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# On Martingale Selectors of Cone-Valued Processes

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**Abstract** In this note we discuss a result of Guasoni, Rásonyi, and Schachermayer on the existence of martingale selectors for a class of continuous cone-valued processes. The setting includes that arising in models of financial markets with transaction costs.

**Key words:** cone-valued process, martingale selector, transaction costs, Dalang–Morton–Willinger theorem, consistent price system.

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## 1 Introduction

Let $C$ be a cone in $\mathbf{R}^d$ containing the vector $\mathbf{1} = (1, ..., 1)$ in its interior. Let $S = (S_t)_{t\in[0,1]}$ be a $\mathbf{R}^d$-valued continuous adapted process with strictly positive components defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, P)$. The random diagonal operators $\Sigma_t : (x^1, ..., x^d) \mapsto (S_t^1 x^1, ..., S_t^d x^d)$ define the cone-valued adapted process $\Sigma C = (\Sigma_t C)_{t\in[0,1]}$. The question is: *when is the set* $\mathcal{M}_0^1(\Sigma C \setminus \{0\})$ *non-empty?* That is, when does there exist an $\mathbf{R}^d$-valued martingale $M$ with $M_t(\omega) \in \Sigma_t(\omega)C \setminus \{0\}$ for all $\omega$ and $t$?

This type of martingale selection problem arises in models of financial markets with constant proportional transaction costs where $S$ is the price process and $C = K^*$, the dual of the solvency cone $K$ (the investor positions are measured in units of a numéraire). In "canonical" notations $\Sigma_t C$ is just $\widehat{K}_t^*$ where $\widehat{K}_t$ is the solvency cone (random because of price movements) when the investor positions are measured in "physical" units. In the theory of markets with transaction costs the martingales evolving in $\widehat{K}^* \setminus \{0\}$ play the role of (densities of) martingale measures, see [4], [5], [6] etc. They are called consistent price systems, [8].

To formulate the result we introduce the following hypotheses.

If $\tau$ and $\sigma$ are two stopping times with values in $[0,1]$ such that $\sigma \geq \tau$, let $A_{\tau,\sigma}$ denote the (random) topological support of the regular conditional distribution $P_{\tau,\sigma}(dx,\omega)$ of $S_\sigma - S_\tau$ with respect to $\mathcal{F}_\tau$.

$\mathbf{H}_1$**:** $0 \in \operatorname{ri}\operatorname{conv} A_{\tau,\sigma}$ a.s. on $\{\tau < 1\}$ for all stopping times $\tau$ and $\sigma$ such that $\sigma \geq \tau$ (ri means: relative interior).

$\mathbf{H}_2$**:** $P(\sup_{\tau\leq r\leq 1}|S_r - S_\tau| \leq \varepsilon|\mathcal{F}_\tau) > 0$ a.s. on $\{\tau < 1\}$ for all $\varepsilon > 0$ and all stopping times $\tau$.

**Theorem 1** *Assume that* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *hold. Then* $\mathcal{M}_0^1(\Sigma C \setminus \{0\}) \neq \emptyset$.

This note can be viewed as a seminar comment to the interesting recent paper [3], where the authors suggested a sufficient condition for the non-emptiness of $\mathcal{M}_0^1(\Sigma C \setminus \{0\})$. Though our formulation sounds slightly more general (as we prefer the Levental–Skorohod type condition, [7]), the arguments follow the same lines. We only take a shortcut, in the proof of the key lemma (interesting on its own), by directly using the Dalang–Morton–Willinger (DMW) theorem, [1], [2], instead of repeating a part of its proof (cf. Lemma 3.3 in [3]).

## 2 Key Lemma

Let $X = (X_n)_{n\geq 0}$ be an $\mathbf{R}^d$-valued discrete-time adapted process on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{G} = (\mathcal{G}_n), P)$. Put $\xi_n = \Delta X_n$, $\Gamma_n := \{\xi_n = 0\}$.

**Lemma 1** *Suppose that the following conditions hold:*
$(i)$ *for each finite* $N$ *the process* $(X_n)_{n\leq N}$ *has the* $NA$*-property;*
$(ii)$ $I_{\Gamma_n} \uparrow 1$ *a.s.;*
$(iii)$ $E(I_{\Gamma_n}|\mathcal{G}_{n-1}) > 0$ *a.s. on* $\Gamma_{n-1}^c$ *for each* $n \geq 1$.
*Then there exists a probability* $Q \sim P$ *such that* $X$ *is a* $Q$*-martingale bounded in* $L^2(Q)$.

*Proof.* By the DMW theorem condition $(i)$ is equivalent to the $NA$-property for each one-step model: the relation $\gamma\xi_n \geq 0$ with $\gamma \in L^0(\mathbf{R}^d, \mathcal{G}_{n-1})$ may hold only if $\gamma\xi_n = 0$. The same theorem asserts that each $\xi_n$ admits an equivalent martingale measure which can be chosen to ensure the integrability of any fixed finite random variable, e.g., $|\xi_n|^2$. In terms of densities this means that there exist $\mathcal{G}_n$-measurable random variables $\bar{\alpha}_n > 0$ such with $E(\bar{\alpha}_n\xi_n|\mathcal{G}_{n-1}) = 0$ and $c_n := E(\bar{\alpha}_n|\xi_n|^2|\mathcal{G}_{n-1}) < \infty$. Normalizing, we can add to this the property $E(\bar{\alpha}_n|\mathcal{G}_{n-1}) = 1$.

We define a $\mathcal{G}_n$-measurable random variable $\alpha_n > 0$ by the formula

$$\alpha_n = I_{\Gamma_{n-1}} + \left[\frac{(1-\delta_n)I_{\Gamma_n}}{E(I_{\Gamma_n}|\mathcal{G}_{n-1})} + \frac{\delta_n\bar{\alpha}_n I_{\Gamma_n^c}}{E(\bar{\alpha}_n I_{\Gamma_n^c}|\mathcal{G}_{n-1})}\right] I_{\Gamma_{n-1}^c\cap A_n} + I_{\Gamma_{n-1}^c\cap A_n^c},$$

where $A_n := \{E(\bar\alpha_n I_{\Gamma_n^c}|\mathcal{G}_{n-1}) > 0\}$ and $\delta_n := 2^{-n}E(\bar\alpha_n I_{\Gamma_n^c}|\mathcal{G}_{n-1})/(1+c_n)$.

Clearly, $E(\alpha_n|\mathcal{G}_{n-1}) = 1$. Noting that $\bar\alpha_n I_{\Gamma_n^c} I_{A_n^c} = 0$ (a.s.), we obtain that $E(\alpha_n \xi_n^2|\mathcal{G}_{n-1}) \le 2^{-n}$ and $E(\alpha_n\xi_n|\mathcal{G}_{n-1}) = 0$.

The process $Z_n := \alpha_1...\alpha_n$ is a martingale. It converges stationarily a.s. to a random variable $Z_\infty > 0$ with $EZ_\infty \le 1$. Since $I_{\Gamma_n} \uparrow 1$ (a.s.) and $Z_\infty I_{\Gamma_n} = Z_n I_{\Gamma_n}$,

$$EZ_\infty = E\lim_n Z_\infty I_{\Gamma_n} = \lim_n EZ_\infty I_{\Gamma_n} = \lim_n EZ_n I_{\Gamma_n} = 1 - \lim_n EZ_n I_{\Gamma_n^c}.$$

It follows that $EZ_\infty = 1$ (i.e. $(Z_n)$ is uniformly integrable martingale). Indeed, $E(\alpha_k I_{\Gamma_k^c}|\mathcal{G}_{k-1} \le 2^{-k}$ and, hence,

$$EI_{\Gamma_n^c} Z_n = E\prod_{k\le n}\alpha_k I_{\Gamma_k^c} \le \prod_{k\le n} 2^{-k} \to 0.$$

Thus, $Q := Z_\infty P$ is a probability measure under which $X$ is a martingale. At last,

$$E_Q X_n^2 = \sum_{k\le n} EZ_k\xi_k^2 \le \sum_{k\le n} 2^{-k} \le 1,$$

i.e. $X_n$ belongs to the unit ball of $L^2(Q)$. □

*Remark.* The condition $(iii)$ cannot be omitted. Indeed, let $X$ be the symmetric random walk starting from zero and stopped at the moment when it arrives to unit. It is already a martingale and the condition $(ii)$ holds. Since $X_\infty = 1$ a.s., the process $X$ cannot be a uniformly integrable martingale with respect to some $Q \sim P$.

## 3 Martingale Selection Theorem: Proof

Fix $\theta > 1$. Define the sequence of stopping times, $\tau_0 = 0$,

$$\tau_n := \inf\{t \ge \tau_{n-1} :\ \max_{i\le d} |\ln S_t^i - \ln S_{\tau_{n-1}}^i| \ge \ln\theta\} \wedge 1, \qquad n \ge 1,$$

and the stopping time $\tau_t := \min\{\tau_n :\ \tau_n > t\}$ for $t \in [0,1[$. Put also $\sigma_t := \max\{\tau_n :\ \tau_n \le t\}$ and $\nu := \max\{n :\ \tau_n < 1\}$. Since the ratios $S_t^i/S_{\sigma_t}^i$ and $S_{\tau_t}^i/S_{\sigma_t}^i$ take values in the interval $[\theta^{-1},\theta]$, we have the bounds

$$\theta^{-2} \le S_{\tau_t}^i/S_t^i \le \theta^2, \qquad i \le d. \tag{1}$$

Set $X_n := S_{\tau_n} I_{\{\tau_n<1\}} + S_{\tau_\nu} I_{\{\tau_n=1\}}$, $\mathcal{G}_n := \mathcal{F}_{\tau_n}$. Suppose that the discrete-time process $X = (X_n)$ satisfies the conditions of the lemma. Then $X$ is a uniformly integrable $Q$-martingale with respect to some probability measure $Q = Z_\infty P$ equivalent to $P$. Consider the continuous-time martingale $\tilde S_t := E_Q(X_\infty|\mathcal{F}_t)$, $t \in [0,1]$. Since $\tilde S_{\tau_n} = X_n$ we have the inequalities

$$\theta^{-1} \le \tilde S_{\tau_n}^i / S_{\tau_n}^i \le \theta$$

where $\tau_n$ can be replaced by $\tau_t$. Using this and the bounds (1) we get

$$\theta^{-3} \le \tilde{S}^i_{\tau_t}/S^i_t \le \theta^3.$$

But $\tilde{S}^i_t/S^i_t = E_Q(\tilde{S}^i_{\tau_t}/S^i_t|\mathcal{F}_t)$ and, therefore, the ratios $\tilde{S}^i_t/S^i_t$ take values in the interval $[\theta^{-3}, \theta^3]$. Thus, for $\theta$ sufficiently close to unit, the $Q$-martingale $\tilde{S}$ evolves in $\Sigma C \setminus \{0\}$ and so does also the $P$-martingale $M := Z\tilde{S}$.

It remains to note that properties $(i)$ and $(iii)$ hold by virtue of $\mathbf{H}_1$ and $\mathbf{H}_2$ while $(ii)$ is always fulfilled for continuous $S$. □

*Remark.* An important part of the paper [3] is devoted to the property of $S$ called "conditional full support", implying $\mathbf{H}_1$ and $\mathbf{H}_2$. This property is shown to hold for a wide class of continuous processes.

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