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ON $\mathfrak{B}$-CONVERGENCE SPACES

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In the present paper closure spaces determined by $\mathfrak{B}$-convergence structures are studied; i.e. we deal with spaces in which the closure is determined by means of a convergence of nets, the domain of which belongs to a previously given class $\mathfrak{B}$ of directed sets. Further, $\mathfrak{B}$-regular spaces and their $\mathfrak{B}$-envelopes are defined and studied.

If $\mathfrak{B}$ is the class of all directed sets, we get the Moore-Smith's convergence ([6], where other references are given, or [3], 35.A.). If $\mathfrak{B}$ contains sets order-isomorphic with ω_0 only, we get the sequential convergence (Frechet's $\mathscr{L}$-spaces, [8], [9], [7], also [3], 35B.).

In Section 1 the $\mathfrak{B}$-convergence classes and determining $\mathfrak{B}$-convergence relations are defined and characterised, the $\mathfrak{B}$-spaces are defined, the property "to be a $\mathfrak{B}$-space" is studied and some examples of non $\mathfrak{B}$-spaces are given for certain classes $\mathfrak{B}$; for example the product of two $\mathfrak{B}$-spaces need not be a $\mathfrak{B}$-space. Finally, a sufficient and necessary condition for a $\mathfrak{B}$-convergence relation to determine a topological space is given.

In Section 2 we deal with some characterisations of the compactness of closure spaces and the continuity of mappings.

In Section 3 the $\mathfrak{B}$-regular spaces are defined (by means of continuous functions) and studied. If $\mathfrak{B}$ contains countable sets only, the $\mathfrak{B}$-regularity coincides with the sequential regularity [8], [9]; if $\mathfrak{B}$ is the class of all directed sets, a space is $\mathfrak{B}$-regular if and only if it is uniformizable (= completely regular). Further we study relations between the $\mathfrak{B}$-modifications and the uniformizable modification and some properties of the class $\mathscr{P}(\mathfrak{B})$ of the uniformizable modifications of $\mathfrak{B}$-spaces.

In the last section the $\mathfrak{B}$-completness and the $\mathfrak{B}$-envelope of a $\mathfrak{B}$-regular $\mathfrak{B}$-space are defined and some properties of them are studied. The $\mathfrak{B}$-envelope is constructed by means of remarkable $\mathfrak{B}$-nets or by means of the Čech-Stone compactification.

The existence of a $\mathfrak{B}$-envelope and its uniqueness (up to a homeomorphism identical over the primary space) are proved.

If $\mathfrak{B}$ contains sets order-isomorphic with ω_0 (or countable sets) only, then the $\mathfrak{B}$-envelope coincides with the sequential envelope $(\sigma_e(\mathscr{P}))$ ([8], [9], [7]). If $\mathfrak{B}$ is the class of all directed sets (or if $\mathfrak{B}$ is sufficiently large – see 4.17), then the $\mathfrak{B}$-envelope coincides with the Čech-Stone compactification $(\beta\mathscr{P})$.

The paper is written in such a way, that familiarity with the referred papers is not necessary for its understanding although it might be helpful (especially with [8] or [9]).

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Some notations. The notation introduced in [3] is used in this paper.

0.1. Ordinal numbers are understood as comprisable ordinals in [6], p. 267, i.e. any ordinal number α is the set of all ordinals less than α. The class of all ordinal numbers will be denoted by Ord and the relation $\in$ on Ord will be denoted by $<$.

The ordinal number α is called a cardinal number iff it is equipollent with none of its elements (i.e. with no ordinal ζ less than α).

0.2. Let $\mathscr{P} = \langle P, u \rangle$ be a closure space. Than we shall denote by χx the local character of $\mathscr{P}$ at x, by $\chi^L \mathscr{P}$ the local character of $\mathscr{P}$ (both [3], 15.B.8.), by $d\mathscr{P}$ the density character of $\mathscr{P}$ ([3], 22A1.), by ψx the pseudocharacter at x, i.e. the least power of a collection $\mathscr{A}$ of neighborhoods of x in $\mathscr{P}$ such that $\bigcap \mathscr{A} = (x)$, by ωx the interior character at x, i.e. the least power of a collection $\mathscr{A}$ of neighborhoods of x in $\mathscr{P}$ such that $\bigcap \mathscr{A} = (x)$ or $\bigcap \mathscr{A}$ is not a neighborhood of x [4].

The subspace of $\langle P, u \rangle$ whose underlying set is Q will be denoted by $\langle Q, u \mid Q \rangle$.

0.3. Let ϱ be a relation. Then the domain-restriction of ϱ to A will be denoted $\varrho \restriction A$ and the range of $\varrho \restriction A$ will be denoted by $\varrho[A]$. Let $\mathscr{N} = \langle N, \varrho \rangle$ be a net. Then $\mathscr{N} \restriction A$ will denote the pair $\langle N \restriction A, \varrho \cap A \times A$.

In this paper the concept of a directed set has the same meaning as the concept of a directed ordered set in [3], i.e. as a directed set in [6] such that $m\varrho n$ & $n\varrho m$ implies $m = n$.

0.4. The class of all directed sets which contain no largest element will be denoted by $\mathfrak{M}$. $\mathfrak{N}$ will denote the class of all monotone ordered sets belonging to $\mathfrak{M}$.

Let α be an infinite set (an infinite ordinal number). The set of all elements $\langle D, \varrho \rangle$ of $\mathfrak{M}$ such that $D \subset \alpha$ will be denoted by $\mathfrak{M}_\alpha$; further $\mathfrak{N}_\alpha = \mathfrak{M}_\alpha \cap \mathfrak{N}$.

If α is a infinite regular cardinal number (i.e. every cofinal subset of α is order-isomorphic with α [4]), then $\mathfrak{N}_\alpha^r$ will denote the set of all $\leqq$-cofinal subsets of α (which are ordered by $\leqq$); further $\Theta = \mathfrak{N}_{\omega_0}^r$ and $\mathfrak{N}^r = \bigcup \{\mathfrak{N}_\alpha^r \mid \alpha$ is an infinite regular cardinal number$\}$.

0.5. Definition. A class $\mathfrak{V}$ will be called cofinal-closed iff $\mathfrak{V}$ is non-empty subclass of $\mathfrak{M}$ and if any ϱ-cofinal subset $\langle E, \varrho \cap E \times E\rangle$ of each element $\langle D, \varrho\rangle$ of $\mathfrak{V}$ is an element of $\mathfrak{V}$.

In the sequel, all classes denoted by $\mathfrak{V}$ are assumed to be cofinal-closed.

0.6. A directed net $\langle N, \varrho\rangle$ (often denoted only by N) will be called $\mathfrak{V}$-net iff its domain $\langle \mathbf{D}N, \varrho\rangle$ (often denoted only by $\mathbf{D}N$) belongs to $\mathfrak{V}$. We will say that N is a $\mathfrak{V}$-subnet of M iff N is a $\mathfrak{V}$-net and $\mathbf{D}N$ is a cofinal subset of $\mathbf{D}M$ and $N = M \upharpoonright \mathbf{D}N$. We will say that N is generalized $\mathfrak{V}$-subnet of M iff N is a $\mathfrak{V}$-net and N is a generalized subnet of M.

0.7. Remark. If M is a $\mathfrak{V}$-net, the condition "N is a $\mathfrak{V}$-net" in the definition of $\mathfrak{V}$-subnet is satisfied automatically. In general, a subnet of a $\mathfrak{V}$-net need not be a $\mathfrak{V}$-net; if N is a generalized subnet of a $\mathfrak{V}$-net, nor the net $N \circ h$ need not be a $\mathfrak{V}$-net for any bijective mapping h onto $\mathbf{D}N$.

1

1.1. Proposition. The classes $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{M}_\alpha$, $\mathfrak{N}_\alpha$, $\mathfrak{N}_\alpha^r$ are cofinal-closed.

Each element of $\mathfrak{M}$ (resp. of $\mathfrak{N}$) whose cardinallity is less or equal to the cardinal number α is order-isomorphic with an element of $\mathfrak{M}_\alpha$ (resp. of $\mathfrak{N}_\alpha$).

The easy proof is omitted.

1.2. Lemma. *If a net N converges to a point x in a closure space $\mathscr{P}$, and its domain and some element $\langle E, \sigma\rangle$ of $\mathfrak{V}$ are order-isomorphic, then there exists a $\mathfrak{V}$-net M ranging in $\mathbf{E}N$ which converges to x in $\mathscr{P}$. ($M = N \circ h$, where h is an order-isomorphism of $\langle E, \sigma\rangle$ onto $\mathbf{D}N$.)*

1.3. Lemma. *Let be $\alpha \geqq \chi^L \mathscr{P}|$. Then the point x is an accumulation point of the $\mathfrak{M}_\alpha$-net N in $\mathscr{P}$ if and only if a generalized $\mathfrak{M}_\alpha$-subnet M of N converges to x in $\mathscr{P}$.*

The proof is analogous to the proof in [3], 15B.22., $\mathscr{U}$ can be chosen such that card $\mathscr{U} \leqq \alpha$, therefore card $\mathbf{D}M \leqq$ card $\mathscr{U}$. card $\mathbf{D}N \leqq \alpha$; further see 1.2..

1.4. Definition. A relation $\mathscr{C}$ ranging in a set such that $\mathbf{D}\mathscr{C}$ is a class of $\mathfrak{V}$-nets, will be called the $\mathfrak{V}$-convergence relation.

The $\mathfrak{V}$-convergence class of a closure space $\mathscr{P}$ (denoted also by $\mathfrak{V}$-Lim $\mathscr{P}$) is the $\mathfrak{V}$-convergence relation consisting of all pairs $\langle N,x\rangle$ such that N converges to a point x in $\mathscr{P}$.

A $\mathfrak{V}$-convergence class is a $\mathfrak{V}$-convergence class of some closure space.

Remarks. *The $\mathfrak{V}$-convergence class of a space $\mathscr{P}$ is a set if and only if $\mathfrak{V}$ is a set. If $\mathfrak{V}$ is a set, then* card $\mathscr{C}$ = card $\mathfrak{V}$. card $|\mathscr{P}|$.

The same is satisfied for each $\mathfrak{B}$*-convergence structure* (see 1.12.) *such that* $\mathbf{E}\mathscr{C} = |\mathscr{P}|$.

1.5. Definition. A determining $\mathfrak{B}$-convergence relation for a closure space $\langle P, u\rangle$ is a subclass $\mathscr{C}$ of $\mathfrak{B}$-Lim $\langle P, u\rangle$ such that for every subset A of P a point x belongs to uA only if there exists a $\mathfrak{B}$-net N ranging in A with $\langle N, x\rangle \in \mathscr{C}$; we will say also that the space $\langle P, u\rangle$ is determined by $\mathscr{C}$.

A determining $\mathfrak{B}$-convergence relation is a determining $\mathfrak{B}$-convergence relation for some closure space.

A closure space $\langle P, u\rangle$ will be called $\mathfrak{B}$-space and a closure u will be called $\mathfrak{B}$-closure, iff $\langle P, u\rangle$ is determined by a determining $\mathfrak{B}$-convergence relation.

1.6. Remarks. (a) *Let* $\mathfrak{B}$ *be a given class. Then the class of all* $\mathfrak{B}$*-spaces is hereditary and closed under sums.*

(b) *If* $\mathfrak{B}' \subset \mathfrak{B}$ *and* $\mathscr{P}$ *is a* $\mathfrak{B}'$*-space, then* $\mathscr{P}$ *is a* $\mathfrak{B}$*-space.*

Proofs are easy and are omitted.

(c) The "Θ-space" is the same as the "S-space" in [3] (because a Θ-net is the same as a sequence).

If $\mathfrak{B}$ contains only countable sets only, then $\mathscr{P}$ is a $\mathfrak{B}$-space if and only if $\mathscr{P}$ is a S-space. The proof is based on the following proposition: Every countable directed set has a cofinal subsequence.

1.7. Lemma. *Let* $\mathscr{P}$ *be a closure space. If for each point* x *of* $\mathscr{P}$ *there exists some local base at* x *directed by* $\supset$ *and an element of* $\mathfrak{B}$ *which are order-isomorphic, then* $\mathscr{P}$ *is a* $\mathfrak{B}$*-space.*

If α is a cardinal number such that $\alpha \geqq \chi^L\mathscr{P}$, then $\mathscr{P}$ is an $\mathfrak{M}_\alpha$-space.

Proof. The first proposition is a corollary of 1.2 (analogously as in [3], the proof of 15 B.4. or in [6], p. 66), the second one is a corollary of the first.

Proposition. *Let* $\chi x = \omega x \leqq \alpha$ *for each point* x *of a space* $\mathscr{P}$*. Then for each point* x *of* $\mathscr{P}$ *there exists a monotone local base at* x *and* $\mathscr{P}$ *is a* $\mathfrak{N}_\alpha$*-space.*

Proof. Since ωx is a regular cardinal, the local base at x of cardinallity χx can be regularly ordered.

1.8. Proposition. *Let* $\langle P, u\rangle$ *be a* $T_1 - \mathfrak{B}$*-space, let each element* $\langle D, \varrho\rangle$ *of* $\mathfrak{B}$ *contains a* ϱ*-cofinal subset whose power is less than* α*. Then* $\omega x < \alpha$ *for each point* x *of* $\mathscr{P}$*.*

Proof. If x is isolated, then $\omega x = 1 < \alpha$. In the other case x belongs to $uA - A$ for some subset A of $\mathscr{P}$ and there exists a $\mathfrak{B}$-net N ranging in A which converges to x in $\langle P, u\rangle$. Let us denote $\mathscr{U}$ the family $\{U_n = P - (Nn) \mid n \in E\}$, where E is a cofinal subset of $\mathbf{D}N$ with card $E < \alpha$. Then card $\mathscr{U} < \alpha$ and $\bigcap \mathscr{U} = P - N[E]$ is not a neighborhood of x in $\langle P, u \rangle$.

Corollaries. *If $\mathscr{P}$ is a $T_1 - \mathfrak{M}_\alpha$-space, then $\omega x \leqq \alpha$ holds for each point x of $\mathscr{P}$.*

If there exists a cardinal number α such that card $D < \alpha$ *for each element $\langle D, \varrho \rangle$ of $\mathfrak{V}$ (in particular, if $\mathfrak{V}$ is a set), then there exists a normal topological space which is not a $\mathfrak{V}$-space.*

The examples 1.9 a, b show also, that this condition is not necessary.

1.9. Examples. Let $\alpha > \beta$ be infinite regular cardinal numbers. Let $\mathfrak{V}$ consist of directed sets cardinality of which is less than α and of all monotone ordered sets (or, let $\mathfrak{V}$ satisfy $\mathfrak{N}_\alpha^r \cup \mathfrak{N}_\beta^r \subset \mathfrak{V} \subset \mathfrak{N} \cup \mathbf{E}\{\mathfrak{M}_\gamma \mid \gamma < \alpha\}$).

(a) The product $\mathscr{P}$ of two (even normal and compact) $\mathfrak{V}$-spaces, T_α and T_β (see [3], 29 B. 7) is not a $\mathfrak{V}$-space.

(b) Let $|\mathscr{Q}| = \alpha \times \beta \cup (\alpha, \beta)$, $\alpha \times \beta$ is relatively discrete in $\mathscr{Q}$ and local base at (α, β) in $\mathscr{Q}$ is the relativization of the one in $\mathscr{P}$. Then $\mathscr{Q}$ is a hereditarily normal, non $\mathfrak{V}$-space. Further, $\chi^L \mathscr{P} = \chi^L \mathscr{Q} = \alpha$ and $\psi x \leqq \beta$ for each $x \in |\mathscr{P}|$; hence the term "character" cannot be replaced by the term "pseudocharacter"in Lemma 1.7.

(c) The following example shows that the term "interior character" cannot be replaced by the term "pseudocharacter" in 1.8. Let R be a set of power α, x an element of R, let $\beta < \alpha$ be infinite cardinal numbers; let a subset A of R be closed iff x belongs to A or card $A < \beta$. Then R with this topology is an $\mathfrak{M}_\beta$-space and $\psi(x) = \alpha > \beta$.

1.10. Definition. The coarsest $\mathfrak{V}$-closure finer than a closure u is called the $\mathfrak{V}$-modification of u.

1.11. Theorem. *Let $\mathscr{C}$ be the $\mathfrak{V}$-convergence class of $\langle P, u \rangle$. Then $\mathscr{C}$ is a determining $\mathfrak{V}$-convergence relation for $\langle P, v \rangle$ if and only if v is the $\mathfrak{V}$-modification of u.*

The proof is an application of definitions 1.5 and 1.10.

Corollary. *Let $\mathfrak{V}$ be a given class. Then there exists the bijective correspondence between $\mathfrak{V}$-covergence classes and $\mathfrak{V}$-spaces such that each $\mathfrak{V}$-space is determined by the corresponding $\mathfrak{V}$-convergence class.*

Let $\mathscr{C}_1$ be the $\mathfrak{V}$-convergence class of a space $\langle P_1, u_1 \rangle$, let $\mathscr{C}_2$ be a determining $\mathfrak{V}$-convergence relation for a space $\langle P_2, u_2 \rangle$. Then $\mathscr{C}_1 \subset \mathscr{C}_2$ if and only if u_2 is finer than the relativization of u_1 to P_2.

1.12. Definition. The $\mathfrak{V}$-convergence structure is a $\mathfrak{V}$-convergence relation such that the following conditions are satisfied:

(1) $\mathbf{E}N \subset \mathbf{E}\mathscr{C}$ for each net $N \in \mathbf{D}\mathscr{C}$.

(2) If N is a constant $\mathfrak{B}$-net ranging in (x), then $\langle N, x \rangle \in \mathscr{C}$.

(3) If $\langle N, x \rangle \in \mathscr{C}$ and M is a $\mathfrak{B}$-subnet of N, then $\langle M, x \rangle \in \mathscr{C}$.

1.13. Theorem. *The conditions* (1), (2′) *and* (3′) *are sufficient and necessary for a $\mathfrak{B}$-convergence relation $\mathscr{C}$ to be determining*:

(2′) *If $w \in \mathbf{E}\mathscr{C}$, then $\langle N, x \rangle \in \mathscr{C}$ for some constant $\mathfrak{B}$-net ranging in (x).*

(3′) *If $\langle M, x \rangle \in \mathscr{C}$ and $C_1 \cup C_2 = \mathbf{E}M$, then $\langle N, x \rangle \in \mathscr{C}$ for some $i \in (1, 2)$ and for some $\mathfrak{B}$-net N ranging in C_i.*

The condition (3) *in* 1.12 *is sufficient for* (3′), *but it is not necessary in* 1.13.

Proof. Necessity of (1) and (2′) is obvious. If $\langle N, x \rangle \in \mathscr{C}$ and $\mathbf{E}N = A_1 \cup A_2$, then x belongs to $u\mathbf{E}N = uA_1 \cup uA_2$ and $\langle M, x \rangle \in \mathscr{C}$ for some i and some net M ranging in A.

Let the conditions be satisfied. Let us define an operation u as in [6]: if $A \subset P$, then $x \in uA$ if $\langle N, x \rangle \in \mathscr{C}$ for some net N ranging in A. Obviously, $u\emptyset = \emptyset$ and $A \subset B$ implies $uA \subset uB$; $A \subset uA$ is implied by (2′). If $x \in u(A_1 \cup A_2)$, then $\langle N, x \rangle \in \in \mathscr{C}$ for some net N ranging in $A_1 \cup A_2$, hence $\langle M, x \rangle \in \mathscr{C}$ for some $i \in (1, 2)$ and some net M ranging in $A_i \cap \mathbf{E}N \subset A_i$ by (3′), therefore $x \in uA_i$ for this i.

(3) $\Rightarrow$ (3′): If $\langle N, x \rangle \in \mathscr{C}$ and $\mathbf{E}N = A_1 \cup A_2$, then $D_i = N^{-1}[A_i]$ is cofinal in DN for some i; for this i, $N_i = N \upharpoonright D_i$ is a $\mathfrak{B}$-subnet of N and $\langle N_i, x \rangle \in \mathscr{C}$ by (3).

1.14. Corollary. *Every $\mathfrak{B}$-convergence structure is a determining $\mathfrak{B}$-convergence relation.*

1.15. Theorem. *The following conditions are sufficient and necessary for $\mathscr{C}$ to be a $\mathfrak{B}$-convergence class*:

(0) *$\mathscr{C}$ is a $\mathfrak{B}$-convergence structure.*

(4) *If N is a $\mathfrak{B}$-net ranging in $\mathbf{E}\mathscr{C}$ and $x \in \mathbf{E}\mathscr{C}$, and every $\mathfrak{B}$-subnet M of N has a generalized $\mathfrak{B}$-subnet S with $\langle S, x \rangle \in \mathscr{C}$, then $\langle N, x \rangle \in \mathscr{C}$.*

(5) *If $x \in \mathbf{E}\mathscr{C}$ and N is a $\mathfrak{B}$-net ranging in $\mathbf{E}\mathscr{C}$ such that for every cofinal subset D of $\mathbf{D}N$ there exists a net N_D ranging in $N[D]$ with $\langle N_D, x \rangle \in \mathscr{C}$, then there exists a generalised subnet M of N with $\langle M, x \rangle \in \mathscr{C}$.*

Theorem 1.15 remains true, if we omit the word "generalized" in both conditions (4) and (5).

Proof. Let $\mathscr{C}$ be the $\mathfrak{B}$-convergence class of a closure space $\mathscr{P} = \langle P, u \rangle$. It can be easily proved that $\mathscr{C}$ is a $\mathfrak{B}$-convergence structure. Let $x \in \mathbf{E}\mathscr{C}$ and N be a $\mathfrak{B}$-net ranging in $\mathbf{E}\mathscr{C}$ such that $\langle N, x \rangle$ does not belong to $\mathscr{C}$. Then there exists a u-neighborhood U of x and a cofinal subset D of $\mathbf{D}N$ such that $N[D] \subset P - U$; hence no generalized subnet of the $\mathfrak{B}$-net $N \upharpoonright D$ converges to x in the space $\mathscr{P}$ and (4) is proved.

Furthermore, no net M ranging in $N[D] \subset P - U$ converges to x in $\mathscr{P}$; thereby, the necessity of the condition (5′) stronger than (5) is proved.

(5′) If the assumptions of (5) are satisfied, then $\langle N, x\rangle \in \mathscr{C}$.

Let $\mathscr{C}$ be a convergence structure satisfying the conditions (4) and (5). Let $\mathscr{P} = \langle P, u\rangle$ be the closure space determined by $\mathscr{C}$ (1.14). Let S be a $\mathfrak{B}$-net converging to point x in P such that $\langle S, x\rangle \notin \mathscr{C}$. Then by condition (4) there exists a $\mathfrak{B}$-subnet N of S such that $\langle M, x\rangle \in \mathscr{C}$ for no (generalized) $\mathfrak{B}$-subnet M of N. Because N converges to x in $\mathscr{P}$, $x \in u\, N[D]$ and there exists a $\mathfrak{B}$-net N_D ranging in $N[D]$ with $\langle N_D, x\rangle \in \mathscr{C}$ for every cofinal subset D of $\mathbf{D}N$; but this is a contradiction with the condition (5).

1.16. Remarks. The condition (4) corresponds to the condition (c) in [3], 35 A.16. and to the Urysohn's axiom $\mathscr{B}_3$ for sequential classes (see [2] or [8]).

The condition of diagonalization ([3], 35 A.14.) need not be necessary for $\mathscr{C}$ be the $\mathfrak{B}$-convergence class, because the net M from this condition need not be $\mathfrak{B}$-net (and one need not have any generalized $\mathfrak{B}$-subnet).

Convergence of the nets M^a to x need not be trivial, for example if $\mathfrak{B} = \mathfrak{N}$ or $\mathfrak{B} = \bigcup\{\mathfrak{N}_{\aleph_n} \mid n \in \omega_0\}$ or $\mathfrak{B} = \bigcup\{\mathfrak{M}_{\aleph_n} \mid n \in \omega_0\}$ and if P is the disjoint sum of ordered topological spaces $T_{\aleph_\alpha}$ over ω_0 with further point x whose local base consists of all sets residual in every $\aleph_\alpha$ and containing x.

If $\mathfrak{B} = \Theta$, the conditions (0), (4) – i.e. the axioms $\mathscr{B}_1$, $\mathscr{B}_2$ and $\mathscr{B}_3$ – are sufficient in 1.15 ([2]). If $\mathfrak{B}$ consists of countable elements only, sufficiency of (0), (4) can be easily proved.

1.17. Theorem. *A closure space determined by the $\mathfrak{B}$-convergence relation $\mathscr{C}$ is topological if and only if the following condition is satisfied.*

(6) *If $\langle S, x\rangle \in \mathscr{C}$ and $\langle S_m, Sm\rangle \in \mathscr{C}$ for each $m \in \mathbf{D}S$, then $\langle R, x\rangle \in \mathscr{C}$ for some $\mathfrak{B}$-net R ranging in $\bigcup\{\mathbf{E}S_m \mid m \in \mathbf{D}S\}$.*

Proof. Let condition (6) be satisfied and let $x \in uuA$. Then $\langle N, x\rangle \in \mathscr{C}$ for some $\mathfrak{B}$-net N ranging in uA and there exists a $\mathfrak{B}$-net N_m ranging in A with $\langle N_m, Nm\rangle \in \mathscr{C}$ for each $m \in \mathbf{D}N$; hence $\langle M, x\rangle \in \mathscr{C}$ for some net M ranging in $\bigcup\{\mathbf{E}N_m \mid m \in \mathbf{D}N\} \subset A$ by (6) and $x \in uA$.

Let $\langle P, u\rangle$ be topological, $\langle S, x\rangle \in \mathscr{C}$ and $\langle S_m, Sm\rangle \in \mathscr{C}$ for each $m \in \mathbf{D}S$. Let us denote $B = \bigcup\{\mathbf{E}S_m \mid m \in \mathbf{D}S\}$. Then $x \in u\mathbf{E}S \subset uuB = uB$ and hence $\langle M, x\rangle \in \mathscr{C}$ for some net M ranging in B.

1.18. Theorem. *The following conditions are sufficient and necessary for a class $\mathscr{C}$ to be the $\mathfrak{B}$-convergence class of a topological $\mathfrak{B}$-space: $\mathscr{C}$ is a $\mathfrak{B}$-convergence structure,*

$\mathscr{C}$ satisfies conditions (4), (5), (6) *from* 1.15 *and* 1.17.

Proof. The necessity is a corollary of 1.15, 1.14 and 1.17. Sufficiency: $\mathscr{C}$ is a $\mathfrak{B}$-convergence class by 1.15 and a determining $\mathfrak{B}$-convergence relation for a topological space by 1.14 and 1.17, therefore $\mathscr{C}$ is the $\mathfrak{B}$-convergence class of this topological $\mathfrak{B}$-space by 1.11.

1.19. Lemma. *Let $\langle P, u\rangle$ be a $\mathfrak{B}$-space. If a cardinal number α satisfies either α is regular and $\langle D, \varrho\rangle \in \mathfrak{B}$ implies* card $D < \alpha$ *or* $\alpha >$ card P, *then the closure u^α is the topological modification of u.*

Proof. In the case $\alpha >$ card P the lemma is well-known. In the other case let be $x \in u^{\alpha+1}B$. Then some $\mathfrak{B}$-net N ranging in $u^\alpha B$ converges to x in $\langle P, u\rangle$; card $\mathbf{E}N \leqq$ card $\mathbf{D}N < \alpha$ and thus there exists an ordinal number $\zeta < \alpha$ such that $\mathbf{E}N \subset u^\zeta B$. Then $x \in u^{\zeta+1}B \subset u^\alpha B$.

2

2.1. Theorem. *Let $\mathscr{P}$ be either a topological space and $\alpha \geqq \chi^t\mathscr{P}$ or let $\mathscr{P} = \langle P, u\rangle$ be a closure space and $\alpha \geqq$* exp card P. *Let $\mathfrak{B} \supset \mathfrak{M}_\alpha$. Then*

(1) *$\mathscr{P}$ is compact if and only if every $\mathfrak{B}$-net ranging in $|\mathscr{P}|$ has an accumulation point in $\mathscr{P}$.*

(2) *$\mathscr{P}$ is compact if and only if every $\mathfrak{M}_\alpha$-net ranging in $|\mathscr{P}|$ has a convergent in $\mathscr{P}$ generalized $\mathfrak{M}_\alpha$-subnet.*

Proof. First we shall prove the sufficiency of (1). Let $\mathscr{A}$ be a centered collection of subsets of $\mathscr{P}$. If $\mathscr{P}$ is topological and $\mathscr{B}$ is a closed base satisfying card $\mathscr{B} = \chi^t\mathscr{P}$, and sets $\mathscr{C}_a \subset \mathscr{B}$ are chosen so that $\bigcap\mathscr{C}_a = ua$ for each $a \in \mathscr{A}$, then let us denote $\mathscr{C} = \bigcup\{\mathscr{C}_a \mid a \in \mathscr{A}\}$ (hence $\mathscr{C} \subset \mathscr{B}$). In the other case ($\alpha \geqq$ exp card P) let us denote $\mathscr{C} = \mathscr{A}$. In both cases $\mathscr{C}$ is centered and card $\mathscr{C} \leqq \alpha$.

Denote $\mathscr{D}$ the collection of all finite intersections of sets belonging to $\mathscr{C}$. Then $\mathscr{D}$ is centered and directed by the inclusion $\supset$ and card $\mathscr{D} \leqq \alpha$. Let f be an order-isomorphism of $\langle E, \sigma\rangle \in \mathfrak{M}_\alpha$ onto $\langle \mathscr{D}, \supset\rangle$ and let N assign to each $d \in \mathscr{D}$ an element Nd of d. Then $\langle N \circ f, \sigma\rangle$ is a $\mathfrak{M}_\alpha$-net ranging in P and therefore, $\langle N \circ f, \sigma\rangle$ has an accumulation point x in $\mathscr{P}$.

Furthermore, for each set $c \in \mathscr{C}$ the net $N \circ f$ is eventually in c, hence x belongs to uc for each $c \in \mathscr{C}$; the compactness of $\mathscr{P}$ is thus proved (if $\mathscr{P}$ is topological, then $uc = c$ and $x \in \bigcap\mathscr{C} \subset \bigcap\mathscr{C}_a = ua$ for each $a \in \mathscr{A}$). The second implication in (1) is a corollary of 41 A.18. in [3], the proposition (2) is a corollary of (1) and 1.3.

2.2. Notation and definition. Let $\mathscr{B} \subset$ **exp** P and N be a net ranging in P. Then N will be called the $\mathscr{B}$-universal net, if for each set a belonging to $\mathscr{B}$, N is eventually either in a or in $P - a$.

An **exp** P-universal net is called universal ([6], p. 81, in [3] such a net is called the ultranet).

Let $\mathscr{P}$ be a closure space. Then $\mathscr{B}_0$ will denote an open base of $\mathscr{P}$ with power $\chi^t(\mathscr{P})$, $\mathscr{B}_1$ will denote the least subcollection of **exp** $|\mathscr{P}|$ containing $\mathscr{B}_0$ as a subset which is closed under finite intersections and differences.

Remark. $\mathscr{B}_1$ can be easily constructed by means of usual induction. If $\mathscr{P}$ is infinite, then card $\mathscr{B}_1 = \chi^t(\mathscr{P})$.

2.3. Lemma. *Let $\langle P, u\rangle$ be a topological space and let $\alpha \geqq \chi^t\langle P, u\rangle$. Then every $\mathfrak{M}_\alpha$-net ranging in P has a $\mathscr{B}_1$-universal generalized $\mathfrak{M}_\alpha$-subnet.*

Proof. Let $\langle N, \prec\rangle$ be an $\mathfrak{M}_\alpha$-net ranging in P; let us denote $\mathscr{G}$ the maximal subcollection of $\mathscr{B}_1$ closed under finite intersections such that N is frequently in a for each $a \in \mathscr{G}$. ($\mathscr{G}$ exists by Zorn's lemma.) By a contradiction one can prove that, for each $a \in \mathscr{B}_1$, either $a \in \mathscr{G}$ or $P - a \in \mathscr{G}$.

Let us denote f an order-isomorphism of $\langle E, \sigma\rangle \in \mathfrak{M}_\alpha$ onto the product of directed sets $\langle \mathbf{D}N, \prec\rangle \times \langle \mathscr{G}, \supset\rangle$ and let S assign to each pair $\langle p, a\rangle \in \mathbf{D}N \times \mathscr{G}$ an element $S(p, a) \in \mathbf{D}N$ such that $p \prec S\langle p, a\rangle$ and $N \circ S(p, a) \in a$. Then $M = \langle N \circ S \circ f, \sigma\rangle$ is a generalized $\mathfrak{M}_\alpha$-subnet of $\langle N, \prec\rangle$ and a $\mathscr{B}_1$-universal net, because M is eventually in each element of $\mathscr{G}$.

2.4. Theorem. (a) *Let $\alpha \geqq$ exp card $|\mathscr{P}|$ and $\mathfrak{B} \supset \mathfrak{M}_\alpha$. Then the closure space $\mathscr{P}$ is compact if and only if every universal $\mathfrak{B}$-net is convergent in $\mathscr{P}$.*

(b) *Let $\alpha \geqq \chi^t\mathscr{P}$ and $\mathfrak{B} \supset \mathfrak{M}_\alpha$. Then the topological space is compact if and only if every $\mathscr{B}_1$-universal $\mathfrak{B}$-net is convergent in $\mathscr{P}$.*

Proof. An accumulation point of the $\mathscr{B}_1$-universal net is its limit point, because $\mathscr{B}_1 \supset \mathscr{B}_0$. Let the space be not compact. By 2.1 there exists an $\mathfrak{M}_\alpha$-net ranging in $|\mathscr{P}|$ which has no in $\mathscr{P}$ convergent generalized $\mathfrak{M}_\alpha$-subnet; and there exists its generalized $\mathfrak{M}_\alpha$-subnet which is $\mathscr{B}_1$-universal in the case (b) by 2.3 and universal in the case (a) (by the proof of 2.3, where we replace $\mathscr{B}_1$ by **exp** $|\mathscr{P}|$).

2.5. Definition. Let $\alpha < \beta$ be cardinal numbers. Then a topological space $\mathscr{P}$ is called $[\![\alpha, \beta[\![$-compact (resp. $[\![\alpha, \rightarrow[\![$-compact), if every subset E of $\mathscr{P}$ such that card E is regular and $\alpha \leqq$ card $E < \beta$ (resp. $\alpha \leqq$ card E), has a complete accumulation point. (It is little more generally than in [1].) A net $\langle N, \prec\rangle$ is called decreasing, if $m \prec n$ implies $Nm \supset Nn$.

2.6. Theorem. *The following condition is sufficient and necessary for a topological space $\mathscr{P}$ to be $[\![\alpha, \beta[\![$-compact (resp. $[\![\alpha, \rightarrow[\![$-compact): Every $\mathfrak{N}^r_\gamma$-net ranging in $|\mathscr{P}|$ such that γ is a regular cardinal number and $\alpha \leqq \gamma < \beta$ (resp. $\alpha \leqq \gamma$) has an accumulation point in $\mathscr{P}$.*

Corollary. *A topological space $\mathscr{P}$ is compact if and only if every $\mathfrak{N}^r$-net ranging in $|\mathscr{P}|$ has an accumulation point in $\mathscr{P}$.*

Proof of 2.6. $\langle P, u\rangle$ is $[\![\alpha, \beta[\![$-compact if and only if every decreasing N^r_γ-net $\{U_n \mid n \in D\}$ of closed non-empty subsets of P such that $\alpha \leqq \gamma < \beta$ and γ is regular, satisfies $\bigcap\{U_n \mid n \in D\} \neq \emptyset$ (see [1] p. 22).

Let γ be a regular cardinal number such that $\alpha \leqq \gamma < \beta$. Let $\langle N, \prec\rangle$ be an $\mathfrak{N}^r_\gamma$-net ranging in P; for each $m \in \mathbf{D}N$ let us denote $B_m = u\mathbf{E}\{Nn \mid n \succ m\}$. Then $\langle\{B_m \mid m \in \mathbf{D}N\}, \prec\rangle$ is a decreasing $\mathfrak{N}^r_\gamma$-net of non-empty closed sets and hence there exists a point belonging to $\bigcap\{B_m \mid m \in \mathbf{D}N\}$; this point is obviously an accumulation point of the net $\langle N, \prec\rangle$.

On the other hand, let U be a decreasing $\mathfrak{N}^r_\gamma$-net of closed non-empty sets. Let $\mathbf{D}N = \mathbf{D}U$ and let N assign to each $n \in \mathbf{D}U$ a point Nn belonging to Un; then N is an $\mathfrak{N}^r_\gamma$-net and thus N has an accumulation point x. Because N is eventually in every Un, x belongs to $uUn = Un$ for each $n \in \mathbf{D}U$.

2.7. Lemma. *The following condition is sufficient and necessary for a mapping f on a $\mathfrak{B}$-space $\mathscr{P}$ into a closure space $\mathscr{Q}$ to be continuous. For each point x of $\mathscr{P}$ and for every $\mathfrak{B}$-net N converging to x in $\mathscr{P}$ the net $f \circ N$ converges to fx in $\mathscr{Q}$.*

Proof. Let this condition be satisfied, let $A \subset P$ and $x \in uA$. Let $\mathscr{P} = \langle P, u\rangle$, $\mathscr{Q} = \langle Q, v\rangle$. Then a $\mathfrak{B}$-net N ranging in A converges to x in $\mathscr{P}$, the net $f \circ N$ converges to fx in $\mathscr{Q}$; therefore $fx \in v\mathbf{E}f \circ N \subset v f[A]$. The second implication is well-known.

2.8. Remark. The assumption "$\mathscr{P}$ is a $\mathfrak{B}$-space" is essential; if $\mathscr{P}$ is a semiuniformizable non $\mathfrak{B}$-space and $\mathscr{Q}$ is a non-accrete space, then there exists a subspace $\mathscr{R}$ of $\mathscr{P}$ such that the condition in 2.7 is not sufficient for the continuity of a mapping on $\mathscr{R}$ into $\mathscr{Q}$.

Indeed, let us choose a set A and a point $x \in uA$ so that no $\mathfrak{B}$-net ranging in A converges to x in $\mathscr{P}$. Let us denote $|\mathscr{R}| = A \cup (x)$ and choose a function f on $|\mathscr{R}|$ such that $f[A] = y, fx = z$ for some points y, z of $\mathscr{Q}$ satisfying $z \notin v(y)$.

3

3.1. Propositions. (a) *Let $\mathscr{C}$ be the $\mathfrak{B}$-convergence class of a closure space $\mathscr{P}$ or a determining $\mathfrak{B}$-convergence relation for $\mathscr{P}$. Then $\mathscr{P}$ is a T_1-space if and only if $\mathscr{C}$ is single-valued at every constant net* (*as* 35 B.7. *in* [3]).

The $\mathfrak{B}$-convergence class of a separated space is single-valued [3].

(b) *Let $\mathscr{P}$ be a closure space, $\mathfrak{B} \supset \mathfrak{M}_{\chi L \mathscr{P}}$, let the $\mathfrak{B}$-convergence class of $\mathscr{P}$ is single-valued. Then $\mathscr{P}$ is separated.*

"$\mathfrak{B} \supset \mathfrak{M}_{\chi L \mathscr{P}}$" *can be replaced by this weaker condition: for every pair $\langle x, y\rangle$ of points of $\mathscr{P}$ the product of some local base at x and at y directed by $\supset$ is order-isomorphic with some element of $\mathfrak{B}$.*

Proofs are easy and omitted.

3.2. Example. Let $\alpha > \beta$ and $\mathfrak{B}$ be the same as in 1.9a. Let $\mathscr{P}$ be the product of spaces T_α, T_β, let $|\mathscr{Q}| = |\mathscr{P}| \cup (x) \cup (y)$, let $\mathscr{P}$ be a subspace of $\mathscr{Q}$, let $U \subset |\mathscr{Q}|$ be a neighborhood of x (resp. of y) iff the projection of $|\mathscr{P}| - U$ into α (resp. into β) is bounded in α (resp. in β). Then $\mathscr{Q}$ is not separated and its $\mathfrak{B}$-convergence class is single-valued.

3.3. Notation and definition. $\mathbf{I}$ will denote the closed unit interval $[\![0, 1]\!]$ with its usual topology. $\mathscr{F}(\mathscr{P})$ will denote the collection of all continuous functions on the closure space $\mathscr{P}$ into $\mathbf{I}$.

A closure space $\mathscr{P}$ will be called $\mathfrak{B}$-regular, if for each point x of $\mathscr{P}$ and for every $\mathfrak{B}$-net N ranging in $|\mathscr{P}|$ which does not converge to x in $\mathscr{P}$ there exists a function $f \in \mathscr{F}(\mathscr{P})$ such that the net $f \circ N$ does not converge to fx in $\mathbf{I}$.

A closure u (on an underlying set P) will be called $\mathfrak{B}$-regular iff $\langle P, u\rangle$ is $\mathfrak{B}$-regular.

3.4. In 3.4 we will study a dependence of the definition in 3.3 and the definition (r) analogous to the definition of sequential regularity of convergence spaces in [8].

Definition (identical with the analogous definition in [8] for $\mathfrak{B} = \Theta$). $\langle P, \mathscr{C}, u\rangle$ will be called the $\mathfrak{B}$-convergence space, if $\mathscr{C}$ is a determining $\mathfrak{B}$-convergence relation for the closure space $\langle P, u\rangle$.

Definition (r). A $\mathfrak{B}$-convergence space $\langle P, \mathscr{C}, u\rangle$ will be called $\mathfrak{B}$-regular, if for each point $x \in P$ and for every $\mathfrak{B}$-net N ranging in P no subnet M of which satisfies $\langle M, x\rangle \in \mathscr{C}$, there exists a function $f \in \mathscr{F}\langle P, u\rangle$ such that the net $f \circ N$ does not converge to fx in $\mathbf{I}$.

Proposition. *A $\mathfrak{B}$-convergence space $\langle P, \mathscr{C}, u\rangle$ is $\mathfrak{B}$-regular if and only if $\langle P, u\rangle$ is $\mathfrak{B}$-regular and $\mathscr{C}$ satisfies the condition* (5) *from* 1.15 *without the word "generalized".*

Remark. If $\mathfrak{B}$ contains countable sets only, then every determining $\mathfrak{B}$-convergence relation satisfies the condition (5).

Proof of the proposition. Let $\langle P, u\rangle$ be a $\mathfrak{B}$-regular space and let $\mathscr{C}$ satisfies (5). Let N be a $\mathfrak{B}$-net ranging in P such that $f \circ N$ converges to fx in $\mathbf{I}$ for each $f \in \mathscr{F}\langle P, u\rangle$. Then N converges to x in $\langle P, u\rangle$, $x \in u\, N[D]$ for every cofinal subset D of $\mathbf{D}N$, hence some subnet M of N satisfies $\langle M, x\rangle \in \mathscr{C}$ by (5).

Let $\langle P, \mathscr{C}, u\rangle$ be a $\mathfrak{B}$-regular $\mathfrak{B}$-convergence space. Obviously, $\langle P, u\rangle$ is $\mathfrak{B}$-regular. Let N be a $\mathfrak{B}$-net ranging in P such that for each cofinal subset D of $\mathbf{D}N$ there exists a net N_D ranging in $N[D]$ with $\langle N_D, x\rangle \in \mathscr{C}$. Then N converges to x in $\langle P, u\rangle$ by the condition (5') in 1.15, $f \circ N$ converges to fx in $\mathbf{I}$ for each $f \in \mathscr{F}\langle P, u\rangle$, hence there exists a subnet M of N such that $\langle M, x\rangle \in \mathscr{C}$ by (r).

3.5. Propositions. (a) *Let $\mathfrak{B}$ be a given class. Then the class of all $\mathfrak{B}$-regular spaces is hereditary and closed under sums and products.*

(b) *If $\mathfrak{W} \subset \mathfrak{B}$, then every $\mathfrak{B}$-regular space is a $\mathfrak{W}$-regular space.*

(c) *If $\mathfrak{B}$ contains countable sets only, then the space $\mathscr{P}$ is $\mathfrak{B}$-regular if and only if it is Θ-regular.*

A proof of (c) is based on the same proposition as 1.6c, a proof of $\mathfrak{B}$-regularity of products is analogous to that in [9]; the other proofs are easy and are omitted.

3.6. Proposition. *A $\mathfrak{B}$-regular T_0-space is functionally separated and hence separated.* (Obvious.)

More generally, if $\mathscr{P} = \langle P, u\rangle$ is a $\mathfrak{B}$-regular space, then $fx \neq fy$ for some $f \in \mathscr{F}(\mathscr{P})$, whenever $x \in P - u(y)$; the relation $\sigma = \{\langle x, y\rangle \mid x \in u(y)\}$ is an equivalence and the quotient space $\mathscr{P}/\sigma$ is a $\mathfrak{B}$-regular separated space. (Easy.)

3.7. Corollary. *A $\mathfrak{B}$-regular compact space is uniformizable.*

Proof. The quotient space $\mathscr{P}/\sigma$ is separated and evidently also compact, hence uniformizable. Thus $\mathscr{P}$ is uniformizable by 28 A.9 in [3].

3.8. Lemma. *A uniformizable space is $\mathfrak{B}$-regular for every class $\mathfrak{B}$.*

Proof. If a $\mathfrak{B}$-net M does not converge to x in a uniformizable space $\mathscr{P}$, there exists a open neighborhood U and a $\mathfrak{B}$-subnet N of M ranging in $P - V$ and a function $f \in \mathscr{F}(\mathscr{P})$ so that $fx = 1$ and $\mathbf{E} f \circ N \subset f[P - V] = (0)$.

Remark. In 4.16 we will prove that an $\mathfrak{M}_\alpha$-regular space is uniformizable, if $\alpha \geqq \geqq \exp d\mathscr{P}$.

3.9. Theorem. *Let $\mathscr{P}$ be a $\mathfrak{B}$-regular space.*

If there exists a local base at x in $\mathscr{P}$, directed by the inclusion $\supset$, which is order-isomorphic to some element of $\mathfrak{B}$, then x is an R-point, i.e. there exists a local base at x consisting of closed sets ([4], 5.2).

If $\mathfrak{B} \supset \mathfrak{M}_{\chi_x}$, then x is an R-point.

If $\mathfrak{B} \supset \mathfrak{M}_{\chi L\mathscr{P}}$, then $\mathscr{P}$ is regular.

Proof. Let x be not an R-point in a space $\mathscr{P} = \langle P, u\rangle$. Then there exists a u-neighborhood U of x such that $uV - U$ is non-empty for any u-neighborhood V of x. Let $\mathbf{D}N$ be a local base at x considered in assumption and let $NV \in uV - U$ for each $V \in \mathbf{D}N$. Then N does not converge to x in $\mathscr{P}$ and this is a contradiction with the $\mathfrak{B}$-regularity of $\mathscr{P}$ and with the regularity of $\mathbf{I}$.

Other propositions are corollaries of the first one.

3.10. Example. The regular space on which each continuous function is constant [10] is not $\mathfrak{B}$-regular for any $\mathfrak{B}$.

Θ-modification of the product $2^{\aleph_1}$ is a Θ-regular non-regular closure space [8].

3.11. Proposition. (a) *A closure u is $\mathfrak{B}$-regular if and only if its $\mathfrak{B}$-modification is $\mathfrak{B}$-regular.*

(b) *If $\mathfrak{B} \cap \mathfrak{W}$ is not empty, then the $\mathfrak{B}$-modification of a $\mathfrak{W}$-regular closure is $\mathfrak{B} \cap \mathfrak{W}$-regular.*

(c) *The $\mathfrak{B}$-modification of a uniformizable closure is $\mathfrak{B}$-regular.*

Proofs are easy and are omitted.

3.12. Lemma. *Let $\langle P, u \rangle$ be a $\mathfrak{B}$-regular space. Then there exists the uniformizable modification $\tilde{u}$ of u (i.e. the finest uniformizable closure coarser than u) and the following conditions are satisfied.*

(a) $\mathscr{F}\langle P, u \rangle = \mathscr{F}\langle P, \tilde{u} \rangle$,

(b) *$x \in \tilde{u}A$ if and only if, for each $f \in \mathscr{F}\langle P, u \rangle$, $f[A] = 0$ implies $fx = 0$.*

Proof. Lemma is a corollary of the analogous theorem in [7] and of 3.6 for T_0-spaces, in the other case we apply in addition quotient spaces and 3.6.

3.13. Theorem. *Let $\tilde{u}$ be the uniformizable modification of u, let v be the $\mathfrak{B}$-modification of u.*

(a) *If u is $\mathfrak{B}$-regular, then v is finer than u.*

(b) *If u is a $\mathfrak{B}$-closure, than v is coarser than u.*

(c) *$v = u$ if and only if u is a $\mathfrak{B}$-regular $\mathfrak{B}$-closure.*

Proof. (a) Let $x \in vA$; then some $\mathfrak{B}$-net N ranging in A converges to x in $\langle P, u \rangle$, for each $f \in \mathscr{F}\langle P, u \rangle = \mathscr{F}\langle P, \tilde{u} \rangle$ the net $f \circ N$ converges to fx in $\mathbf{I}$. Because u is $\mathfrak{B}$-regular, N converges to x in $\langle P, u \rangle$; thus $x \in uA$.

(b) follows from definitions, (c) is a corollary of (a), (b), 3.11c.

3.14. Definition and proposition. Let $\mathfrak{B}$ be a given class. Let us denote $\mathbf{R}(\mathfrak{B})$ the class of all $\mathfrak{B}$-regular $\mathfrak{B}$-spaces, and $\mathbf{P}(\mathfrak{B})$ the class of all uniformizable spaces whose closures are uniformizable modifications of $\mathfrak{B}$-closures. Let us denote $\prec$ a relation such that $\mathbf{D} \prec = \mathbf{E} \prec$ is the class of all closure spaces and $\langle P_1, u_1 \rangle \prec \langle P_2, u_2 \rangle$ iff $P_1 = P_2$ and u_1 is finer than u_2. Then $\mathbf{P}(\mathfrak{B})$ and $\mathbf{R}(\mathfrak{B})$ are $(\prec, \prec)$-isomorphic; a mapping h which assigns the space $\langle P, v \rangle$ to each $\langle P, u \rangle \in \mathbf{P}(\mathfrak{B})$ so that v is the $\mathfrak{B}$-modification of u, is $(\prec, \prec)$-isomorphism of $\mathbf{P}(\mathfrak{B})$ onto $\mathbf{R}(\mathfrak{B})$ and h^{-1} assigns the space $\langle P, \tilde{u} \rangle$ to each $\langle P, u \rangle \in \mathbf{R}(\mathfrak{B})$ (by 3.13c, 3.11c).

3.15. Theorem. *The following condition is necessary and sufficient for $\mathbf{P}(\mathfrak{B})$ to be the class of all uniformizable spaces: Each closure space is a $\mathfrak{B}$-space.*

Proof. Let $\langle P, u \rangle$ be not a $\mathfrak{B}$-space. Then for a subset A of P and a point x belonging to uA no $\mathfrak{B}$-net ranging in A converges to x in $\langle P, u \rangle$. Let us denote B the set $A \cup (x)$ and v the closure on B so that A is the relatively discrete subspace of $\langle B, v \rangle$ and the local base at x is the same in $\langle B, v \rangle$ as in $\langle B, u \mid B \rangle$. Then $\langle B, v \rangle$ is uniformizable non-$\mathfrak{B}$-space and the only $\mathfrak{B}$-closure finer than v is discrete, hence $\langle B, v \rangle$ does not belong to $\mathbf{P}(\mathfrak{B})$.

Corollary. $\mathbf{P}(\mathfrak{N} \cup \mathfrak{B}) \neq \mathbf{P}(\mathfrak{M})$ *for any set* $\mathfrak{B}$.

Remark. The uniformizable space in the example 3 in [7] not belonging to $\mathbf{P} = \mathbf{P}(\Theta) = \mathbf{P}(\mathfrak{M}_{\aleph_0})$ does not belong nor to $\mathbf{P}(\mathfrak{N})$; it belongs to the class $\mathbf{P}(\mathfrak{M}_{\exp \aleph_0})$ (because it is an $\mathfrak{M}_{\exp \aleph_0}$-space).

4

4.1. Definition. A net N will be called remarkable in a closure space $\mathscr{P}$ iff the net $f \circ N$ is convergent in I for any bounded continuous function f on $\mathscr{P}$.

If a net M converges to k in the space $\mathbf{I}$, then we shall write $k = \lim M$.

Proposition. *Any $\mathfrak{B}$-net remarkable in a $\mathfrak{B}$-regular space $\mathscr{P}$ is either convergent in $\mathscr{P}$ or totally divergent in $\mathscr{P}$ (i.e. none of its generalized subnets is convergent in $\mathscr{P}$).*

The proof is analogous (and for $\mathfrak{B} = \Theta$ the same) to that in [7] and is easy.

4.2. Definition. A $\mathfrak{B}$-complete space is a $\mathfrak{B}$-regular $\mathfrak{B}$-space $\mathscr{P}$ such that any in $\mathscr{P}$ remarkable $\mathfrak{B}$-net is convergent in $\mathscr{P}$.

4.3. Definition. Let $\mathscr{P} = \langle P, u \rangle$ and $\mathscr{Q} = \langle Q, u \rangle$ be $\mathfrak{B}$-regular $\mathfrak{B}$-spaces. We say that $\mathscr{Q}$ is a $\mathfrak{B}$-envelope of $\mathscr{P}$, iff the following conditions are satisfied (relative to $\mathscr{P}$ and $\mathscr{Q}$).

(λ_0) $\mathscr{P}$ is a subspace of $\mathscr{Q}$ and $v(x) = (x)$ for each point x belonging to $Q - P$.

(λ_1) $v^\alpha P = Q$ for some ordinal number α.

(λ_2) Any bounded continuous function on $\mathscr{P}$ has a continuous domain-extension to $\mathscr{Q}$.

(λ_3) If $\mathscr{Q}$ is a subspace of a $\mathfrak{B}$-regular $\mathfrak{B}$-space $\mathscr{R}$ and the conditions (λ_1), (λ_2) and (λ_0) are satisfied relative to $\mathscr{P}$ and $\mathscr{R}$, then $R = Q$.

4.4. Lemma. *Let $\mathscr{Q}$ be a $\mathfrak{B}$-complete space and let the conditions (λ_0), (λ_1), (λ_2) are satisfied relative to $\mathscr{P}$ and $\mathscr{Q}$. Then $\mathscr{Q}$ is a $\mathfrak{B}$-envelope of the space $\mathscr{P}$.*

Proof. $\mathscr{P}$ is a $\mathfrak{B}$-regular $\mathfrak{B}$-space by (λ_0), 1.6a, 3.5a. Let $\mathscr{Q} = \langle Q, v \rangle$ be a subspace of a $\mathfrak{B}$-regular $\mathfrak{B}$-space $\mathscr{R} = \langle R, w \rangle$ such that $R \supset Q$ and (λ_0), (λ_1), (λ_2) are satisfied relative to $\mathscr{P}$ and $\mathscr{R}$. Let us denote $\gamma_x = \min \{\zeta \in \mathsf{Ord} \mid x \in w^\zeta P\}$ for each $x \in R - Q$ and $\gamma = \min \{\gamma_x \mid x \in R - Q\}$. γ is obviously isolated. Let us choose $x \in R - Q$ and a $\mathfrak{B}$-net N ranging in $w^{\gamma-1}P \subset Q$ such that $x \in w^\gamma P$ and N converges to x in $\mathscr{R}$.

For each $f \in \mathscr{F}(\mathscr{Q})$ a continuous extension $\tilde{f}$ of the function $f \restriction P$ to $\mathscr{R}$ satisfies $f = \tilde{f} \restriction Q$ by (λ_1) and the net $f \circ N = \tilde{f} \circ N$ is convergent in $\mathbf{I}$. Thus the $\mathfrak{B}$-net N is remarkable in $\mathscr{Q}$ and hence convergent in $\mathscr{Q}$. If we denote its limit point in $\mathscr{Q}$ by y, N converges to y also in $\mathscr{R}$, therefore $y \in w(x) = (x)$ by 3.6 and (λ_0), but this is a contradiction $(x \in R - Q,\ y \in Q)$.

4.5. Lemma. *Let $\mathscr{P} = \langle P, u\rangle$ be a $\mathfrak{B}$-regular $\mathfrak{B}$-space. Then there exists a transfinite sequence of $\mathfrak{B}$-regular $\mathfrak{B}$-spaces $\{\mathscr{P}_\zeta = \langle P_\zeta, u_\zeta\rangle \mid \zeta \in \mathsf{Ord}\}$ such that $\mathscr{P}_0 = \mathscr{P}$ and the following conditions are satisfied for any ordinal numbers ζ, η.*

(a) *If $\eta \leqq \zeta$, then $\mathscr{P}_\eta$ is a subspace of $\mathscr{P}_\zeta$ and $u_\zeta(x) = u_\eta(x)$ for each point x of $\mathscr{P}_\eta$.*

(b) $u_\zeta^\zeta P = P_\zeta$.

(c) *If $\eta \leqq \zeta$, then every bounded continuous function on $\mathscr{P}_\eta$ has a (unique) continuous extension to $\mathscr{P}_\zeta$.*

(d) *If $\eta < \zeta$, then every $\mathfrak{B}$-net remarkable in $\mathscr{P}_\eta$ is convergent in $\mathscr{P}_\zeta$.*

Proof. Let $\mathscr{P}_\eta$ be defined for each $\eta < \zeta$. If $\zeta = \varkappa + 1$, let us denote $\mathscr{M}_\varkappa$ the maximal subset of the class of all totally divergent $\mathfrak{B}$-nets remarkable in $\mathscr{P}_\varkappa$ such that $\lim f \circ M \neq \lim f \circ N$ for some $f \in \mathscr{F}(\mathscr{P}_\varkappa)$ provided M and N are different elements of $\mathscr{M}_\varkappa$; let $\mathscr{P}_\zeta = \mathscr{P}_\varkappa \cup \mathscr{M}_\varkappa$, let $\mathscr{F}_\zeta$ be the set of all extensions $\tilde{f}$ of functions $f \in \mathscr{F}(\mathscr{P}_\varkappa)$ to $\mathscr{P}_\zeta$ such that $\tilde{f}N = \lim f \circ N$ for each $N \in \mathscr{M}_\varkappa$.

If ζ is a limit ordinal number, let us put $P_\zeta = \bigcup\{P_\eta \mid \eta \in \zeta\}$, let $\mathscr{F}_\zeta$ be the set of all extensions $\tilde{f}$ of functions $f \in \mathscr{F}(\mathscr{P}_0)$ to $\mathscr{P}_\zeta$ such that, for each $\eta < \zeta$ and for each $x \in P_\eta$, $\tilde{f}x = f_\eta x$, where f_η is a extension of f to P_η continuous in $\mathscr{P}_\eta$ (see 4.5c for $0 \leqq \eta < \zeta$).

In both cases we can easily prove that the class $\mathscr{C}_\zeta$ consisting of all pairs $\langle N, x\rangle$ such that N is a $\mathfrak{B}$-net ranging in P_ζ and the net $\tilde{f} \circ N$ converges to $\tilde{f}_x$ in $\mathbf{I}$ for each $\tilde{f} \in \mathscr{F}_\zeta$, is the $\mathfrak{B}$-convergence class (by 1.15) and the space $\langle P_\zeta, u_\zeta\rangle$ determined by $\mathscr{C}_\zeta$ is $\mathfrak{B}$-regular and satisfies all conditions in 4.5.

4.6. Theorem. *A space $\mathscr{Q}$ is a $\mathfrak{B}$-envelope of a $\mathfrak{B}$-regular $\mathfrak{B}$-space $\mathscr{P}$ if and only if $\mathscr{Q}$ is $\mathfrak{B}$-complete and the conditions (λ_0), (λ_1), (λ_2) are satisfied relative to $\mathscr{P}$ and $\mathscr{Q}$.*

Proof. Let $\mathscr{Q} = \langle Q, v\rangle$ be a $\mathfrak{B}$-envelope of a space $\mathscr{P} = \langle P, u\rangle$, let $\mathscr{Q}_1 = \langle Q_1, v_1\rangle$ be constructed from $\mathscr{Q}$ in the same way as the space $\langle P_1, u_1\rangle$ from $\mathscr{P}$ in 4.5. Then $v^\alpha P = Q$ for some ordinal number α by (λ_1) and $Q_1 \supset v_1^{\alpha+1}P \supset v_1 v^\alpha P = v_1 Q = Q_1$ by (a), (b) in 4.5, hence (λ_1) is satisfied relative to $\mathscr{P}$ and $\mathscr{Q}_1$. The condition (λ_0) (resp.

(λ_2)) relative to $\mathscr{P}$ and $\mathscr{Q}_1$ is implied by the same condition relative to $\mathscr{P}$ and $\mathscr{Q}$ and by (a) (resp. by (c)) in 4.5; therefore $Q_1 = Q$ and the space $\mathscr{Q}$ is $\mathfrak{B}$-complete by 4.5d.

The other implication is 4.4.

4.7. Corollary. *Let* $\{\mathscr{P}_\zeta \mid \zeta \in \mathrm{Ord}$ *be the transfinite sequence from* 4.5. *Then the space* $\mathscr{P}_\zeta$ *is a* $\mathfrak{B}$*-envelope of* $\mathscr{P}$ *if and only if* $\mathscr{P}_{\zeta+1} = \mathscr{P}_\zeta$.

Remark. The existence of such ordinal number will be proved in 4.13. If $\mathscr{P}_{\zeta+1} = \mathscr{P}_\zeta$ and $\eta > \zeta$, then $\mathscr{P}_\eta = \mathscr{P}_\zeta$.

4.8. Lemma. *Let the conditions* $(\lambda_0), (\lambda_1), (\lambda_2)$ *be satisfied relative to a space* $\mathscr{P} = \langle P, u\rangle$ *and a* $\mathfrak{B}$*-regular* $\mathfrak{B}$*-space* $\mathscr{Q} = \langle Q, v\rangle$ (*in particular, let* $\mathscr{Q}$ *be a* $\mathfrak{B}$*-envelope of* $\mathscr{P}$). *Let* $\tilde{u}$ (*resp.* $\tilde{v}$) *be the uniformizable modification of* u (*resp. of* v). *Then a space* $\mathscr{R}$ *is the Čech-Stone compactification of the space* $\langle Q, \tilde{v}\rangle$ *if and only if* $|\mathscr{R}| \supset Q$ *and* $\mathscr{R}$ *is the Čech-Stone compactification of* $\langle P, \tilde{u}\rangle$.

Proof. Let $\mathscr{R} = \langle R, w\rangle$ be the Čech-Stone compactification of $\langle Q, \tilde{v}\rangle$. $\langle P, \tilde{u}\rangle$ is a subspace of $\langle Q, \tilde{v}\rangle$ by (λ_0), (λ_2) and 3.12, thus $\langle P, \tilde{u}\rangle$ is a subspace of $\mathscr{R}$. Each function belonging to $\mathscr{F}\langle P, \tilde{u}\rangle = \mathscr{F}(\mathscr{P})$ has a continuous extension to $\mathscr{Q}$ (by (λ_2)), hence to $\mathscr{R}$. Because $\mathscr{R}$ is topological, Q is dense in $\mathscr{R}$ and $v^\alpha P = Q$ for some $\alpha \in \mathrm{Ord}$, $wP = ww^\alpha P \supset wv^\alpha P = wQ = R$ is satisfied.

Let $\mathscr{R}$ be the Čech-Stone compactification of $\langle P, \tilde{u}\rangle$ and $R \supset Q$. Then $wQ \supset wP = R$, hence Q is dense in $\mathscr{R}$. If $f \in \mathscr{F}\langle Q, \tilde{v}\rangle$ then $f \upharpoonright P \in \mathscr{F}\langle P, \tilde{u}\rangle = \mathscr{F}(\mathscr{P})$ and the continuous extension g of the function $f \upharpoonright P$ to $\mathscr{R}$ satisfies $f = g \upharpoonright Q \in \mathscr{F}\langle Q, w \mid Q\rangle$. On the other hand, if $f \in \mathscr{F}\langle Q, w \mid Q\rangle$ then $f \upharpoonright P$ is an element of $\mathscr{F}\langle P, u\rangle = \mathscr{F}(\mathscr{P})$ and has the continuous extension g to $\langle Q, v\rangle$, hence $f = g$ is an element of $\mathscr{F}\langle Q, v\rangle = \mathscr{F}\langle Q, \tilde{v}\rangle$. Therefore $\langle Q, \tilde{v}\rangle$ is a subspace of $\mathscr{R}$, because $\mathscr{R}$ and $\langle Q, \tilde{v}\rangle$ are uniformizable.

4.9. Lemma 4.8 can be generalized in this way:

We say that a closure $\hat{v}$ *is the* $\mathfrak{B}$*-regular modification of a closure* v, *if* $\hat{v}$ *is the finest* $\mathfrak{B}$*-regular coarser than* v.

Proposition. *If* v *is a* $\mathfrak{W}$*-closure and* $\tilde{v}$ *is the uniformizable modification of* v, *then the* $\mathfrak{W}$*-regular modification* $\hat{v}$ *of* v *is the* $\mathfrak{W}$*-modification of* $\tilde{v}$ (by 3.13; for $\mathfrak{W} = \Theta$ it is in [7]).

If $\mathfrak{B} \subset \mathfrak{W}$ *then Lemma* 4.8 *remains true, if we replace uniformizable modifications by* $\mathfrak{W}$*-regular modifications and Čech-Stone compactifications by* $\mathfrak{W}$*-envelopes.*

Proof. Because $w^\gamma Q = R$ for some ordinal number γ and $v^\alpha P = Q$ for some ordinal number α, $w^{\alpha+\gamma}P \supset w^\gamma v^\alpha P = w^\gamma Q = R$ holds for these α, γ. The other parts of the proof are the same as in 4.8; the identity of the collections of all bounded continuous functions is sufficient for the identity of the corresponding $\mathfrak{W}$-regular $\mathfrak{W}$-spaces (by 3.13).

4.10. Definition. The $\mathfrak{B}$-cube $\langle C, \bar{v} \rangle$ of a closure space $\mathscr{P}$ is the $\mathfrak{B}$-modification of the cube $\langle C, \bar{w} \rangle = \mathsf{I}^{\mathscr{F}(\mathscr{P})}$.

4.11. Lemma. *A closure space $\mathscr{P}$ is a $\mathfrak{B}$-regular $T_0 - \mathfrak{B}$-space if and only if $\mathscr{P}$ is homeomorph to some subspace of a $\mathfrak{B}$-cube of $\mathscr{P}$.*

Proof. "If" follows from 3.11c and 1.6; on the other hand the evaluation mapping $\varphi = \{\{fx \mid f \in \mathscr{F}(\mathscr{P})\} \mid x \in |\mathscr{P}|\}$ is a homeomorphism of $\mathscr{P}$ into $\langle C, \bar{v} \rangle$ by 3.6 and 2.7.

4.12. Lemma. *Let $\mathscr{P}$ be a $\mathfrak{B}$-regular $T_0 - \mathfrak{B}$-space, let φ_0 be an evaluation mapping of $\mathscr{P}$ into the $\mathfrak{B}$-cube $\langle C, \bar{v} \rangle$ of $\mathscr{P}$, let $\{\mathscr{P}_\zeta = \langle P_\zeta, u_\zeta \rangle \mid \zeta \in \mathrm{Ord}\}$ be the transfinite sequence from 4.5. Then for each ordinal number ζ there exists a unique homeomorphism φ_ζ of $\mathscr{P}_\zeta$ into $\langle C, \bar{v} \rangle$ such that following conditions are satisfied.*

(i) $\varphi_\eta = \varphi_\zeta \upharpoonright P_\eta$, *if ordinal numbers η, ζ satisfy $\eta \leqq \zeta$.*

(ii) $\varphi_\zeta[P_\zeta] = v_\zeta\, \varphi[P_0]$ *for each ordinal number ζ.*

Proof. For each $\zeta \in \mathrm{Ord}$ every function $g \in \mathscr{F}(\mathscr{P})$ has a unique extension $h_\zeta g$ belonging to $\mathscr{F}(\mathscr{P}_\zeta)$ (by 4.5c). Because P_ζ is a $\mathfrak{B}$-regular $T_0 - \mathfrak{B}$-space and the mapping h_ζ on $\mathscr{F}(\mathscr{P})$ is a bijective mapping onto $\mathscr{F}(\mathscr{P}_\zeta)$, the mapping $\varphi_\zeta = \{h_\zeta g \mid \mid g \in \mathscr{F}(\mathscr{P})\}$ is an evaluation mapping of $\mathscr{P}_\zeta$ into $\langle C, \bar{v} \rangle$, hence a homeomorphism by 4.11.

Conditions (i) and (ii) can be easy proved by 4.5b, c.

Remark. 4.11 and 4.12 remain true, if the condition "$\mathscr{P}$ is a T_0-space" is omitted and the term "homeomorphism" is replaced by the term "quotient mapping under σ_ζ", where $\sigma_\zeta = \{\langle x, y \rangle \mid x \in u_0(y)$ or $x = y \in P_\zeta$.

Obvious (3.6).

4.13. Theorem. *Let $\mathscr{P}$ be a $\mathfrak{B}$-regular $\mathfrak{B}$-space and let $\{\mathscr{P}_\xi \mid \xi \in \mathrm{Ord}\}$ be the transfinite sequence from 4.5. Then $\mathscr{P}_\gamma$ is the $\mathfrak{B}$-envelope of the space $\mathscr{P}$ for some ordinal number γ.*

$\mathscr{P}_\alpha$ is a $\mathfrak{B}$-envelope of $\mathscr{P}$, if any of the following conditions for α is satisfied.

(a) card $\alpha >$ exp card $\mathscr{F}(\mathscr{P})$.

(b) card $\alpha >$ exp exp $d\mathscr{P}$.

(c) *α is a regular cardinal number and every directed set belonging to $\mathfrak{B}$ contains a cofinal subset whose cardinallity is less than α.*

Proof. Let $\langle C, \bar{v} \rangle$ be the $\mathfrak{B}$-cube of $\mathscr{P}$. The closure $\bar{v}^\alpha$ is the topological modification of $\bar{v}$ by 1.19 (in the case (a) card $C = (\exp \aleph_0)^{\mathrm{card}\,\mathscr{F}(\mathscr{P})} = \exp \mathrm{card}\, \mathscr{F}(\mathscr{P}) < \alpha$; the condition (b) implies (a), because card $\mathscr{F}(\mathscr{P}) \leqq \exp d(\mathscr{P})$ [5]). Hence the following is satisfied by 4.12 $\varphi_{\alpha+1}[P_\alpha] = \varphi_\alpha[P_\alpha] = \bar{v}^\alpha \varphi[P] = \bar{v}^{\alpha+1} \varphi[P] = \varphi_{\alpha+1}[P_{\alpha+1}]$, hence $P_\alpha = P_{\alpha+1}$. (If $x \in P_{\alpha+1} - P_\alpha$, then $\varphi_{\alpha+1} y = \varphi_{\alpha+1} x$ for some $y \in P_\alpha$; but $y \in \bar{v}(x) = (x)$ by 3.6 and that is a contradiction), therefore the space $\mathscr{P}_\alpha$ is the $\mathfrak{B}$-envelope of $\mathscr{P}$ by 4.7.

4.14. Theorem. *Let $\mathscr{Q} = \langle Q, v \rangle$ and $\mathscr{R} = \langle R, w \rangle$ be $\mathfrak{B}$-envelopes of a space $\mathscr{P} = \langle P, u \rangle$. Then there exists a unique homeomorphism of $\mathscr{Q}$ onto $\mathscr{R}$ identical on the set P.*

Proof. Let us denote $\tilde{u}, \tilde{v}, \tilde{w}$ the uniformizable modifications of $\mathfrak{B}$-regular closures u, v, w. Let $\beta\tilde{\mathscr{Q}}$ resp. $\beta\tilde{\mathscr{R}}$ be the Čech-Stone compactifications of $\langle Q, \tilde{v} \rangle$ resp. $\langle R, \tilde{w} \rangle$. Then $\beta\tilde{\mathscr{Q}}$ and $\beta\tilde{\mathscr{R}}$ are Čech-Stone compactifications of the space $\langle P, \tilde{u} \rangle$ by 4.8 and there exists a unique homeomorphism f of $\beta\tilde{\mathscr{Q}}$ onto $\beta\tilde{\mathscr{R}}$ identical on P (by [3], 41D.)

First we prove this lemma: $f[v^\zeta P] = w^\zeta P$ holds for any ordinal number ζ. Let this lemma be true for $\varkappa \in \mathsf{Ord}$ and let x belong to $v^{\varkappa+1}P - P$. Then a $\mathfrak{B}$-net N ranging in $v^\varkappa P$ converges to the point x in $\mathscr{Q}$; this net N converges to x also in $\langle Q, \tilde{v} \rangle$ and in $\beta\tilde{\mathscr{Q}}$, hence the net $f \circ N$ converges to fx in $\beta\tilde{\mathscr{R}}$ by 2.7.

Let $g \in \mathscr{F}(\mathscr{R})$. Then $g \in \mathscr{F}\langle R, \tilde{v} \rangle$ by 3.12a, there exists its extension $\tilde{g}$ to $\beta\tilde{R}$; the net $\tilde{g} \circ f \circ N = g \circ f \circ N$ is convergent in I and $\mathsf{E} f \circ N \subset f[v^\varkappa P] = w^\varkappa P \subset R$. Thus the net $f \circ N$ is remarkable in $\mathscr{R}$ and hence convergent in $\mathscr{R}$ by 4.6 to some point $y \in R$. Then $f \circ N$ converges to y also in $\langle R, \tilde{w} \rangle$ and in $\beta\tilde{\mathscr{R}}$.

Because the point fx does not belong to P, the set (fx) is closed in $\beta\tilde{\mathscr{R}}$ and therefore $fx = y$ belongs to $w \mathsf{E} f \circ N \subset w^{\varkappa+1}P$. Because f is a homeomorphism, the lemma is thus proved for $\varkappa + 1$.

If ζ is a non-isolated ordinal number and the lemma is true for all $\varkappa < \zeta$, then $w^\zeta P = \bigcup\{f[v^\varkappa P] \mid \varkappa < \zeta\} = f[v^\zeta P]$. This finishes the proof of the lemma.

By 4.13 there exists ordinal number γ, δ so that $v^\gamma P = Q$ and $w^\delta P = R$. If $\alpha \geqq \gamma$ and $\alpha \geqq \delta$, then $f[Q] = f[v^\alpha P] = w^\alpha[P] = R$, thus $f \upharpoonright Q$ is a homeomorphism of $\langle Q, \tilde{v} \rangle$ onto $\langle R, \tilde{w} \rangle$ and hence a homeomorphism of $\mathscr{Q}$ onto $\mathscr{R}$ by 3.13 (the $\mathfrak{B}$-modification is a topological property).

4.15. Theorem. *Let $\mathscr{P} = \langle P, u \rangle$ be a $\mathfrak{B}$-regular $\mathfrak{B}$-space, let $\tilde{u}$ be the uniformizable modification of u, $\langle Q, w \rangle$ the Čech-Stone compactification of $\langle P, \tilde{u} \rangle$, v the $\mathfrak{B}$-modification of w. Let $v^{\alpha+1}P = v^\alpha P$ hold. Then the space $\langle v^\alpha P, v \mid v^\alpha P \rangle$ is a $\mathfrak{B}$-envelope of the space $\mathscr{P}$.*

Remark. If α satisfies the condition (a) or (b) or (c) in 4.13, then the condition $v^{\alpha+1}P = v^\alpha P$ is satisfied.

Proof. Let $\mathscr{P}$ be a T_0-space. The evaluation mapping φ is a homeomorphism of $\mathscr{P}$ into the $\mathfrak{B}$-cube $\langle C, \bar{v} \rangle$ of $\mathscr{P}$ by 4.11. Let us denote $H = \varphi[P]$. Then φ is a homeomorphism of $\langle P, \tilde{u} \rangle$ onto $\langle H, \bar{w} \mid H \rangle$ ($\bar{w}$ is the topology of the cube C). The space $\langle \bar{w}H, \bar{w} \mid \bar{w}H \rangle$ is the Čech-Stone compactification of $\langle H, \bar{w} \mid H \rangle$, hence there exists a homeomorphism f of $\langle Q, w \rangle$ onto $\langle \bar{w}H, \bar{w} \mid \bar{w}H \rangle$ such that $f \upharpoonright P = \varphi$.

Let $\zeta = \alpha$ or $\zeta = \alpha + 1$. $g_\zeta = f \upharpoonright v^\zeta P$ is a homeomorphism of $\langle v^\zeta P, v \mid v^\zeta P \rangle$ onto $\langle \bar{v}^\zeta H, \bar{v} \mid \bar{v}^\zeta H \rangle$, because the $\mathfrak{B}$-modification is a topological property. φ_ζ from 4.12 is a homeomorphism of $\mathscr{P}$ onto $\langle \bar{v}^\zeta H, \bar{v} \mid \bar{v}^\zeta H \rangle$. Hence the mapping $h_\zeta = g_\xi^{-1} \circ \varphi_\zeta$ is a homeomorphism of $\mathscr{P}_\zeta$ onto $\langle v^\zeta P, v \mid v^\zeta P \rangle$.

Thus $P_{\alpha+1} = P_\alpha$ is satisfied and $\mathscr{P}_\alpha$ is a $\mathfrak{B}$-envelope of $\mathscr{P}$ by 4.7. Hence also $\langle v^\alpha P, v \mid v^\alpha P \rangle$ is a $\mathfrak{B}$-envelope of $\mathscr{P}$, because the property "to be a $\mathfrak{B}$-envelope of" is topological and the mapping $h_\alpha \upharpoonright P = \varphi^{-1} \circ \varphi$ is identical.

To prove 4.15 for non-T_0-space $\mathscr{P}$, we apply the preceding for the quotient spaces under σ (where $\mathbf{D}\sigma$ is P or Q or $v^\alpha P$ and $\langle x, y \rangle \in \sigma$ iff $x \in u(y)$ or $x = y \in \mathbf{D}\sigma$) generated by the canonical mapping.

4.16. Theorem. *Let $\mathscr{P}$ be a $\mathfrak{B}$-regular space. Let us denote A_0 the collection of all finite subsets of the set* card $\mathscr{F}(\mathscr{P})$. *Let a $\subset$-cofinal subcollection A of A_0 exist such that the product of directed sets $\langle A, \subset \rangle \times \langle \omega_0, \leqq \rangle$ and some element of $\mathfrak{B}$ are order-isomorphic. Then $\mathscr{P}$ is a uniformizable $\mathfrak{B}$-space and the $\mathfrak{B}$-envelope of $\mathscr{P}$ coincides with the Čech-Stone compactification of $\mathscr{P}$.*

4.17. Corollary. *Let $\mathscr{P}$ be a $\mathfrak{B}$-regular space and either $\alpha = \exp d\mathscr{P}$ or $\alpha =$* $=$ card $\mathscr{F}(\mathscr{P})$; *let $\mathfrak{B} \supset \mathfrak{M}_\alpha$. Then $\mathscr{P}$ is a uniformizable $\mathfrak{B}$-space and the $\mathfrak{B}$-envelope of $\mathscr{P}$ coincides with the Čech-Stone compactification of $\mathscr{P}$.*

Proof of 4.16. Let $\mathscr{P} = \langle P, u \rangle$, let us denote $\tilde{u}$ the uniformizable modification of u and $\langle Q, w \rangle$ a Čech-Stone compactification of $\langle P, \tilde{u} \rangle$. First we prove that the cube $\langle C, \overline{w} \rangle$ of $\mathscr{P}$ is a $\mathfrak{B}$-space. Let us denote F_0 the collection of all finite subsets of $\mathscr{F}(\mathscr{P})$. By the assumption there exists a $\subset$-cofinal subset F of F_0 such that the product $\langle F, \subset \rangle \times \langle \omega_0, \leqq \rangle$ and an element $\langle E, \sigma \rangle$ of $\mathfrak{B}$ are order-isomorphic and $\emptyset \notin F$. For each $G \in F$, $n \in \omega_0$, $j \in [\![0, 1]\!]$ let us denote $U^f_{G,n} j = [\![0, 1]\!]$ if $f \in \mathscr{F}(\mathscr{P}) - G$, $U^f_{G,n} j =]\!] j - 1/(n+2), j + 1/(n+2) [\![\cap [\![0, 1]\!]$ if f belongs to G.

Then the collection $\mathscr{G} = \{ \Pi\{ U^f_{G,n} z_f \mid f \in \mathscr{F}(\mathscr{P})\} \mid G \in F, n \in \omega_0 \}$ is a local base at the point $z = \{ z_f \mid f \in \mathscr{F}(\mathscr{P})\}$ in the space $\langle C, \overline{w} \rangle$. We can easily verify that the directed sets $\langle G, \supset \rangle$ and $\langle F, \subset \rangle \times \langle \omega_0, \leqq \rangle$ and hence $\langle G, \supset \rangle$ and $\langle E, \sigma \rangle$ are order-isomorphic, thus $\langle C, \overline{w} \rangle$ is a $\mathfrak{B}$-space by 1.2.

Consequently, its subspace $\langle f[Q], \overline{w} \mid f[Q] \rangle$ (where f is the homeomorphism from the proof of 4.15), the space $\langle Q, w \rangle$ homeomorph with $\langle f[Q], \overline{w} \mid f[Q]$ and the subspace $\langle P, \tilde{u} \rangle$ of $\langle Q, w \rangle$ are $\mathfrak{B}$-spaces. Therefore $\tilde{u} = u$ by 3.13a and $\mathscr{P} =$ $= \langle P, \tilde{u} \rangle$ is a uniformizable $\mathfrak{B}$-space. Hence the space $\langle Q, w \rangle = \langle wP, w \mid wP \rangle$ is a $\mathfrak{B}$-envelope of $\mathscr{P}$ by 4.15.

By 4.14 any $\mathfrak{B}$-envelope of $\mathscr{P}$ is its Čech-Stone compactification.

The corollary 4.17 follows from 4.16, because card $(A \times \omega_0) \leqq$ card $\mathscr{F}(\mathscr{P}) \,.\, \aleph_0 \leqq$ $\leqq \alpha$.

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