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On $\mathbb{A}$-numerical radius equalities and inequalities for certain operator matrices

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Abstract
The main goal of this article is to establish several new $\mathbb{A}$-numerical radius equalities for $n\times n$ circulant, skew circulant, imaginary circulant, imaginary skew circulant, tridiagonal, and anti-tridiagonal operator matrices, where $\mathbb{A}$ is the $n\times n$ diagonal operator matrix whose diagonal entries are positive bounded operator A. Some special cases of our results lead to the results of earlier works in the literature, which shows that our results are more general. Further, some pinching type $\mathbb{A}$-numerical radius inequalities for $n\times n$ block operator matrices are given. Some equality conditions are also given. We also provide a concluding section, which may lead to several new problems in this area.

Keywords $\mathbb{A}$-numerical radius · Positive operator · Semi-inner product · Inequality · Circulant operator matrix · Tridiagonal operator matrix

1 Introduction and preliminaries

The operator matrices such as circulant, reverse circulant, symmetric circulant, k-circulant, Toeplitz matrices etc. [13, 24] play a crucial role in pure as well as applied mathematical researches such as graph theory, image processing, block filtering design, signal processing, regular polygon solutions, encoding, control and system theory, network, etc. The norm estimation for the operator matrices [6, 27] is extensively carried out in the past and it is widely used in operator theory, quantum information theory, mathematical physics, numerical analysis, etc. The norms of some

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circulant type matrices were determined by various mathematicians. For instance, Li et al. [28], gave four kinds of norms for circulant and left circulant matrices involving special numbers. Bose et al. [12], discussed the convergence in probability and the convergence in distribution of the spectral norms of scaled Toeplitz, circulant, reverse (left) circulant, symmetric circulant, and k-circulant matrices. Works on norm equalities and inequalities of special kind of operator matrices can be found in the literature [2, 7, 25, 26]. Jiang and Xu [23] explored special cases for norm equalities and inequalities, such as usual operator norm and Schatten p-norms. Several norm equalities and inequalities for the circulant, skew circulant, and w-circulant operator matrices were studied [4, 23].

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on $\mathcal{H}$. Let $\mathbb{H}=\bigoplus_{i=1}^n \mathcal{H}$ be the direct sum of n copies of $\mathcal{H}$. If T_{ij}, $1\le i,j\le n$ are operators in $\mathcal{B}(\mathcal{H})$, then operator matrix $\mathbb{T}=[T_{i,j}]$ can be defined on $\mathbb{H}$ by

$$\mathbb{T}x=\begin{bmatrix}\sum_{j=1}^n T_{1j}x_j\\ \vdots\\ \sum_{j=1}^n T_{nj}x_j\end{bmatrix}$$

for every vector $x=[x_1,\dots,x_n]^{\mathrm{T}}\in\mathbb{H}$. If $S_i\in\mathcal{B}(\mathcal{H}), i=1,\dots,n$, then their direct sum, $\bigoplus_{i=1}^n S_i$, (which is an $n\times n$ block diagonal operator matrix), is given by

$$\bigoplus_{i=1}^n S_i=\begin{bmatrix}S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_n\end{bmatrix}.$$

If $T_i\in\mathcal{B}(\mathcal{H}), i=1,\dots,n$, then the circulant operator matrix $\mathbb{T}_{\mathrm{circ}}=\ \mathrm{circ}\,(T_1,\dots,T_n)$ is the $n\times n$ matrix whose first row has entries $T_1,\dots,T_n$ and the other rows are obtained by successive cyclic permutations of these entries, i.e., $\mathbb{T}_{\mathrm{circ}}=\begin{bmatrix}T_1 & T_2 & T_3 & \cdots & T_n\\ T_n & T_1 & T_2 & \cdots & T_{n-1}\\ T_{n-1} & T_n & T_1 & \ddots & T_{n-2}\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ T_2 & T_3 & \cdots & T_n & T_1\end{bmatrix}$. The skew circulant operator matrix $\mathbb{T}_{\mathrm{scirc}}=\ \mathrm{scirc}\,(T_1,\dots,T_n)$ is the $n\times n$ circulant matrix followed by a change in sign to all the elements below the main diagonal. Thus, $\mathbb{T}_{\mathrm{scirc}}=\begin{bmatrix}T_1 & T_2 & T_3 & \cdots & T_n\\ -T_n & T_1 & T_2 & \cdots & T_{n-1}\\ -T_{n-1} & -T_n & T_1 & \ddots & T_{n-2}\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ -T_2 & -T_3 & \cdots & -T_n & T_1\end{bmatrix}$. It is well-known that every skew circulant operator matrix is unitarily equivalent to a circulant operator matrix. Details discussion on circulant, skew-circulant and their properties are given in [13].

If $T_i \in \mathcal{B}(\mathcal{H}), i = 1, \ldots, n$, then the imaginary circulant operator matrix $\mathbb{T}_{\mathrm{circ}_i} = \mathrm{circ}_i(T_1, \ldots, T_n)$ is the $n \times n$ matrix whose first row has entries $T_1, \ldots, T_n$ and the other rows are obtained by successive cyclic permutations of i-multiples of these entries, i.e., $\mathbb{T}_{\mathrm{circ}_i} = \begin{bmatrix} T_1 & T_2 & T_3 & \cdots & T_n \\ iT_n & T_1 & T_2 & \cdots & T_{n-1} \\ iT_{n-1} & iT_n & T_1 & \ddots & T_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ iT_2 & iT_3 & \cdots & iT_n & T_1 \end{bmatrix}$. Every imaginary circulant operator matrix is unitarily equivalent to a circulant operator matrix. The imaginary skew circulant operator matrix $\mathbb{T}_{\mathrm{scirc}_i} = \mathrm{scirc}_i(T_1, \ldots, T_n)$ is the $n \times n$ imaginary circulant followed by a change in sign to all the elements below the main diagonal. Thus, $\mathbb{T}_{\mathrm{scirc}_i} = \begin{bmatrix} T_1 & T_2 & T_3 & \cdots & T_n \\ -iT_n & T_1 & T_2 & \cdots & T_{n-1} \\ -iT_{n-1} & -iT_n & T_1 & \ddots & T_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -iT_2 & -iT_3 & \cdots & -iT_n & T_1 \end{bmatrix}$.

Here $\overline{L}$ is the norm closure of the linear subspace L in the norm topology of $\mathcal{H}$, P_M is the orthogonal projection onto the closed linear subspace M of $\mathcal{H}$, I is identity operator and O is the null operator on $\mathcal{H}$, respectively. For any $A \in \mathcal{B}(\mathcal{H})$, the range, null space and adjoint of A are denoted by $\mathcal{R}(A)$, $\mathcal{N}(A)$ and A^*, respectively. An operator $A \in \mathcal{B}(\mathcal{H})$ is called *positive* if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and is called *strictly positive* if $\langle Ax, x\rangle > 0$ for all non-zero $x \in \mathcal{H}$. We denote a positive (strictly positive) operator A by $A \geq O$ $(A > O)$. Throughout this paper, we assume that $A \in \mathcal{B}(\mathcal{H})$ is a positive operator, and $\mathbb{A} \in \mathcal{B}(\bigoplus_{i=1}^n \mathcal{H})$ is an $n \times n$ diagonal operator matrix whose diagonal entries are positive operator A. Then, any such A defines a positive semidefinite sesquilinear form:

$$\langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \quad \langle x, y\rangle_A = \langle Ax, y\rangle, \quad x, y \in \mathcal{H}.$$

Let $\|\cdot\|_A$ denote the seminorm on $\mathcal{H}$ induced by $\langle \cdot, \cdot \rangle_A$, i.e. $\|x\|_A = \sqrt{\langle x, x\rangle_A}$ for all $x \in \mathcal{H}$. Note that $\|x\|_A = 0$ if and only if $x \in \mathcal{N}(A)$, and $\|x\|_A$ is a norm if and only if A is one-to-one (or $A > O$). Also, $(\mathcal{H}, \|\cdot\|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed in $\mathcal{H}$.

The A-operator seminorm of $T \in \mathcal{B}(\mathcal{H})$, denoted by $\|T\|_A$, is defined as follows:

$$\|T\|_A = \sup_{x \in \overline{\mathcal{R}(A)},\ x \neq 0} \frac{\|Tx\|_A}{\|x\|_A} < \infty.$$

An equivalent definition of $\|T\|_A$, is given in [42]. The set of all bounded linear operators on $\mathcal{H}$ whose A-operator seminorm is finite is denoted by $\mathcal{B}^A(\mathcal{H})$. It is known that $\mathcal{B}^A(\mathcal{H})$ is not a subalgebra of $\mathcal{B}(\mathcal{H})$, and $\|T\|_A = 0$ if and only if $T^*AT = O$. For $T \in \mathcal{B}^A(\mathcal{H})$, we have

$$\|T\|_A = \sup\left\{ |\langle Tx, y\rangle_A| : x, y \in \overline{\mathcal{R}(A)},\ \|x\|_A = \|y\|_A = 1 \right\}.$$

The operator T is called *A-positive* if $AT \geq O$. Note that if T is A-positive, then

$$\|T\|_A = \sup\left\{ \langle Tx, x\rangle_A \,:\, x \in \overline{\mathcal{R}(A)},\ \ \|x\|_A = 1 \right\}.$$

An operator $X \in \mathcal{B}(\mathcal{H})$ is called an *A-adjoint operator* of $T \in \mathcal{B}(\mathcal{H})$ if $\langle Tx, y\rangle_A = \langle x, Xy\rangle_A$ for every $x, y \in \mathcal{H}$, i.e., if $AX = T^*A$. By [14, 29], the existence of an A-adjoint operator is not guaranteed. An operator $T \in \mathcal{B}(\mathcal{H})$ can have none, one or many A-adjoints. An A-adjoint of an operator $T \in \mathcal{B}(\mathcal{H})$ exists if and only if $\mathcal{R}(T^*A) \subseteq \mathcal{R}(A)$. $\mathcal{B}_A(\mathcal{H})$ is the set of all operators which admit A-adjoints and $\mathcal{B}_A(\mathcal{H})$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ which is neither closed nor dense in $\mathcal{B}(\mathcal{H})$. Moreover, the following inclusions $\mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}^A(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ hold with equality if A is injective and has a closed range.

If $T \in \mathcal{B}_A(\mathcal{H})$, the reduced solution of the equation $AX = T^*A$ is a distinguished A-adjoint operator of T, which is denoted by $T^{\#_A}$ ([1, 31]). Note that $T^{\#_A} = A^{\dagger}T^*A$, where $A^{\dagger}$ is the Moore-Penrose inverse [33] of A. Recall that $A^{\dagger} : \mathcal{R}(A) \bigoplus \mathcal{R}(A)^{\perp} \longrightarrow \mathcal{H}$ is the unique operator satisfying $AA^{\dagger}A = A$, $A^{\dagger}AA^{\dagger} = A^{\dagger}$, $A^{\dagger}A = P_{\overline{\mathcal{N}(A)^{\perp}}}$, $AA^{\dagger} = P_{\overline{\mathcal{R}(A)}}|_{\mathcal{R}(A)\bigoplus\mathcal{R}(A)^{\perp}}$. If $T \in \mathcal{B}_A(\mathcal{H})$, then $AT^{\#_A} = T^*A$, $\mathcal{R}(T^{\#_A}) \subseteq \overline{\mathcal{R}(A)}$ and $\mathcal{N}(T^{\#_A}) = \mathcal{N}(T^*A)$ ([14]). An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *A-selfadjoint* if AT is selfadjoint, i.e., if $AT = T^*A$. Observe that if T is A-selfadjoint, then $T \in \mathcal{B}_A(\mathcal{H})$. In general $T \neq T^{\#_A}$. For $T \in \mathcal{B}_A(\mathcal{H}), T = T^{\#_A}$ if and only if T is A-selfadjoint and $\mathcal{R}(T) \subseteq \overline{\mathcal{R}(A)}$. If $T \in \mathcal{B}_A(\mathcal{H})$, then $T^{\#_A} \in \mathcal{B}_A(\mathcal{H})$, $(T^{\#_A})^{\#_A} = P_{\overline{\mathcal{R}(A)}}TP_{\overline{\mathcal{R}(A)}}$, and $\left((T^{\#_A})^{\#_A}\right)^{\#_A} = T^{\#_A}$. Also, $T^{\#_A}T$ and $TT^{\#_A}$ are A-positive operators, and

$$\|T^{\#_A}T\|_A = \|TT^{\#_A}\|_A = \|T\|_A^2 = \|T^{\#_A}\|_A^2. \tag{1}$$

An operator $U \in \mathcal{B}_A(\mathcal{H})$ is said to be *A-unitary* if $\|Ux\|_A = \|U^{\#_A}x\|_A = \|x\|_A$ for all $x \in \mathcal{H}$. For $T, S \in \mathcal{B}_A(\mathcal{H})$, we have $(TS)^{\#_A} = S^{\#_A}T^{\#_A}$, $(T+S)^{\#_A} = T^{\#_A} + S^{\#_A}$, $\|TS\|_A \leq \|T\|_A\|S\|_A$ and $\|Tx\|_A \leq \|T\|_A\|x\|_A$ for all $x \in \mathcal{H}$. The real and imaginary parts of an operator $T \in \mathcal{B}_A(\mathcal{H})$ are $Re_A(T) = \frac{T+T^{\#_A}}{2}$ and $Im_A(T) = \frac{T-T^{\#_A}}{2i}$.

$w(T)$ is the *numerical radius* of $T \in \mathcal{B}(\mathcal{H})$, which is defined as

$$w(T) = \sup\left\{ |\langle Tx, x\rangle| \,:\, x \in \mathcal{H}, \|x\| = 1 \right\}.$$

It is well-known that $w(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, and is equivalent to the usual operator norm $\|T\| = \sup\{\|Tx\| : x \in \mathcal{H}, \|x\| = 1\}$. In fact, for every $T \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{2}$$

Extensive studies on different generalizations, refinements and applications of numerical radius inequalities have been conducted [3, 21, 22, 30, 38–40]. Saddi [36] introduced the *A-numerical radius* of T for $T \in \mathcal{B}(\mathcal{H})$, which is denoted as $w_A(T)$, and is defined as follows:

$$w_A(T) = \sup\left\{ |\langle Tx, x\rangle_A| \,:\, x \in \mathcal{H}, \|x\|_A = 1 \right\}.$$

It then follows that

$$w_A(T) = w_A(T^{\#_A}) \text{ for any } T \in \mathcal{B}_A(\mathcal{H}). \tag{3}$$

If $T \in \mathcal{B}_A(\mathcal{H})$ and U is A-unitary, then $w_A(U^{\#_A}TU) = w_A(T)$. Zamani [42] developed a new formula for computing the numerical radius of $T \in \mathcal{B}_A(\mathcal{H})$:

$$w_A(T) = \sup_{\theta\in\mathbb{R}} \left\| \frac{e^{i\theta}T + (e^{i\theta}T)^{\#_A}}{2} \right\|_A.$$

The inequality (2) is also studied using A-numerical radius of T, and is given as

$$\frac{1}{2}\|T\|_A \le w_A(T) \le \|T\|_A. \tag{4}$$

Furthermore, if T is A-selfadjoint, then $w_A(T) = \|T\|_A$. Moslehian et al. [32] pursued the study of A-numerical radius and established some A-numerical radius inequalities. Bhunia et al. [11] obtained several $\mathbb{A}$-numerical radius inequalities. Further generalizations and refinements of A-numerical radius inequalities are discussed in [8, 9, 15, 34]. Many studies on $\mathbb{A}$-numerical radius inequalities are given in [15–20, 35, 37, 41].

In this aspect, the rest of the paper is organized as follows. Inspired by the work of Bani-Domi and Kittaneh [4], we establish certain $\mathbb{A}$-numerical radius equalities for circulant, skew circulant, imaginary circulant, and imaginary skew circulant operator matrices in Sect. 2. Some special cases of our result have been given in this section. In Sect. 3, we apply these $\mathbb{A}$-numerical radius equalities to obtain pinching type $\mathbb{A}$-numerical radius inequalities for $n \times n$ block operator matrices. Some equality condition are also given. In Sect. 4, we extend some recent results of Bani-Domi et al. [5] to the semi-Hilbert space operators. In particular, we obtain certain $\mathbb{A}$-numerical radius equalities and pinching type inequalities for $n \times n$ tridiagonal and anti-tridiagonal operator matrices. Finally, we end up with a conclusion section, which may spark new problems for future research interest.

We need the following lemmas to prove our results. The first lemma is already proved by Bhunia et al. [11] for the case strictly positive operator A. Very recently the same result proved by Rout et al. [35] by dropping the assumption A is strictly positive is stated next for our purpose. For usual numerical radius versions of Lemmas 1.1–1.3, one may consult [4, 21].

Lemma 1.1 [35, *Lemma* 2.4] *Let* $T_1, T_2 \in \mathcal{B}_A(\mathcal{H})$. *Then*

(i) $w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & O \\ O & T_2 \end{bmatrix}\right) = \max\{w_A(T_1), w_A(T_2)\}$.

(ii) $w_{\mathbb{A}}\left(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix}\right) = w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_1 & O \end{bmatrix}\right)$.

(iii) $w_{\mathbb{A}}\left(\begin{bmatrix} O & T_1 \\ e^{i\theta}T_2 & O \end{bmatrix}\right) = w_{\mathbb{A}}\left(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix}\right)$ *for any* $\theta \in \mathbb{R}$.

(iv) $w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix}\right) = \max\{w_A(T_1+T_2), w_A(T_1-T_2)\}$. *In particular*, $w_{\mathbb{A}}\left(\begin{bmatrix} O & T_1 \\ T_1 & O \end{bmatrix}\right) = w_A(T_1)$.

Lemma 1.2 [35, *Lemma* 2.9] *Let* $T_1, T_2 \in \mathcal{B}_A(\mathcal{H})$. *Then*

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_2 & -T_1 \\ T_1 & T_2 \end{bmatrix}\right) = \max\{w_A(T_1+iT_2), w_A(T_1-iT_2)\}.$$

The following lemma was proved by Rout et al. [35].

Lemma 1.3 [35, *Lemma* 2.2] *Let* $T_1, T_2, T_3, T_4 \in \mathcal{B}_A(\mathcal{H})$. *Then*

(i) $w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & O \\ O & T_4 \end{bmatrix}\right) \le w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right)$.

(ii) $w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \le w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right)$.

Lemma 1.4 [10, *Lemma* 3.1] *Let* $T_{ij} \in \mathcal{B}_A(\mathcal{H})$, $1 \le i,j \le n$. *Then*

$$\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}^{\#_{\mathbb{A}}} = \begin{bmatrix} T_{11}^{\#_A} & T_{21}^{\#_A} & \cdots & T_{n1}^{\#_A} \\ T_{12}^{\#_A} & T_{22}^{\#_A} & \cdots & T_{n2}^{\#_A} \\ \vdots & \vdots & \ddots & \vdots \\ T_{1n}^{\#_A} & T_{2n}^{\#_A} & \cdots & T_{nn}^{\#_A} \end{bmatrix}.$$

Part (i) of Lemma 1.1 can be generalized as follows.

Lemma 1.5 [34, *Theorem* 3.5] *Let* $T_i \in \mathcal{B}_A(\mathcal{H})$, $1 \le i \le n$. *Then*

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & & \cdots & O \\ O & T_2 & & O \\ \vdots & & \ddots & \vdots \\ O & & \cdots & T_n \end{bmatrix}\right) = \max\{w_A(T_1), \ldots, w_A(T_n)\}.$$

2 $\mathbb{A}$-numerical radius equalities for circulant and skew circulant operator matrices

The aim of this section is to discuss certain $\mathbb{A}$-numerical radius equalities for circulant, skew circulant, imaginary circulant, and imaginary skew circulant operator matrices. The very first result is a formula for the $\mathbb{A}$-numerical radius of a circulant operator matrix.

Theorem 2.1 *Let* $T_i \in \mathcal{B}_A(\mathcal{H})$ *for* $1 \le i \le n$. *Then*

$$w_{\mathbb{A}}(\mathbb{T}_{\text{circ}}) = \max\left\{ w_A\left(\sum_{i=1}^{n} \omega^{k(1-i)} T_i\right) : k = 0, 1, \dots, n-1 \right\},$$

where $\omega = e^{\frac{2\pi i}{n}}$.

Proof Let $\mathbb{T}_{\text{circ}} = \begin{bmatrix} T_1 & T_2 & T_3 & \cdots & T_n \\ T_n & T_1 & T_2 & \cdots & T_{n-1} \\ T_{n-1} & T_n & T_1 & \ddots & T_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ T_2 & T_3 & \cdots & T_n & T_1 \end{bmatrix}$, let $1, \omega, \omega^2, \dots, \omega^{n-1}$ be n roots of unity with $\omega = e^{\frac{2\pi i}{n}}$ and $\mathbb{U} = \frac{1}{\sqrt{n}}\begin{bmatrix} I & I & I & \cdots & I \\ I & \omega I & \omega^2 I & \cdots & \omega^{n-1} I \\ I & \omega^2 I & \omega^4 I & \cdots & \omega^{n-2} I \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I & \omega^{n-1} I & \omega^{n-2} I & \cdots & \omega I \end{bmatrix}$.

It can be observed that $\bar{\omega} = \omega^{n-1}, \bar{\omega}^2 = \omega^{n-2}, \cdots, \bar{\omega}^k = \omega^{n-k}$, $k = 0, 1, \dots, n-1$, so $\mathbb{U}^{\#_{\mathbb{A}}} = \frac{1}{\sqrt{n}}\begin{bmatrix} P_{\overline{\mathcal{R}(A)}} & P_{\overline{\mathcal{R}(A)}} & P_{\overline{\mathcal{R}(A)}} & \cdots & P_{\overline{\mathcal{R}(A)}} \\ P_{\overline{\mathcal{R}(A)}} & \omega^{n-1} P_{\overline{\mathcal{R}(A)}} & \omega^{n-2} P_{\overline{\mathcal{R}(A)}} & \cdots & \omega P_{\overline{\mathcal{R}(A)}} \\ P_{\overline{\mathcal{R}(A)}} & \omega^{n-2} P_{\overline{\mathcal{R}(A)}} & \omega^{n-4} P_{\overline{\mathcal{R}(A)}} & \cdots & \omega^2 P_{\overline{\mathcal{R}(A)}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{\overline{\mathcal{R}(A)}} & \omega P_{\overline{\mathcal{R}(A)}} & \omega^2 P_{\overline{\mathcal{R}(A)}} & \cdots & \omega^{n-1} P_{\overline{\mathcal{R}(A)}} \end{bmatrix}$ and $\mathbb{U}\mathbb{U}^{\#_{\mathbb{A}}} = \begin{bmatrix} P_{\overline{\mathcal{R}(A)}} & O & \cdots & O \\ O & P_{\overline{\mathcal{R}(A)}} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & P_{\overline{\mathcal{R}(A)}} \end{bmatrix} = \mathbb{U}^{\#_{\mathbb{A}}}\mathbb{U}$. We set the direct sum $\bigoplus_{i=1}^{n} P_{\overline{\mathcal{R}(A)}}$ for $n \times n$ block diagonal operator matrix $\begin{bmatrix} P_{\overline{\mathcal{R}(A)}} & O & \cdots & O \\ O & P_{\overline{\mathcal{R}(A)}} & \cdots & O \\ \vdots & \vdots & \vdots & \vdots \\ O & O & \cdots & P_{\overline{\mathcal{R}(A)}} \end{bmatrix}$. Now, for $x = [x_1, x_2, \dots, x_n]^{\mathrm{T}} \in \bigoplus_{i=1}^{n} \mathcal{H}$, we have

$$\begin{aligned} \|\mathbb{U}x\|_{\mathbb{A}}^2 = \langle \mathbb{U}x, \mathbb{U}x\rangle_{\mathbb{A}} = \langle \mathbb{U}^{\#_{\mathbb{A}}}\mathbb{U}x, x\rangle_{\mathbb{A}} &= \left\langle \bigoplus_{i=1}^{n} P_{\overline{\mathcal{R}(A)}}x, x \right\rangle_{\mathbb{A}} \\ &= \left\langle \bigoplus_{i=1}^{n} AP_{\overline{\mathcal{R}(A)}}x, x \right\rangle \\ &= \left\langle \bigoplus_{i=1}^{n} AA^{\dagger}Ax, x \right\rangle \\ &= \left\langle \bigoplus_{i=1}^{n} Ax, x \right\rangle \\ &= \|x\|_{\mathbb{A}}^2. \end{aligned}$$

So, $\|\mathbb{U}x\|_{\mathbb{A}} = \|x\|_{\mathbb{A}}$. Similarly, it can be proved that $\|\mathbb{U}^{\#_{\mathbb{A}}}x\|_{\mathbb{A}} = \|x\|_{\mathbb{A}}$. Thus, $\mathbb{U}$ is an $\mathbb{A}$-unitary operator. Now, using Lemma 1.4 we have,

$$\begin{aligned}
\mathbb{U}\mathbb{T}_{\text{circ}}^{\#_{\mathbb{A}}}\mathbb{U}^{\#_{\mathbb{A}}} &= \bigoplus_{k=0}^{n-1}\sum_{i=1}^{n} P_{\overline{\mathcal{R}(A)}}\omega^{k(i-1)}T_i^{\#_A} \\
&= \bigoplus_{k=0}^{n-1}\sum_{i=1}^{n} \omega^{k(i-1)}T_i^{\#_A} \qquad \text{as } \mathcal{R}(T_i^{\#_A}) \subseteq \overline{\mathcal{R}(A)} \\
&= \left(\bigoplus_{k=0}^{n-1}\sum_{i=1}^{n} \bar{\omega}^{k(i-1)}T_i\right)^{\#_{\mathbb{A}}} \\
&= \left(\bigoplus_{k=0}^{n-1}\sum_{i=1}^{n} \omega^{k(1-i)}T_i\right)^{\#_{\mathbb{A}}}.
\end{aligned}$$

Using the fact that $w_{\mathbb{A}}(\mathbb{T}) = w_{\mathbb{A}}(\mathbb{U}\mathbb{T}\mathbb{U}^{\#_{\mathbb{A}}})$ for any $\mathbb{T} \in \mathcal{B}_A(\mathcal{H})$, we get

$$\begin{aligned}
w_{\mathbb{A}}(\mathbb{T}_{\text{circ}}) = w_{\mathbb{A}}(\mathbb{T}_{\text{circ}}^{\#_{\mathbb{A}}}) = w_{\mathbb{A}}(\mathbb{U}\mathbb{T}_{\text{circ}}^{\#_{\mathbb{A}}}\mathbb{U}^{\#_{\mathbb{A}}}) &= w_{\mathbb{A}}\left(\left(\bigoplus_{k=0}^{n-1}\sum_{i=1}^{n} \omega^{k(1-i)}T_i\right)^{\#_{\mathbb{A}}}\right) \\
&= w_{\mathbb{A}}\left(\left(\bigoplus_{k=0}^{n-1}\sum_{i=1}^{n} \omega^{k(1-i)}T_i\right)\right) \\
&= \max\left\{ w_A\left(\sum_{i=1}^{n} T_i\right), w_A\left(\sum_{i=1}^{n} \omega^{(1-i)}T_i\right), \ldots, w_A\left(\sum_{i=1}^{n} \omega^{(n-1)(1-i)}T_i\right)\right\} \\
&= \max\left\{ w_A\left(\sum_{i=1}^{n} \omega^{k(1-i)}T_i\right) : k = 0, 1, \ldots, n-1\right\},
\end{aligned}$$

where the last equality follows from Lemma 1.5. □

As a special case of Theorem 2.1, we have part (iv) of Lemma 1.1.

Our next result is an estimate for $\mathbb{A}$-numerical radius of skew circulant operator matrices.

Theorem 2.2 *Let $T_i \in \mathcal{B}_A(\mathcal{H})$ for $1 \le i \le n$. Then*

$$w_{\mathbb{A}}(\mathbb{T}_{\text{scirc}}) = \max\left\{ w_A\left(\sum_{i=1}^{n}(\sigma\omega^k)^{1-i}T_i\right) : k = 0, 1, \ldots, n-1\right\},$$

where $\sigma = e^{\pi i/n}$ and $\omega = e^{2\pi i/n}$.

Proof The n roots of the equation $z^n=-1$ are $\sigma, \sigma\omega, \sigma\omega^2, \ldots, \sigma\omega^{n-1}$. Let $\mathbb{T}_{\text{scirc}} = \begin{bmatrix} T_1 & T_2 & T_3 & \cdots & T_n \\ -T_n & T_1 & T_2 & \cdots & T_{n-1} \\ -T_{n-1} & -T_n & T_1 & \ddots & T_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -T_2 & -T_3 & \cdots & -T_n & T_1 \end{bmatrix}$ and

$$\mathbb{U} = \frac{1}{\sqrt{n}} \begin{bmatrix} I & \sigma I & \sigma^2 I & \cdots & \sigma^{n-1} I \\ (\sigma\omega) I & (\sigma\omega)^2 I & (\sigma\omega)^3 I & \cdots & (\sigma\omega)^n I \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\sigma\omega^{n-2})^{n-2} I & (\sigma\omega^{n-2})^{n-1} I & (\sigma\omega^{n-2})^{n} I & \cdots & (\sigma\omega^{n-2})^{2n-1} I \\ (\sigma\omega^{n-1})^{n-1} I & (\sigma\omega^{n-1})^{n} I & (\sigma\omega^{n-1})^{n+1} I & \cdots & (\sigma\omega^{n-1})^{2n-2} I \end{bmatrix}.$$

Using a similar argument as used in the Theorem 2.1, we can show that $\mathbb{U}$ is $\mathbb{A}$-unitary.

Now, using Lemma 1.4, we have

$$\mathbb{U}\mathbb{T}_{\text{scirc}}^{\#_{\mathbb{A}}}\mathbb{U}^{\#_{\mathbb{A}}} = \left(\bigoplus_{k=0}^{n-1} \sum_{i=1}^{n} (\sigma\omega^k)^{1-i} T_i \right)^{\#_{\mathbb{A}}}.$$

Using the property $w_{\mathbb{A}}(\mathbb{T}) = w_{\mathbb{A}}(\mathbb{U}\mathbb{T}\mathbb{U}^{\#_{\mathbb{A}}})$ for any $\mathbb{T} \in \mathcal{B}_A(\mathcal{H})$, we get

$$\begin{aligned} w_{\mathbb{A}}(\mathbb{T}_{\text{scirc}}) = w_{\mathbb{A}}(\mathbb{T}_{\text{scirc}}^{\#_{\mathbb{A}}}) = w_{\mathbb{A}}(\mathbb{U}\mathbb{T}_{\text{scirc}}^{\#_{\mathbb{A}}}\mathbb{U}^{\#_{\mathbb{A}}}) &= w_{\mathbb{A}}\left(\left(\bigoplus_{k=0}^{n-1} \sum_{i=1}^{n} (\sigma\omega^k)^{1-i} T_i \right)^{\#_{\mathbb{A}}} \right) \\ &= w_{\mathbb{A}}\left(\bigoplus_{k=0}^{n-1} \sum_{i=1}^{n} (\sigma\omega^k)^{1-i} T_i \right) \\ &= \max \left\{ w_A\left(\sum_{i=1}^{n} (\sigma\omega^k)^{1-i} T_i \right) : k = 0, 1, \ldots, n-1 \right\}, \end{aligned}$$

where the last equality follows from Lemma 1.5. □

As a special case of the above theorem we have the following corollary which is already proved in [35].

Corollary 2.3 *Let $T_1, T_2 \in \mathcal{B}_A(\mathcal{H})$. Then*

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ -T_2 & T_1 \end{bmatrix} \right) = \max\{w_A(T_1 + iT_2), w_A(T_1 - iT_2)\}.$$

Theorem 2.4 provides $\mathbb{A}$-numerical radius equalities for imaginary circulant operator matrices.

Theorem 2.4 *Let $T_i \in \mathcal{B}_A(\mathcal{H})$ for $1 \le i \le n$. Then*

$$w_{\mathbb{A}}(\mathbb{T}_{\mathrm{circ}_i}) = \max\left\{ w_A\left(\sum_{i=1}^{n}(\alpha\omega^k)^{i-1}T_i\right) : k = 0, 1, \ldots, n-1 \right\},$$

where $\alpha = e^{\pi i/2n}$ and $\omega = e^{2\pi i/n}$.

Proof The n roots of the equation $z^n = i$ are $\alpha, \alpha\omega, \alpha\omega^2, \ldots, \alpha\omega^{n-1}$. Let $\mathbb{T}_{\mathrm{circ}_i} = \begin{bmatrix} T_1 & T_2 & T_3 & \cdots & T_n \\ iT_n & T_1 & T_2 & \cdots & T_{n-1} \\ iT_{n-1} & iT_n & T_1 & \ddots & T_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ iT_2 & iT_3 & \cdots & iT_n & T_1 \end{bmatrix}$ and

$$\mathbb{U} = \frac{1}{\sqrt{n}}\begin{bmatrix} I & I & I & \cdots & I \\ \alpha I & \alpha\omega I & \alpha\omega^2 I & \cdots & \alpha\omega^{n-1}I \\ \alpha^2 I & (\alpha\omega)^2 I & (\alpha\omega^2)^2 I & \cdots & (\alpha\omega^{n-1})^2 I \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha^{n-1}I & (\alpha\omega)^{n-1}I & (\alpha\omega^2)^{n-1}I & \cdots & (\alpha\omega^{n-1})^{n-1}I \end{bmatrix}.$$

Using a similar argument as used in the Theorem 2.1, we can show that $\mathbb{U}$ is $\mathbb{A}$-unitary.

Now, using Lemma 1.4, we have

$$\mathbb{U}^{\#_{\mathbb{A}}}\mathbb{T}_{\mathrm{circ}_i}^{\#_{\mathbb{A}}}\mathbb{U} = \left(\bigoplus_{k=0}^{n-1}\sum_{i=1}^{n}(\alpha\omega^k)^{i-1}T_i\right)^{\#_{\mathbb{A}}}.$$

Using the property $w_{\mathbb{A}}(\mathbb{T}) = w_{\mathbb{A}}(\mathbb{U}^{\#_{\mathbb{A}}}\mathbb{T}\mathbb{U})$ for any $\mathbb{T} \in \mathcal{B}_A(\mathcal{H})$, we get

$$\begin{aligned} w_{\mathbb{A}}(\mathbb{T}_{\mathrm{circ}_i}) = w_{\mathbb{A}}(\mathbb{T}_{\mathrm{circ}_i}^{\#_{\mathbb{A}}}) = w_{\mathbb{A}}(\mathbb{U}^{\#_{\mathbb{A}}}\mathbb{T}_{\mathrm{circ}_i}^{\#_{\mathbb{A}}}\mathbb{U}) &= w_{\mathbb{A}}\left(\bigoplus_{k=0}^{n-1}\sum_{i=1}^{n}(\alpha\omega^k)^{i-1}T_i\right)^{\#_{\mathbb{A}}} \\ &= w_{\mathbb{A}}\left(\bigoplus_{k=0}^{n-1}\sum_{i=1}^{n}(\alpha\omega^k)^{i-1}T_i\right) \\ &= \max\left\{ w_A\left(\sum_{i=1}^{n}(\alpha\omega^k)^{i-1}T_i\right) : k = 0, 1, \ldots, n-1 \right\}, \end{aligned}$$

where the last equality follows from Lemma 1.5. □

As a special case of the above theorem we have the following corollary.

Corollary 2.5 *Let $T_1, T_2 \in \mathcal{B}_A(\mathcal{H})$. Then*

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ iT_2 & T_1 \end{bmatrix}\right) = \max\left\{ w_A\left(T_1 + \frac{1+i}{\sqrt{2}}T_2\right), w_A\left(T_1 - \frac{1+i}{\sqrt{2}}T_2\right) \right\}.$$

In Theorem 2.6, we give an estimate for imaginary skew circulant operator matrices.

Theorem 2.6 *Let $T_i \in \mathcal{B}_A(\mathcal{H})$ for $1 \leq i \leq n$. Then*

$$w_{\mathbb{A}}(\mathbb{T}_{\mathrm{scirc}_i}) = \max\left\{ w_A\left(\sum_{i=1}^{n} (\beta\omega^k)^{1-i} T_i \right) \,:\, k = 0, 1, \ldots, n-1 \right\},$$

where $\beta = e^{\frac{-\pi i}{2n}}$ and $\omega = e^{2\pi i/n}$.

Proof The n roots of the equation $z^n = -i$ are $\beta, \beta\omega, \beta\omega^2, \ldots, \beta\omega^{n-1}$.
Let

$$\mathbb{T}_{\mathrm{scirc}_i} = \begin{bmatrix} T_1 & T_2 & T_3 & \cdots & T_n \\ -iT_n & T_1 & T_2 & \cdots & T_{n-1} \\ -iT_{n-1} & -iT_n & T_1 & \ddots & T_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -iT_2 & -iT_3 & \cdots & -iT_n & T_1 \end{bmatrix}$$

and

$$\mathbb{U} = \frac{1}{\sqrt{n}} \begin{bmatrix} I & I & I & \cdots & I \\ \beta I & \beta\omega I & \beta\omega^2 I & \cdots & \beta\omega^{n-1} I \\ \beta^2 I & (\beta\omega)^2 I & (\beta\omega^2)^2 I & \cdots & (\beta\omega^{n-1})^2 I \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta^{n-1} I & (\beta\omega)^{n-1} I & (\beta\omega^2)^{n-1} I & \cdots & (\beta\omega^{n-1})^{n-1} I \end{bmatrix}.$$

The rest of the proof follows using a similar method as used in Theorem 2.4. □

As a special case of the above theorem, we have the following corollary.

Corollary 2.7 *Let $T_1, T_2 \in \mathcal{B}_A(\mathcal{H})$. Then*

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ -iT_2 & T_1 \end{bmatrix} \right) = \max\left\{ w_A\left(T_1 + \frac{1-i}{\sqrt{2}} T_2 \right), w_A\left(T_1 - \frac{1-i}{\sqrt{2}} T_2 \right) \right\}.$$

3 Pinching type $\mathbb{A}$-numerical radius inequalities for operator matrices

The pinching type inequalities are among the most inequalities of operator matrices. Very recently, Rout et al. [35] established some pinching type $\mathbb{A}$-numerical radius inequalities (see Lemma 1.3). For usual pinching type numerical radius inequalities, one may consult [21]. Our goal of this section is to establish certain pinching type $\mathbb{A}$-numerical radius inequalities for $n \times n$ block operator matrices.

Theorem 3.1 *Let $\mathbb{T} = [T_{ik}]$ be an operator matrix where $T_{ik} \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\frac{1}{n} w_{\mathbb{A}}\left(\bigoplus_{k=0}^{n-1} \sum_{i=1}^{n} \omega^{k(i-1)} S_i \right) \leq w_{\mathbb{A}}(\mathbb{T}),$$

where

$$S_i = \sum_{j=1}^{n} T_{jj+i-1}^{\#_A}, \text{ with } \; T_{n+i} = T_i (\text{we could say that the subscripts are modulo } n),$$
$$i = 1, 2, \dots, n \; \text{ and } \; \omega = e^{2\pi i/n}.$$

Proof Let $\mathbb{M}_{k+1,k+n+1} = [m_{rs}]$ be the $n \times n$ operator matrix with

$$m_{rs} = \begin{cases} I; & \text{for } r+s = k+1 \text{ or } r+s = k+n+1, \\ O; & \text{otherwise.} \end{cases} \tag{5}$$

Then we need to prove that $\mathbb{M}_{k+1,k+n+1}$ is an $\mathbb{A}$-unitary operator for all $k = 1, 2, \dots, n$, i.e.,

$$\mathbb{M}_{2,n+2} = \begin{bmatrix} I & O & O & \cdots & O & O \\ O & O & O & \cdots & O & I \\ O & O & O & \cdots & I & O \\ \vdots & \vdots & \vdots & .\cdot^{\cdot} & \vdots & \vdots \\ O & O & I & \cdots & O & O \\ O & I & O & \cdots & O & O \end{bmatrix},$$

$$\mathbb{M}^{\#_{\mathbb{A}}}_{2,n+2} = \begin{bmatrix} P_{\overline{\mathcal{R}(A)}} & O & O & \cdots & O & O \\ O & O & O & \cdots & O & P_{\overline{\mathcal{R}(A)}} \\ O & O & O & \cdots & P_{\overline{\mathcal{R}(A)}} & 0 \\ \vdots & \vdots & \vdots & .\cdot^{\cdot} & \vdots & \vdots \\ O & O & P_{\overline{\mathcal{R}(A)}} & \cdots & O & O \\ O & P_{\overline{\mathcal{R}(A)}} & O & \cdots & O & O \end{bmatrix} \quad as \;\; \mathcal{N}(A)^{\perp} = \overline{\mathcal{R}(A^*)} \;\; and \;\; \mathcal{R}(A^*) = \mathcal{R}(A).$$

Now, $\mathbb{M}_{2,n+2}\mathbb{M}^{\#_{\mathbb{A}}}_{2,n+2} = \begin{bmatrix} P_{\overline{\mathcal{R}(A)}} & O & \cdots & O \\ O & P_{\overline{\mathcal{R}(A)}} & \cdots & O \\ \vdots & \vdots & \vdots & \vdots \\ O & O & \cdots & P_{\overline{\mathcal{R}(A)}} \end{bmatrix} = \mathbb{M}^{\#_{\mathbb{A}}}_{2,n+2}\mathbb{M}_{2,n+2}.$

For $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in \bigoplus_{i=1}^{n} \mathcal{H}$, we have

$$\begin{aligned}
\|\mathbb{M}_{2,n+2}x\|_{\mathbb{A}}^2 = \langle \mathbb{M}_{2,n+2}x, \mathbb{M}_{2,n+2}x\rangle_{\mathbb{A}} &= \langle \mathbb{M}_{2,n+2}^{\#_{\mathbb{A}}}\mathbb{M}_{2,n+2}x, x\rangle_{\mathbb{A}} \\
&= \left\langle \bigoplus_{i=1}^{n} P_{\overline{\mathcal{R}(A)}}x, x \right\rangle_{\mathbb{A}} \\
&= \left\langle \bigoplus_{i=1}^{n} AP_{\overline{\mathcal{R}(A)}}x, x \right\rangle \\
&= \left\langle \bigoplus_{i=1}^{n} AA^{\dagger}Ax, x \right\rangle \\
&= \left\langle \bigoplus_{i=1}^{n} Ax, x \right\rangle \\
&= \|x\|_{\mathbb{A}}^2.
\end{aligned}$$

So, $\|\mathbb{M}_{2,n+2}x\|_{\mathbb{A}} = \|x\|_{\mathbb{A}}$. Similarly, it can be proved that $\|\mathbb{M}_{2,n+2}^{\#_{\mathbb{A}}}x\|_{\mathbb{A}} = \|x\|_{\mathbb{A}}$. Thus, $\mathbb{M}_{2,n+2}$ is an $\mathbb{A}$-unitary operator. Similarly, it can be shown that other operator matrices $\mathbb{M}_{k+1,k+n+1}$ are $\mathbb{A}$-unitary operators for all $k = 1, 2, \ldots, n$.

Thus, using Lemma 1.4, we get

$$\mathbb{M}_{2,n+2}\mathbb{T}^{\#_{\mathbb{A}}}\mathbb{M}_{2,n+2}^{\#_{\mathbb{A}}} = \begin{bmatrix} T_{11}^{\#_A}P_{\overline{\mathcal{R}(A)}} & T_{n1}^{\#_A}P_{\overline{\mathcal{R}(A)}} & \cdots & T_{21}^{\#_A}P_{\overline{\mathcal{R}(A)}} \\ T_{1n}^{\#_A}P_{\overline{\mathcal{R}(A)}} & T_{nn}^{\#_A}P_{\overline{\mathcal{R}(A)}} & \cdots & T_{2n}^{\#_A}P_{\overline{\mathcal{R}(A)}} \\ \vdots & \vdots & \ddots & \vdots \\ T_{12}^{\#_A}P_{\overline{\mathcal{R}(A)}} & T_{n2}^{\#_A}P_{\overline{\mathcal{R}(A)}} & \cdots & T_{22}^{\#_A}P_{\overline{\mathcal{R}(A)}} \end{bmatrix}.$$

Similarly,

$$\mathbb{M}_{3,n+3} = \begin{bmatrix} O & I & O & \cdots & O \\ I & O & O & \cdots & O \\ O & O & O & \cdots & I \\ \vdots & \vdots & \vdots & \iddots & \vdots \\ O & O & I & \cdots & O \end{bmatrix},$$

$$\mathbb{M}_{3,n+3}^{\#_{\mathbb{A}}} = \begin{bmatrix} O & P_{\overline{\mathcal{R}(A)}} & O & \cdots & O \\ P_{\overline{\mathcal{R}(A)}} & O & O & \cdots & O \\ O & O & O & \cdots & P_{\overline{\mathcal{R}(A)}} \\ \vdots & \vdots & \vdots & \iddots & \vdots \\ O & O & P_{\overline{\mathcal{R}(A)}} & \cdots & O \end{bmatrix},$$

and $\mathbb{M}_{3,n+3}\mathbb{T}^{\#_{\mathbb{A}}}\mathbb{M}_{3,n+3}^{\#_{\mathbb{A}}} = \begin{bmatrix} T_{22}^{\#_A}P_{\overline{\mathcal{R}(A)}} & T_{12}^{\#_A}P_{\overline{\mathcal{R}(A)}} & \cdots & T_{32}^{\#_A}P_{\overline{\mathcal{R}(A)}} \\ T_{21}^{\#_A}P_{\overline{\mathcal{R}(A)}} & T_{11}^{\#_A}P_{\overline{\mathcal{R}(A)}} & \cdots & T_{31}^{\#_A}P_{\overline{\mathcal{R}(A)}} \\ T_{2n}^{\#_A}P_{\overline{\mathcal{R}(A)}} & T_{1n}^{\#_A}P_{\overline{\mathcal{R}(A)}} & \cdots & T_{3n}^{\#_A}P_{\overline{\mathcal{R}(A)}} \\ \vdots & \vdots & \ddots & \vdots \\ T_{23}^{\#_A}P_{\overline{\mathcal{R}(A)}} & T_{13}^{\#_A}P_{\overline{\mathcal{R}(A)}} & \cdots & T_{33}^{\#_A}P_{\overline{\mathcal{R}(A)}} \end{bmatrix}$. Proceeding in this way, we have

$$\mathbb{M}_{n,2n} = \begin{bmatrix} O & O & \cdots & O & I & O \\ O & O & \cdots & I & O & O \\ \vdots & \iddots & \vdots & \vdots & \vdots & \vdots \\ I & O & \cdots & O & O & O \\ O & O & \cdots & O & O & I \end{bmatrix},$$

$$\mathbb{M}_{n,2n}^{\#_{\mathbb{A}}} = \begin{bmatrix} O & O & \cdots & O & P_{\overline{\mathcal{R}(A)}} & O \\ O & O & \cdots & P_{\overline{\mathcal{R}(A)}} & O & O \\ \vdots & \iddots & \vdots & \vdots & \vdots & \vdots \\ P_{\overline{\mathcal{R}(A)}} & O & \cdots & O & O & O \\ O & O & \cdots & O & O & P_{\overline{\mathcal{R}(A)}} \end{bmatrix},$$

$$\mathbb{M}_{n,2n}\mathbb{T}^{\#_{\mathbb{A}}}\mathbb{M}_{n,2n}^{\#_{\mathbb{A}}} = \begin{bmatrix} T_{n-1n-1}^{\#_A}P_{\overline{\mathcal{R}(A)}} & \cdots & T_{2n-1}^{\#_A}P_{\overline{\mathcal{R}(A)}} & T_{1n-1}^{\#_A}P_{\overline{\mathcal{R}(A)}} & T_{nn-1}^{\#_A}P_{\overline{\mathcal{R}(A)}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T_{n-11}^{\#_A}P_{\overline{\mathcal{R}(A)}} & \cdots & T_{21}^{\#_A}P_{\overline{\mathcal{R}(A)}} & T_{11}^{\#_A}P_{\overline{\mathcal{R}(A)}} & T_{n1}^{\#_A}P_{\overline{\mathcal{R}(A)}} \\ T_{n-1n}^{\#_A}P_{\overline{\mathcal{R}(A)}} & \cdots & T_{2n}^{\#_A}P_{\overline{\mathcal{R}(A)}} & T_{1n}^{\#_A}P_{\overline{\mathcal{R}(A)}} & T_{nn}^{\#_A}P_{\overline{\mathcal{R}(A)}} \end{bmatrix},$$

$$\mathbb{M}_{n+1,2n+1} = \begin{bmatrix} O & O & \cdots & O & I \\ O & O & \cdots & I & O \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ O & I & \cdots & O & O \\ I & O & \cdots & O & O \end{bmatrix},$$

$$\mathbb{M}^{\#_{\mathbb{A}}}_{n+1,2n+1} = \begin{bmatrix} O & O & \cdots & O & P_{\overline{\mathcal{R}(A)}} \\ O & O & \cdots & P_{\overline{\mathcal{R}(A)}} & O \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ O & P_{\overline{\mathcal{R}(A)}} & \cdots & O & O \\ P_{\overline{\mathcal{R}(A)}} & O & \cdots & O & O \end{bmatrix},$$

and $\mathbb{M}_{n+1,2n+1}\mathbb{T}^{\#_{\mathbb{A}}}\mathbb{M}^{\#_{\mathbb{A}}}_{n+1,2n+1} = \begin{bmatrix} T^{\#_A}_{nn}P_{\overline{\mathcal{R}(A)}} & \cdots & T^{\#_A}_{2n}P_{\overline{\mathcal{R}(A)}} & T^{\#_A}_{1n}P_{\overline{\mathcal{R}(A)}} \\ T^{\#_A}_{nn-1}P_{\overline{\mathcal{R}(A)}} & \cdots & T^{\#_A}_{2n-1}P_{\overline{\mathcal{R}(A)}} & T^{\#_A}_{1n-1}P_{\overline{\mathcal{R}(A)}} \\ \vdots & \vdots & \ddots & \vdots \\ T^{\#_A}_{n1}P_{\overline{\mathcal{R}(A)}} & \cdots & T^{\#_A}_{21}P_{\overline{\mathcal{R}(A)}} & T^{\#_A}_{11}P_{\overline{\mathcal{R}(A)}} \end{bmatrix}$. So,

$$\sum_{k=1}^{n} \mathbb{M}_{k+1,k+n+1}\mathbb{T}^{\#_{\mathbb{A}}}\mathbb{M}^{\#_{\mathbb{A}}}_{k+1,k+n+1} = \begin{bmatrix} S_1 & S_2 & \cdots & S_{n-1} & S_n \\ S_n & S_1 & \cdots & S_{n-2} & S_{n-1} \\ S_{n-1} & S_n & \ddots & S_{n-3} & S_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ S_2 & S_3 & \cdots & S_n & S_1 \end{bmatrix} = \mathbb{S}_{\mathrm{circ}}(S_1, \dots, S_n),$$

where

$$S_i = \sum_{j=1}^{n} T^{\#_A}_{jj+i-1}, \text{ with } T_{n+i} = T_i (\text{we could say that the subscripts are modulo } n),$$
$$i = 1, 2, \dots, n \text{ and } \omega = e^{2\pi i/n}.$$

Let $\mathbb{U} = \frac{1}{\sqrt{n}} \begin{bmatrix} I & I & I & I & \cdots & I \\ I & \omega I & \omega^2 I & \omega^3 I & \cdots & \omega^{n-1} I \\ I & \omega^2 I & \omega^4 I & \omega^6 I & \cdots & \omega^{n-2} I \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ I & \omega^{n-1} I & \omega^{n-2} I & \omega^{n-3} I & \cdots & \omega I \end{bmatrix}$. Then, using the proof of Theorem 2.1, we get

$$\mathbb{U}\mathbb{S}^{\#_{\mathbb{A}}}_{\mathrm{circ}}\mathbb{U}^{\#_{\mathbb{A}}} = \left(\bigoplus_{k=0}^{n-1} \sum_{i=1}^{n} \omega^{k(1-i)} S_i \right)^{\#_{\mathbb{A}}}. \tag{6}$$

Using the property $w_{\mathbb{A}}(\mathbb{U}\mathbb{T}\mathbb{U}^{\#_{\mathbb{A}}}) = w_{\mathbb{A}}(\mathbb{T})$ and triangle inequality, we get

$$\begin{aligned}
w_{\mathbb{A}}\left(\bigoplus_{k=0}^{n-1}\sum_{i=1}^{n}\omega^{k(1-i)}S_i\right) &= w_{\mathbb{A}}\left(\left(\bigoplus_{k=0}^{n-1}\sum_{i=1}^{n}\omega^{k(1-i)}S_i\right)^{\#_{\mathbb{A}}}\right)\\
&= w_{\mathbb{A}}\left(\mathbb{U}\mathbb{S}_{\text{circ}}^{\#_{\mathbb{A}}}\mathbb{U}^{\#_{\mathbb{A}}}\right)\\
&= w_{\mathbb{A}}(\mathbb{S}_{\text{circ}}^{\#_{\mathbb{A}}})\\
&= w_{\mathbb{A}}(\mathbb{S}_{\text{circ}})\\
&= w_{\mathbb{A}}\left(\sum_{k=1}^{n}\mathbb{M}_{k+1,k+n+1}\mathbb{T}^{\#_{\mathbb{A}}}\mathbb{M}_{k+1,k+n+1}^{\#_{\mathbb{A}}}\right)\\
&\le n w_{\mathbb{A}}(\mathbb{T}^{\#_{\mathbb{A}}})\\
&= n w_{\mathbb{A}}(\mathbb{T}).
\end{aligned}$$

Hence, $\frac{1}{n}w_{\mathbb{A}}\left(\bigoplus_{k=0}^{n-1}\sum_{i=1}^{n}\omega^{k(1-i)}S_i\right) \le w_{\mathbb{A}}(\mathbb{T})$. □

Corollary 3.2 *Let $\mathbb{T} = [T_{ik}]$ be an operator matrix where $T_{ik} \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\frac{1}{n}\max\left\{w_A\left(\sum_{i=1}^{n}\omega^{k(1-i)}S_i\right) : k = 0, 1, \dots, n-1\right\} \le w_{\mathbb{A}}(\mathbb{T}).$$

As a special case of the above corollary, we have

$$\frac{1}{2}\max\{w_A(T_{11}+T_{22}+T_{21}+T_{12}), w_A(T_{11}+T_{22}-T_{21}-T_{12})\} \le w_{\mathbb{A}}\left(\begin{bmatrix} T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix}\right).$$

We remark here that the $\mathbb{A}$-numerical radius inequality in Theorem 3.1 is sharp.

Remark 3.3 Let $T_i \in \mathcal{B}_A(\mathcal{H})$ for $1 \le i \le n$. If $\mathbb{T}$ is a circulant operator matrix, then the inequality in Theorem 3.1 becomes equality. So, the $\mathbb{A}$-numerical radius inequality in Theorem 3.1 is sharp.

Remark 3.4 Based on Theorem 2.2, one can employ a similar argument as used in Theorem 3.1 to obtain a pinching type inequality analogous to that in Theorem 3.1.

Theorem 3.5 *Let $\mathbb{T} = [T_{jk}]$ be an operator matrix where $T_{jk} \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\frac{1}{n}w_{\mathbb{A}}\left(\bigoplus_{k=0}^{n-1}\sum_{j=1}^{n}(\alpha\omega^k)^{j-1}S_j\right) \le w_{\mathbb{A}}(\mathbb{T}),$$

where

$$S_j = e^{\frac{-i\pi}{2}} \sum_{k=1}^{j-1} T^{\#_A}_{kk+n-j+1} + \sum_{k=j}^{n} T^{\#_A}_{kk+n-j+1}, \quad \text{with}$$
$$T_{n+j} = T_j (\text{we could say that the subscripts are modulo } n), j = 1, \dots, n,$$

$\alpha = e^{\pi i/2n}$ *and* $\omega = e^{2\pi i/n}$.

Proof Let $\mathbb{M}_{k,n-k} = [m_{rs}]$ be the $n \times n$ operator matrix with

$$m_{rs} = \begin{cases} I \ ; \ s-r=k, \\ e^{\frac{i\pi}{2}} I; \ r-s=n-k, \\ O; \ \text{otherwise.} \end{cases} \tag{7}$$

It is not difficult to show that $\mathbb{M}_{k,n-k}$ is an $\mathbb{A}$-unitary operator for all $k = 1, 2, \dots, n$. Now,

$$\sum_{k=1}^{n} \mathbb{M}_{k,n-k} \mathbb{T}^{\#_{\mathbb{A}}} \mathbb{M}^{\#_{\mathbb{A}}}_{k,n-k} = \begin{bmatrix} S_1 & S_2 & \cdots & S_{n-1} & S_n \\ iS_n & S_1 & \cdots & S_{n-2} & S_{n-1} \\ iS_{n-1} & iS_n & \ddots & S_{n-3} & S_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ iS_2 & iS_3 & \cdots & iS_n & S_1 \end{bmatrix} = \mathbb{S}_{\text{circ}_i}(S_1, S_2, \dots, S_n),$$

where

$$S_j = e^{\frac{-i\pi}{2}} \sum_{k=1}^{j-1} T^{\#_A}_{kk+n-j+1} + \sum_{k=j}^{n} T^{\#_A}_{kk+n-j+1}, \text{with}$$
$$T_{n+j} = T_j (\text{we could say that the subscripts are modulo } n), \quad j = 1, \dots, n.$$

Let

$$\mathbb{U} = \frac{1}{\sqrt{n}} \begin{bmatrix} I & I & I & \cdots & I \\ \alpha I & \alpha\omega I & \alpha\omega^2 I & \cdots & \alpha\omega^{n-1} I \\ \alpha^2 I & (\alpha\omega)^2 I & (\alpha\omega^2)^2 I & \cdots & (\alpha\omega^{n-1})^2 I \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha^{n-1} I & (\alpha\omega)^{n-1} I & (\alpha\omega^2)^{n-1} I & \cdots & (\alpha\omega^{n-1})^{n-1} I \end{bmatrix}.$$

Then, using the proof of Theorem 2.4, we get

$$\mathbb{U}^{\#_{\mathbb{A}}} \mathbb{S}^{\#_{\mathbb{A}}}_{\text{circ}_i} \mathbb{U} = \left(\bigoplus_{k=0}^{n-1} \sum_{j=1}^{n} (\alpha\omega^k)^{j-1} S_j \right)^{\#_{\mathbb{A}}}. \tag{8}$$

Using the property $w_{\mathbb{A}}(\mathbb{U}^{\#_{\mathbb{A}}} \mathbb{T} \mathbb{U}) = w_{\mathbb{A}}(\mathbb{T})$ and triangle inequality, we get

$$
\begin{aligned}
w_{\mathbb{A}}\left(\left(\bigoplus_{k=0}^{n-1}\sum_{j=1}^{n}(\alpha\omega^k)^{j-1}S_j\right)^{\#_{\mathbb{A}}}\right) &= w_{\mathbb{A}}\left(\bigoplus_{k=0}^{n-1}\sum_{j=1}^{n}(\alpha\omega^k)^{j-1}S_j\right)\\
&= w_{\mathbb{A}}\left(\mathbb{U}^{\#_{\mathbb{A}}}\mathbb{S}_{\mathrm{circ}_i}^{\#_{\mathbb{A}}}\mathbb{U}\right)\\
&= w_{\mathbb{A}}(\mathbb{S}_{\mathrm{circ}_i}^{\#_{\mathbb{A}}})\\
&= w_{\mathbb{A}}(\mathbb{S}_{\mathrm{circ}_i})\\
&= w_{\mathbb{A}}\left(\sum_{k=1}^{n}\mathbb{M}_{k,n-k}\mathbb{T}^{\#_{\mathbb{A}}}\mathbb{M}_{k,n-k}^{\#_{\mathbb{A}}}\right)\\
&\le n w_{\mathbb{A}}(\mathbb{T}^{\#_{\mathbb{A}}})\\
&= n w_{\mathbb{A}}(\mathbb{T}).
\end{aligned}
$$

Hence, $\frac{1}{n}w_{\mathbb{A}}\left(\bigoplus_{k=0}^{n-1}\sum_{j=1}^{n}(\alpha\omega^k)^{j-1}S_j\right) \le w_{\mathbb{A}}(\mathbb{T})$. □

Corollary 3.6 *Let $\mathbb{T} = [T_{jk}]$ be an operator matrix where $T_{jk} \in \mathcal{B}_A(\mathcal{H})$. Then*

$$
\frac{1}{n}\max\left\{ w_{\mathbb{A}}\left(\sum_{j=1}^{n}(\alpha\omega^k)^{j-1}S_j\right) \,:\, k = 0, 1, \dots, n-1 \right\} \le w_{\mathbb{A}}(\mathbb{T}).
$$

As a special case of the above corollary we have

$$
\frac{1}{2}\max\left\{ w_A\left(T_{11}+T_{22}+e^{\frac{-i\pi}{4}}(T_{21}+iT_{12})\right), w_A\left(T_{11}+T_{22}-e^{\frac{-i\pi}{4}}(T_{21}+iT_{12})\right)\right\} \le w_{\mathbb{A}}\left(\begin{bmatrix} T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix}\right).
$$

We remark here that the $\mathbb{A}$-numerical radius inequality in the Theorem 3.5 is sharp.

Remark 3.7 Let $T_i \in \mathcal{B}_A(\mathcal{H})$ for $1 \le i \le n$. If $\mathbb{T}$ is an imaginary circulant operator matrix, then the inequality in Theorem 3.5 becomes equality. So, the $\mathbb{A}$-numerical radius inequality in Theorem 3.5 is sharp.

Remark 3.8 Based on Theorem 2.6, one can employ a similar argument as used in Theorem 3.5 to obtain a pinching type inequality analogous to that in Theorem 3.5.

4 $\mathbb{A}$-numerical radius equalities for $n \times n$ tridiagonal and anti-tridiagonal operator matrices

In this section, inspired by the work of Bani-Domi et al. [5], we extend some recent results of [5] to semi-Hilbert space. Using a similar analysis as used in the previous section, one can employ certain $\mathbb{A}$-numerical radius equalities for $n \times n$ tridiagonal and anti-tridiagonal operator matrices. We present here the results without proofs.

Theorem 4.1 *Let $T_1, T_2 \in \mathcal{B}_A(\mathcal{H})$ and $\mathbb{T} = \begin{bmatrix} T_1 & T_2 & O & \cdots & 0 \\ T_2 & T_1 & T_2 & \cdots & \vdots \\ O & T_2 & T_1 & \ddots & O \\ \vdots & \vdots & \ddots & \ddots & T_2 \\ O & O & \cdots & T_2 & T_1 \end{bmatrix}$ be an $n \times n$ tridiagonal operator matrix. Then*

$$w_{\mathbb{A}}(\mathbb{T}) = \max\left\{ w_A\left(T_1 + \left(2\cos\frac{k\pi}{n+1}\right)T_2\right) : k = 1, \dots, n \right\}.$$

Remark 4.2 By setting $\mathbb{A} = I$ in Theorem 4.1, we get a recent result proved by Bani-Domi et al. [5].

Some special cases of Theorem 4.1 are described in the following table.

Results on $\mathbb{A}$-numerical radius	Results on usual numerical radius
Letting $n = 2$ in Theorem 4.1, we have $w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix}\right) = \max\{w_A(T_1 + T_2), w_A(T_1 - T_2)\}$, see [Lemma 1.1].	$w\left(\begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix}\right) = \max\{w(T_1 + T_2), w(T_1 - T_2)\}$, see [21, Lemma 2.1].
If $T_1 = O$ in Theorem 4.1, we have $w_{\mathbb{A}}(\mathbb{T}) = 2\max\left\{\left\vert\cos\frac{k\pi}{n+1}\right\vert w_A(T_2) : k = 1, \dots, n\right\}$. In particular, $w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_2 & O \end{bmatrix}\right) = w_A(T_2)$, [see Lemma 1.1].	$w(\mathbb{T}) = 2\max\left\{\left\vert\cos\frac{k\pi}{n+1}\right\vert w(T_2) : k = 1, \dots, n\right\}$ see [5]. In particular, $w\left(\begin{bmatrix} O & T_2 \\ T_2 & O \end{bmatrix}\right) = w(T_2)$, see [21, Lemma 2.1].
If $T_2 = O$ in Theorem 4.1, we have $w_{\mathbb{A}}(\mathbb{T}) = w_A(T_1)$, see [Lemma 1.1].	$w(\mathbb{T}) = w(T_1)$ see [5].
If $T_1 = T_2$ in Theorem 4.1, we have $w_{\mathbb{A}}(\mathbb{T}) = \max\left\{\left\vert 1 + 2\cos\frac{k\pi}{n+1}\right\vert w_A(T_1) : 1 \le k \le n\right\}$.	$w(\mathbb{T}) = \max\left\{\left\vert 1 + 2\cos\frac{k\pi}{n+1}\right\vert w(T_1) : 1 \le k \le n\right\}$, see [5].
If $T_2 = iT_1$ in Theorem 4.1, we have $w_{\mathbb{A}}(\mathbb{T}) = \max\left\{w_A\left(\left(1 + \left(2\cos\frac{k\pi}{n+1}\right)i\right)T_1\right) : 1 \le k \le n\right\}$.	$w(\mathbb{T}) = \max\left\{w\left(\left(1 + \left(2\cos\frac{k\pi}{n+1}\right)i\right)T_1\right) : 1 \le k \le n\right\}$, see [5].

Theorem 4.3 *Let $T_1, T_2 \in \mathcal{B}_A(\mathcal{H})$, $\mathbb{T} = \begin{bmatrix} T_1 & \alpha^{n-2}T_2 & O & \cdots & O \\ T_2 & T_1 & \alpha^{n-2}T_2 & \cdots & \vdots \\ O & T_2 & T_1 & \ddots & O \\ \vdots & \vdots & \ddots & \ddots & \alpha^{n-2}T_2 \\ O & O & \cdots & T_2 & T_1 \end{bmatrix}$ be an $n \times n$ tridiagonal operator matrix, and let $\alpha = e^{\frac{\pi i}{2n}}$. Then*

$$w_{\mathbb{A}}(\mathbb{T}) = \max\left\{ w_A\left(T_1 + \left(2\alpha^{n-1}\cos\frac{k\pi}{n+1}\right)T_2\right) : k = 1, \dots, n \right\}.$$

Theorem 4.4 *Let* $T_1, T_2 \in \mathcal{B}_A(\mathcal{H})$, $\mathbb{T} = \begin{bmatrix} T_1 & \omega^{n-1}T_2 & O & \cdots & O \\ \omega T_2 & T_1 & \omega^{n-1}T_2 & \cdots & \vdots \\ O & \omega T_2 & T_1 & \ddots & O \\ \vdots & \vdots & \ddots & \ddots & \omega^{n-1}T_2 \\ O & O & \cdots & \omega T_2 & T_1 \end{bmatrix}$ *be an* $n \times n$ *tridiagonal operator matrix, and let* $\omega = e^{\frac{2\pi i}{n}}$. *Then*

$$w_{\mathbb{A}}(\mathbb{T}) = \max\left\{ w_A\left(T_1 + \left(2\cos\frac{k\pi}{n+1}\right)T_2\right) : k = 1, \dots, n \right\}.$$

Theorem 4.5 *Let* $T_1, T_2 \in \mathcal{B}_A(\mathcal{H})$, *and*

$$\mathbb{T} = \begin{bmatrix} O & \cdots & O & T_2 & T_1 \\ \vdots & \iddots & T_2 & T_1 & T_2 \\ O & \iddots & T_1 & T_2 & O \\ T_2 & \iddots & \iddots & \iddots & \vdots \\ T_1 & T_2 & O & \cdots & O \end{bmatrix}$$

be an $n \times n$ *anti-tridiagonal operator matrix. Then*

$$w_{\mathbb{A}}(\mathbb{T}) = \max\left\{ w_A\left((-1)^{k+1}\left[T_1 + \left(2\cos\frac{k\pi}{n+1}\right)T_2\right]\right) : k = 1, \dots, n \right\}.$$

Remark 4.6 By setting $\mathbb{A} = I$ in Theorem 4.3, 4.4, and 4.5, we get usual numerical radius equalities proved by Bani-Domi et al. [5] very recently.

Finally, at the end of this section, we establish certain pinching inequalities for $n \times n$ tridiagonal and anti-tridiagonal operator matrices.

Theorem 4.7 *Let* $T_1, T_2, T_3 \in \mathcal{B}_A(\mathcal{H})$ *and* $\mathbb{T} = \begin{bmatrix} T_1 & T_2 & O & \cdots & O \\ T_3 & T_1 & T_2 & \cdots & \vdots \\ O & T_3 & T_1 & \ddots & O \\ \vdots & \vdots & \ddots & \ddots & T_2 \\ O & O & \cdots & T_3 & T_1 \end{bmatrix}$ *be an* $n \times n$ *tridiagonal operator matrix. Then*

$$\max\left\{ w_A\left(T_1 + \left(\cos\frac{k\pi}{n+1}\right)(T_2 + T_3)\right) : k = 1, \dots, n \right\} \le w_{\mathbb{A}}(\mathbb{T}).$$

Theorem 4.8 *Let $T_1, T_2, T_3 \in \mathcal{B}_A(\mathcal{H})$, and*

$$\mathbb{T} = \begin{bmatrix} O & \cdots & O & T_2 & T_1 \\ \vdots & \iddots & T_2 & T_1 & T_3 \\ O & \iddots & T_1 & T_3 & O \\ T_2 & \iddots & \iddots & \iddots & \vdots \\ T_1 & T_3 & O & \cdots & O \end{bmatrix}$$

be an $n \times n$ anti-tridiagonal operator matrix. Then

$$\max\left\{ w_A\left((-1)^{k+1}\left[T_1 + \left(\cos\frac{k\pi}{n+1}\right)(T_2+T_3)\right]\right) : k = 1, \dots, n \right\} \leq w_{\mathbb{A}}(\mathbb{T}).$$

Remark 4.9 The inequalities in Theorems 4.7 and 4.8 become equalities if $T_2 = T_3$. So, the $\mathbb{A}$-numerical radius inequalities in Theorems 4.7 and 4.8 are sharp. For the usual numerical radius, the usual operator norm and the Schatten p-norms, one can visit [5].

5 Concluding remarks

In this paper, we have presented different $\mathbb{A}$-numerical radius equalities and inequalities, which depend on the nice structure of circulant, skew circulant, imaginary circulant, imaginary skew circulant, tridiagonal, and anti-tridiagonal operator matrices. By employing similar analysis to different special operator matrices, it is possible to obtain further $\mathbb{A}$-numerical radius equalities and inequalities. We now conclude the article by remarking that further study on this topic may develop an interesting area for future research.

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