbf{e}_p \wedge t} \leq y\}/y$ is bounded and continuous for $y \in [0, \infty)$, where its value at $y = 0$ is defined as the right limit*

$$\lim_{y \downarrow 0} \frac{\mathbb{P}\{M_{\mathbf{e}_p \wedge t} \leq y\}}{y} = \frac{1}{d_H} \bigg(\int_{(0,t)} p e^{-ps} \bar{\nu}_L(s) \,\mathrm{d}s + e^{-pt} \bar{\nu}_L(t) + \kappa(0,0) \bigg).$$

Proof. From Proposition 5.2, it is only left to justify the limit of $\mathbb{P}\{M_{\mathbf{e}_p \wedge t} \leq y\}/y$ as $y \downarrow 0$. By dominated convergence theorem and Proposition 5.2 again, we have

$$\begin{aligned} \lim_{y \downarrow 0} \frac{\mathbb{P}\{M_{\mathbf{e}_p \wedge t} \leq y\}}{y} &= \int_0^t p e^{-ps} \lim_{y \downarrow 0} \frac{\mathbb{P}\{M_s \leq y\}}{y} \,\mathrm{d}s + e^{-pt} \lim_{y \downarrow 0} \frac{\mathbb{P}\{M_t \leq y\}}{y} \\ &= \int_0^t p e^{-ps} \lim_{y \downarrow 0} p_s^M(y) \,\mathrm{d}s + e^{-pt} \lim_{y \downarrow 0} p_t^M(y) \\ &= \frac{1}{d_H} \bigg(\int_{(0,t)} p e^{-ps} \bar{\nu}_L(s) \,\mathrm{d}s + e^{-pt} \bar{\nu}_L(t) + \kappa(0,0) \bigg), \end{aligned}$$

which ends the proof. □

Now we are ready to present the main result of this subsection.

Theorem 5.2. *Consider the Lévy model (4.3). If $\tilde{X}$ has unbounded variation and satisfies Assumptions 4.1 and 5.1, for any $q > 0$, we have*

$$\mathbb{E}\big[e^{-q\eta_b}\big] = e^{-qb} \frac{\bar{\nu}_L(b) + \kappa(0,0)}{\int_{(0,b)} q e^{-qt} \bar{\nu}_L(t) \,\mathrm{d}t + e^{-qb} \bar{\nu}_L(b) + \kappa(0,0)}.$$

Proof. It is clear that the Lévy model (4.3) creeps upward as its spectrally negative component $\tilde{X}$ does and its upward jumps follow a compound Poisson structure. Moreover, since $\tilde{X}$ satisfies Assumption 5.1, by Remark 5.2, we see that all the conditions of Proposition 5.2 are satisfied.

For the finite measure $\mu_\varepsilon(\mathrm{d}y)$ defined in (5.9), it is straightforward to verify from Theorem 4.2 that, for any $s \geq 0$,

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}_+} e^{-sy} \mu_\varepsilon(\mathrm{d}y) &= \lim_{\varepsilon \downarrow 0} \frac{W^{(q+\lambda')\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)} \big(\mathbb{E}\big[e^{-q\tau_\varepsilon - sY_{\tau_\varepsilon}}\big] - \mathbb{E}\big[e^{-q\tau_\varepsilon - (s+1)Y_{\tau_\varepsilon}}\big] \big) \\ &= 1 = \int_{\mathbb{R}_+} e^{-sy} \delta_0(\mathrm{d}y). \end{aligned}$$

It follows from Proposition A.1 that $\mu_\varepsilon(\mathrm{d}y)$ weakly converges to the Dirac measure $\delta_0(\mathrm{d}y)$ as $\varepsilon \downarrow 0$. Moreover, by Corollary 5.1, we know that the function $\mathbb{P}\{M_{\mathbf{e}_p \wedge t} \leq y\}/(1 - e^{-y})$ is also bounded and continuous for $y \in [0, \infty)$, where its value at $y = 0$ is defined by the limit as $y \downarrow 0$.

From (5.8) and Corollary 5.1, we have

$$\lim_{\varepsilon\downarrow 0}\frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)}f_\varepsilon^{(p)}(b)=\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}_+}\frac{\mathbb{P}\{M_{\mathbf{e}_p\wedge b}\le y\}}{1-e^{-y}}\mu_\varepsilon(\mathrm{d}y)$$
$$=\lim_{y\downarrow 0}\frac{\mathbb{P}\{M_{\mathbf{e}_p\wedge b}\le y\}}{1-e^{-y}}\tag{5.13}$$
$$=\frac{1}{d_H}\left(\int_{(0,b)}pe^{-pt}\bar{\nu}_L(t)\,\mathrm{d}t+e^{-pb}\bar{\nu}_L(b)+\kappa(0,0)\right).$$

It follows from (5.6), (5.13) and Theorem 4.2 that

$$\mathbb{E}\big[e^{-q\eta_b}\big]=e^{-qb}\frac{\lim_{\varepsilon\downarrow 0}\frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)}f_\varepsilon^{(0)}(b)}{\lim_{\varepsilon\downarrow 0}\frac{W^{(q+\lambda^+)\prime}(\varepsilon)}{W^{(q+\lambda^+)}(\varepsilon)}(1-\mathbb{E}[e^{-q\tau_\varepsilon}]+f_\varepsilon^{(q)}(b))}$$
$$=e^{-qb}\frac{\bar{\nu}_L(b)+\kappa(0,0)}{\int_{(0,b)}qe^{-qt}\bar{\nu}_L(t)\,\mathrm{d}t+e^{-qb}\bar{\nu}_L(b)+\kappa(0,0)},$$

which ends the proof. □

In general, the function $\bar{\nu}_L$ and $\kappa(0,0)$ are only implicitly known via (2.10) and Wiener–Hopf factorization. When X has no positive jumps, we can express $\mathbb{E}[e^{-q\eta_b}]$ explicitly in terms of p_t^X.

Corollary 5.2. *Let X be a spectrally negative Lévy process with unbounded variation and satisfies Assumptions 4.1 and 5.1. For any $q>0$, we have*

$$\mathbb{E}\big[e^{-q\eta_b}\big]=e^{-qb}\frac{\int_{(b,\infty)}\frac{1}{s}p_s^X(0)\,\mathrm{d}s}{\int_{(0,b)}qe^{-qt}\int_{(t,\infty)}\frac{1}{s}p_s^X(0)\,\mathrm{d}s\,\mathrm{d}t+e^{-qb}\int_{(b,\infty)}\frac{1}{s}p_s^X(0)\,\mathrm{d}s}.$$

Proof. By Kendall's identity, for any fixed t, $y>0$, we have

$$\frac{1}{y}\mathbb{P}\{M_t\le y\}=\frac{1}{y}\int_{(t,\infty)}\mathbb{P}\big\{T_y^+\in\mathrm{d}s\big\}=\int_{(t,\infty)}\frac{1}{s}p_s^X(y)\,\mathrm{d}s.$$

It follows from Fourier inversion that, for any $y\in\mathbb{R}$ and $s>0$,

$$0\le\frac{1}{s}p_s^X(y)\le\frac{1}{2\pi s}\int_{\mathbb{R}}\big|e^{-s\Psi(u)}\big|\,\mathrm{d}u.$$

From the proof of Proposition 5 of Chaumont and Małecki [7], we know that for any fixed $t>0$,

$$\frac{1}{2\pi}\int_{(t,\infty)}\frac{1}{s}\int_{\mathbb{R}}\big|e^{-s\Psi(u)}\big|\,\mathrm{d}u\,\mathrm{d}s<\infty.$$

By the dominated convergence theorem and Corollary 5.1, we have

$$\frac{\bar{\nu}_L(t)+\kappa(0,0)}{d_H}=\lim_{y\downarrow 0}\frac{\mathbb{P}\{M_t\le y\}}{y}=\int_{(t,\infty)}\frac{1}{s}p_s^X(0)\,\mathrm{d}s,$$

which completes the proof by using Theorem 5.2. □

6. Examples

Example 6.1. Consider a Brownian motion, i.e. $X_t=\mu t+\sigma W_t$ with $\sigma>0$. For any fixed $t>0$, we have

$$p_t^X(x)=\frac{1}{\sigma\sqrt{2\pi t}}\exp\left\{-\frac{(x-\mu t)^2}{2\sigma^2 t}\right\}.$$

By Remarks 4.1, 5.2 and Corollary 5.2, we have

$$\mathbb{E}\left[e^{-q\eta_b}\right]=\frac{e^{-qb}g(b)}{\int_{(0,b)}qe^{-qt}g(t)\,\mathrm{d}t+e^{-qb}g(b)},$$

where $g(t):=\int_{(t,\infty)}\frac{1}{s}p_s^X(0)\,\mathrm{d}s=\frac{2}{\sigma\sqrt{2\pi t}}e^{-\mu^2t/(2\sigma^2)}-\frac{2\mu}{\sigma}N(-\frac{\mu\sqrt{t}}{\sigma})$ and $N(\cdot)$ is the cumulative distribution function of a standard normal rv.

Example 6.2. Consider a spectrally negative α-stable process with Laplace exponent $\psi(s)=s^\alpha$ with $\alpha\in(1,2)$. For fixed $t>0$, it is well known (e.g., pages 87–88 of Sato [33]) that

$$p_t^X(x)=\frac{1}{\pi}t^{-1/\alpha}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\Gamma(1+n/\alpha)}{n!}\sin\left(\frac{n\pi}{\alpha}\right)\left(t^{-1/\alpha}x\right)^{n-1},$$

where $\Gamma(\cdot)$ is the Gamma function. It follows that

$$\int_{(t,\infty)}\frac{1}{s}p_s^X(0)\,\mathrm{d}s=\frac{\alpha}{\pi}\Gamma\left(\frac{1}{\alpha}\right)\sin\left(\frac{\pi}{\alpha}\right)t^{-1/\alpha}.$$

By Remarks 4.1, 5.2 and Corollary 5.2, we have

$$\mathbb{E}\left[e^{-q\eta_b}\right]=\frac{1}{e^{qb}b^{1/\alpha}\int_{(0,b)}qe^{-qt}t^{-1/\alpha}\,\mathrm{d}t+1}.$$

Example 6.3. Consider a spectrally negative Gamma process with Laplace exponent

$$\psi(s)=sd+\int_{(-\infty,0)}\left(e^{sx}-1\right)\beta|x|^{-1}e^{\alpha x}\,\mathrm{d}x=sd-\beta\log(1+s/\alpha),\qquad s\in\mathbb{H}^+,$$

where $\alpha, \beta > 0$ are constants. From Remark 4.2, we define

$$\varphi(s) := \frac{s\psi'(s) - \psi(s)}{\psi(s)^2} = \frac{\beta \log(1 + s/\alpha) - \beta s/(s + \alpha)}{(sd - \beta \log(1 + s/\alpha))^2}.$$

One can easily verify that, for any fixed $s_0 > \Phi(0)$, we have $\varphi(s_0 + i\cdot) \in L^1(\mathbb{R})$ which implies Assumption 4.1 holds. Using Kendall's identity and

$$p_t^X(x) = \frac{\alpha^{\beta s}}{\Gamma(\beta t)}(sd - y)^{\beta s - 1} e^{-\alpha(sd-y)} 1_{\{y<sd\}}, \qquad x \in \mathbb{R}, t > 0,$$

we have

$$\mathbb{P}\{T_y^+ > t\} = y \int_{(t,\infty)} \frac{1}{s} \frac{\alpha^{\beta s}}{\Gamma(\beta s)} 1_{\{y<sd\}}(sd - y)^{\beta s - 1} e^{-\alpha(sd-y)} \,\mathrm{d}s, \qquad y > 0 \text{ and } t > 0.$$

Hence, by Fubini's theorem followed by some calculations, we can show that

$$\begin{aligned}
&\int_{(0,\infty)} \mathbb{P}\{M_t \le y\} \Pi(-\mathrm{d}y) \\
&\quad = \int_{(0,\infty)} \mathbb{P}\{T_y^+ > t\} \Pi(-\mathrm{d}y) \\
&\quad = \int_{(0,\infty)} y \int_{(t,\infty)} \frac{1}{s} \frac{\alpha^{\beta s}}{\Gamma(\beta s)} 1_{\{y<sd\}}(sd - y)^{\beta s - 1} e^{-\alpha(sd-y)} \,\mathrm{d}s \,\beta \frac{e^{-\alpha y}}{y} \,\mathrm{d}y \\
&\quad = (d\alpha)^{\beta s} \int_{(t,\infty)} \frac{1}{\Gamma(\beta s)} s^{\beta s - 2} e^{-\alpha s d} \,\mathrm{d}s.
\end{aligned}$$

Using Theorem 5.1, one concludes

$$\mathbb{E}\big[e^{-q\eta_b}\big] = \frac{e^{-qb}(d\alpha)^{\beta s} \int_{(b,\infty)} \frac{s^{\beta s-2} e^{-\alpha s d}}{\Gamma(\beta s)} \,\mathrm{d}s}{q + (d\alpha)^{\beta s} \int_{(0,b)} q e^{-qt} \,\mathrm{d}t \int_{(t,\infty)} \frac{s^{\beta s-2} e^{-\alpha s d}}{\Gamma(\beta s)} \,\mathrm{d}s + e^{-qb}(d\alpha)^{\beta s} \int_{(b,\infty)} \frac{s^{\beta s-2} e^{-\alpha s d}}{\Gamma(\beta s)} \,\mathrm{d}s}.$$

Example 6.4. Consider Kou's jump-diffusion model given by

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t^+} J_i^+ - \sum_{j=1}^{N_t^-} J_j^-,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, $N^\pm$ are two independent Poisson processes with arrival rates $\lambda^\pm > 0$ and $J^\pm$ are a sequence of i.i.d. exponentially distributed random variables with mean $1/\eta^\pm > 0$. Its Laplace exponent is given by

$$\psi(s) = \frac{\sigma^2}{2}s^2 + \mu s + \lambda^-\left(\frac{\eta^-}{\eta^- + s} - 1\right) + \lambda^+\left(\frac{\eta^+}{\eta^+ - s} - 1\right), \qquad s \in \left(-\eta^-, \eta^+\right).$$

According to Corollary 1 of Asmussen *et al.* [1] and Section 6.5.4 of Kyprianou [20], it is known that the Laplace exponent of the ascending ladder height is given by

$$\kappa(\alpha,\beta)=\frac{(\beta+\rho_{1,\alpha})(\beta+\rho_{2,\alpha})}{(\beta+\eta^+)},\qquad \alpha,\beta\ge 0,$$

where $\rho_{1,\alpha}$ and $\rho_{2,\alpha}$ (with $\rho_{1,\alpha}<\eta^+<\rho_{2,\alpha}$) are the two distinct nonnegative solutions of $\psi(s)=\alpha$. By Remarks 4.1, 5.2 and Theorem 5.2, one obtains

$$\mathbb{E}\big[e^{-q\eta_b}\big]=e^{-qb}\frac{\bar{\upsilon}_L(b)+\frac{\rho_{1,0}\rho_{2,0}}{\eta^+}}{\int_{(0,b)}qe^{-qt}\bar{\upsilon}_L(t)\,\mathrm{d}t+e^{-qb}\bar{\upsilon}_L(b)+\frac{\rho_{1,0}\rho_{2,0}}{\eta^+}}.$$

Appendix

The following result is from Theorem 5.22 of Kallenberg [17].

Theorem A.1 (Extended continuity theorem). *Let $\mu_1,\mu_2,\dots$ be probability measures on $\mathbb{R}^d$ with characteristic functions $\hat{\mu}_n(t)\to\varphi(t)$ pointwisely for every $t\in\mathbb{R}^d$, where the limit φ is continuous at* 0. *Then μ_n converges weakly to μ for some probability measure μ in $\mathbb{R}^d$ with $\hat{\mu}=\varphi$. A corresponding statement holds for the Laplace transforms of measures on $\mathbb{R}^d_+$.*

Proposition A.1. *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be finite measures on $[0,\infty)$ with Laplace transforms*

$$\hat{\mu}_n(s)=\int_{\mathbb{R}_+}e^{-sy}\mu_n(\mathrm{d}y),$$

for $n\in\mathbb{N}$ and $s\ge 0$. Suppose that $\lim_{n\to\infty}\hat{\mu}_n(s)=\varphi(s)$ for all $s\ge 0$, where $\varphi(\cdot)$ is a positive and continuous function on $[0,\infty)$. Then μ_n weakly converges to μ as $n\to\infty$, for some finite measure μ on $[0,\infty)$, and $\hat{\mu}=\varphi$.

Proof. Since $\lim_{n\to\infty}\hat{\mu}_n(0)=\varphi(0)>0$, we can consider a sequence of probability measures $\nu_n(\mathrm{d}y):=\mu_n(\mathrm{d}y)\hat{\mu}_n(0)^{-1}$. By our assumptions, it is easy to see

$$\lim_{n\to\infty}\hat{\nu}_n(s)=\varphi(s)\varphi(0)^{-1},$$

which is a continuous function at 0. By Theorem A.1, one concludes that $\{\nu_n\}_{n\in\mathbb{N}}$ weakly converges to some probability measure v on $[0,\infty)$ with $\hat{v}(\cdot)=\varphi(\cdot)\varphi(0)^{-1}$. Therefore, by letting $\mu(\cdot):=v(\cdot)\varphi(0)$, we can see that μ_n weakly converges to μ as $n\to\infty$ and $\hat{\mu}=\varphi$. □

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