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ON MATROIDS REPRESENTABLE OVER $GF(3)$ AND OTHER FIELDS

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Abstract. The matroids that are representable over $GF(3)$ and some other fields depend on the choice of field. This paper gives matrix characterisations of the classes that arise. These characterisations are analogues of the characterisation of regular matroids as the ones that can be represented over the rationals by a totally-unimodular matrix. Some consequences of the theory are as follows. A matroid is representable over $GF(3)$ and $GF(5)$ if and only if it is representable over $GF(3)$ and the rationals, and this holds if and only if it is representable over $GF(p)$ for all odd primes p. A matroid is representable over $GF(3)$ and the complex numbers if and only if it is representable over $GF(3)$ and $GF(7)$. A matroid is representable over $GF(3)$, $GF(4)$ and $GF(5)$ if and only if it is representable over every field except possibly $GF(2)$. If a matroid is representable over $GF(p)$ for all odd primes p, then it is representable over the rationals.

1. Introduction

It is a classical (1958) result of Tutte [15, 16] that a matroid is representable over $GF(2)$ and some field whose characteristic is not 2 if and only if it can be represented over the rationals by a totally unimodular matrix, that is, by a matrix over $\mathbf{Q}$ with the property that all of its subdeterminants are in $\{0, 1, -1\}$. This paper focuses on the problem of finding analogues of Tutte's result for the field $GF(3)$. This continues a study begun in [18] where a matrix characterisation of the matroids representable over both $GF(3)$ and the rationals is given. In this paper the techniques of [18] are extended to give matrix characterisations of the matroids representable over $GF(3)$ and any other given field. We now outline some of the highlights of the theory.

A *dyadic–matrix* is a matrix over the rationals with the property that all of its subdeterminants belong to the set $\{0, \pm 2^i : i \text{ an integer}\}$. A *dyadic–matroid* is a matroid that can be represented over the rationals by the columns of a dyadic matrix. A $\sqrt[6]{1}$*–matrix* is a matrix over the complex numbers with the property that all of its non-zero subdeterminants are complex sixth roots of unity. A $\sqrt[6]{1}$*–matroid* is a matroid that can be represented over the complex numbers by the columns of a $\sqrt[6]{1}$–matrix. Most of the work in the paper is dedicated to proving that if M is a 3–connected, ternary matroid that is representable over some field that does not have characteristic 3, then M is either a dyadic matroid or a $\sqrt[6]{1}$–matroid. In combination with other results this yields the following theorems as corollaries.

Theorem 1.1. *The following are equivalent for a matroid M.*

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1. *M is representable over $GF(3)$ and $GF(5)$.*
2. *M is representable over $GF(p)$ for all odd primes p.*
3. *M is representable over $GF(3)$ and the rationals.*
4. *M is representable over $GF(3)$ and the reals.*
5. *M is representable over $GF(3)$ and $GF(q)$ where q is an odd prime power that is congruent to 2 mod 3.*
6. *M is a dyadic matroid.*

Theorem 1.2. *The following are equivalent for a matroid M.*

1. *M is representable over $GF(3)$ and $GF(4)$.*
2. *M is representable over $GF(3)$ and $GF(2^k)$ where k is an even positive integer.*
3. *M is a $\sqrt[6]{1}$–matroid.*

Theorem 1.3. *The following are equivalent for a matroid M.*

1. *M is representable over $GF(3)$ and the complex numbers.*
2. *M is representable over $GF(3)$ and $GF(7)$.*
3. *M is representable over $GF(3)$ and $GF(q)$ where q is an odd prime power that is congruent to 1 mod 3.*
4. *M is either a dyadic matroid, a $\sqrt[6]{1}$–matroid, or can be obtained by taking 2–sums and direct sums of dyadic matroids and $\sqrt[6]{1}$–matroids.*

Let $\mathbf{Q}(\alpha)$ denote the field obtained by extending the rationals by the transcendental α. A matrix over $\mathbf{Q}(\alpha)$ is *near-unimodular* if all of its non-zero subdeterminants are in $\{\pm\alpha^i(\alpha-1)^j : i, j \in \mathbf{Z}\}$. A matroid is *near-regular* if it can be represented over $\mathbf{Q}(\alpha)$ by the columns of a near-unimodular matrix.

Theorem 1.4. *The following are equivalent for a matroid M.*

1. *M is representable over $GF(3)$, $GF(4)$ and $GF(5)$.*
2. *M is representable over $GF(3)$ and $GF(8)$.*
3. *M is representable over all fields except possibly $GF(2)$.*
4. *M is representable over $GF(3)$, $GF(4)$ and the rationals.*
5. *M is near-regular.*

Theorem 1.5. *Let $\mathcal{F}$ be a set of fields containing $GF(3)$, and let $\mathcal{M}$ be the class of matroids representable over all fields in $\mathcal{F}$. Then for some $q \in \{2,3,4,5,7,8\}$, $\mathcal{M}$ is the class of matroids representable over $GF(3)$ and $GF(q)$.*

The above theorems resolve a number of natural conjectures on matroids representable over $GF(3)$ and other fields. In particular it follows from Theorem 1.1 that if a matroid is representable over $GF(p)$ for all odd primes p, then it is representable over the rationals. This is stated as [12, Problem 14.1.11]. It also follows that a problem of Brylawski, stated as [12, Problem 14.1.7], can be answered in the affirmative. It is an easy consequence of these theorems that a ternary matroid that is representable over a field of characteristic 2 is representable over $GF(4)$. A special case of this resolves a conjecture of Zaslavsky [19, Conjecture 8B.1] in the affirmative.

Given the above results it is clear that near-regular matroids, dyadic matroids and $\sqrt[6]{1}$–matroids form significant classes. It would be interesting to know more about their structure. An obvious problem is to try to characterise them by excluded minors. I have no idea how difficult this is likely to be. Near-regular matroids are a particularly natural generalisation of regular matroids—near-regular matroids are exactly the matroids representable over all fields except possibly $GF(2)$, while regular matroids are the ones representable over all fields. It would be of interest to know what results for regular matroids extend to near-regular matroids. Oxley [12, Problem 14.1.10] asks if there is an analogue of Seymour's regular matroid decomposition theorem [14] for the class of matroids representable over all odd primes. This is just the class of dyadic matroids. I believe that this is a very interesting question. As the class of dyadic matroids contains the class of near-regular matroids it is natural to begin by seeking a Seymour type decomposition theorem for near-regular matroids.

One can also speculate on the possibility of characterising the matroids representable over other sets of fields. It will hardly escape the readers' attention that dyadic matroids, $\sqrt[6]{1}$–matroids and near-regular matroids are all defined by reference to certain subgroups of multiplicative groups of fields. It is not hard to see how these notions can be generalised. For a given subgroup G of the multiplicative group of a field $\mathbf{F}$, one can define the class of $(G, \mathbf{F})$–matroids to be those matroids that have a representation over $\mathbf{F}$ in which all non-zero subdeterminants are in G. These classes may reward a general study. It is probably hopelessly optimistic to expect a positive answer to the following question. Given a set S of fields, at least one of which is finite, is it the case that there exists a finite set $\{(G_1, \mathbf{F}_1), \dots, (G_n, \mathbf{F}_n)\}$ with the property that a 3–connected matroid M is representable over all fields in S if and only if for some $i \in \{1, 2, \dots, n\}$, M is a $(G_i, \mathbf{F}_i)$–matroid?

The paper is structured as follows. Section 2 mainly outlines known results that are needed for this paper, particularly results from [18]. Section 3 establishes basic properties of $\sqrt[6]{1}$–matroids, a class that was not discussed in [18]. The matrix result of Section 4 is needed in the proofs in Section 5. This latter section contains the main results, and most of the argument.

2. Preliminaries

Familiarity is assumed with the elements of matroid theory. In particular it is assumed that the reader is familiar with the theory of matroid representations and matroid connectivity. For a good coverage of these topics we refer the reader to Oxley [12]. Terminology in this paper accords with [12] with the exception that we denote the simple matroid canonically associated with a matroid M by $\mathrm{si}(M)$. Note that we regard the ground set of $\mathrm{si}(M)$ to be a subset of the ground set of M rather than a partition of the ground set of M.

3–connected non-binary matroids. Much of the work of [18] is devoted to proving the following fact, [18, Corollary 3.8].

2.1. *Let M be a 3–connected, non-binary matroid with $r(M) \geq 4$. Then there exists an independent triple (a, b, c) of $E(M)$ with the property that $si(M/a)$, $si(M/b)$, $si(M/c)$, $si(M/a, b)$, and $si(M/a, c)$ are all non-binary and 3–connected.*

The above result is also essential in this paper. It is used in the proofs of Theorem 5.1 and Lemma 5.2.

It is assumed that the reader is familiar with Seymour's Splitter Theorem [14]. For a good discussion of this theorem and its consequences see [12, Chapter 11]. One consequence that is used several times in this paper is

2.2. *Let M be a non-binary, 3–connected matroid. If M is not a whirl, then there exists $x \in E(M)$ such that either $M\backslash x$ or M/x is non-binary and 3–connected.*

Weak maps and homomorphisms. Let M and N be matroids on a common ground set E. The identity map on E is a *weak map* from M to N if every independent set in N is also independent in M. In this case, N is a *weak-map image* of M. If, moreover, M and N have the same rank, N is a *rank-preserving* weak-map image of M. A good survey of the theory of weak maps is given in Kung and Nguyen [5]. In general there are few strong results describing the behaviour of weak maps. However the following result for ternary matroids is proved in [13, Theorem 1.1].

2.3. *Let M and N be ternary matroids such that N is a rank-preserving, weak-map image of M. If M is 3–connected, and N is connected and non-binary, then $M = N$.*

It is possible to determine whether one matroid is a weak-map image of another by comparing representations. Let A and B be matrices of the same size, so that their rows and columns are indexed by the same sets. Submatrices A' and B' of A and B respectively are *corresponding submatrices* if their rows and columns are indexed by the same subsets of the index sets of A and B. Lucas [9] proves

2.4. *Let M_1 and M_2 be matroids on a common ground set E represented over fields $\mathbf{F}_1$ and $\mathbf{F}_2$ by the $r \times n$ matrices $[I|A_1]$ and $[I|A_2]$ respectively, where corresponding columns of $[I|A_1]$ and $[I|A_2]$ represent the same elements of E. Then M_2 is a weak-map image of M_1 if and only if the following property holds. If D is a square submatrix of $[I|A_1]$ with $|D| = 0$, and D' is the corresponding submatrix of $[I|A_2]$, then $|D'| = 0$.*

In particular we have

2.5. *$M_1 = M_2$ if and only if the following property holds. For each square submatrix D of $[I|A_1]$ and corresponding submatrix D' of $[I|A_2]$, we have $|D| = 0$ if and only if $|D'| = 0$.*

A necessary condition for 2.4 and 2.5 to hold is for representations of a matroid M to be of the form $[I|A]$, that is, to be in *normal* form. In this paper it is always assumed that representations are in normal form. In accord with standard practice we frequently drop reference to the identity matrix and simply say that M is represented by A. With this convention, one regards the columns and rows of A as representing elements of the ground set of M. The i-th row of A represents the element represented by the i-th column of I.

Matroid representation is usually discussed in terms of representations over fields, although of course it makes sense to represent matroids over integral domains. There is a connection between homomorphisms of integral domains and weak maps. Say $\mathbf{I}_1$ and $\mathbf{I}_2$ are integral domains and let $\varphi : \mathbf{I}_1 \to \mathbf{I}_2$ be a function. For a matrix A over $\mathbf{I}_1$, let $\varphi(A)$ denote the matrix over $\mathbf{I}_2$ whose (i, j)–th entry is $\varphi(a_{ij})$. As an immediate consequence of the definition of determinant and homomorphism we have

2.6. *If D is a square submatrix of A and φ is a homomorphism, then $|\varphi(D)| = \varphi(|D|)$.*

As a direct consequence of 2.4 and 2.6 we have [5, Exercise 9.2].

2.7. *Let M_1 be represented over the integral domain $\mathbf{I}_1$ by the matrix $[I|A]$, let $\varphi : \mathbf{I}_1 \to \mathbf{I}_2$ be a homomorphism, and let M_2 denote the matroid represented over $\mathbf{I}_2$ by $\varphi([I|A])$. Then M_2 is a weak-map image of M_1.*

There is a natural map from the integers to a given field $\mathbf{F}$ defined by sending a positive integer n to the element $\underbrace{1+1+\cdots+1}_{n \text{ terms}}$. The image of a non-positive integer is defined in an obvious way. This map is, of course, a homomorphism. When we regard integers as elements of fields we always mean their images under this homomorphism.

Near-regular matroids. Recall from the introduction that $\mathbf{Q}(\alpha)$ denotes the field obtained by extending the rationals by the transcendental α. Recall also that a matrix is near-unimodular if it has a representation over $\mathbf{Q}(\alpha)$ in which all non-zero subdeterminants are in $\{\pm\alpha^i(\alpha-1)^j : i, j \in \mathbf{Z}\}$, and that a matroid is near-regular if it can be represented over $\mathbf{Q}(\alpha)$ by a near-unimodular matrix. If the matrix B is obtained from the near-unimodular matrix A by multiplying each entry of a row or column by a fixed element of the form $\pm\alpha^i(\alpha-1)^j$, then B is obtained from A by a proper scaling. The following facts are all proved in [18].

2.8. *Let A be a near-unimodular matrix and B be a matrix over $\mathbf{Q}(\alpha)$. If B is obtained from A by a sequence of proper scalings and pivots, then B is near-unimodular.*

2.9. *The class of near-regular matroids is closed under duality, and is closed under the taking of minors, direct sums and 2–sums.*

In general near-regular matroids are not uniquely representable over a given field. Nonetheless, we do have [18, Theorem 5.9].

2.10. *Let M be a ternary, non-binary, 3–connected matroid that has an element $x \in E(M)$ with the property that $M\backslash x$ is non-binary, 3–connected and near-regular. Assume that a near-unimodular representation of $M\backslash x$ extends to a representation of M over $\mathbf{Q}(\alpha)$. If the vector that represents x is scaled to have leading non-zero coefficient 1, then that representation is near-unimodular. Hence M is near-regular.*

2.11. *Let M be a near-regular matroid with an element x such that $M\backslash x$ is 3–connected and non-binary. Then any near-unimodular representation of $M\backslash x$ extends uniquely to a near-unimodular representation of M.*

Let A be a near-unimodular matrix, and $\mathbf{F}$ be a field. For f in $\mathbf{F} - \{0, 1\}$, let $A(f, \mathbf{F})$ denote the matrix over $\mathbf{F}$ obtained by making the substitution $f = \alpha$ in A. It is easily seen that $A(f, \mathbf{F})$ is well-defined. 2.12 below follows from results in [18] and a routine generalisation of [18, Lemma 6.7].

2.12. *Let M be a non-binary, 3–connected, near-regular matroid represented by the near-unimodular matrix A, and let $\mathbf{F}$ be a field. If $f \in \mathbf{F} - \{0, 1\}$, then $A(f, \mathbf{F})$ is a representation of M over $\mathbf{F}$. Moreover, up to equivalence, every representation of M over $\mathbf{F}$ can be obtained in this way.*

An immediate consequence of 2.12 is

2.13. *If M is near-regular, then M is representable over every field except possibly $GF(2)$.*

Dyadic matroids. Recall from the introduction that a matrix over $\mathbf{Q}$ is a dyadic matrix if all non-zero subdeterminants are signed integral powers of 2, and that a matroid is a dyadic matroid if it can be represented over the rationals by a dyadic matrix. If the matrix B is obtained from the dyadic matrix A by multiplying each entry of a given row or column by a fixed signed integral power of 2, then B is obtained from A by a *proper scaling*. The following facts are either proved in [17] or are consequences of results in [7, 8].

2.14. *If the matrix B is obtained from the dyadic matrix A by a sequence of pivots and proper scalings, then B is a dyadic matrix.*

2.15. *The class of dyadic matroids is closed under duality and the taking of minors, direct sums and 2–sums.*

2.16. *If M is a dyadic matroid represented by the dyadic matrix A and $\mathbf{F}$ is a field whose characteristic is not 2, then the matrix obtained by interpreting the entries of A as elements of $\mathbf{F}$ is a representation of M over $\mathbf{F}$. Hence dyadic matroids are representable over any field whose characteristic is not 2.*

2.17. *Near-regular matroids are dyadic matroids.*

3. Sixth Roots of Unity Matroids

Recall that a $\sqrt[6]{1}$–matrix is a matrix over $\mathbf{C}$ with the property that all non-zero subdeterminants are complex sixth roots of unity. A matroid is a $\sqrt[6]{1}$–matroid if it can be represented over $\mathbf{C}$ by a $\sqrt[6]{1}$–matrix. Let r denote a complex root of the polynomial $\alpha^2 - \alpha + 1$. It is easily seen that a matrix over $\mathbf{C}$ is a $\sqrt[6]{1}$–matrix if and only if all subdeterminants are in $\{0, \pm 1, \pm r, \pm(r-1)\}$. Of course $-(r-1)$ is just the other root of $\alpha^2 - \alpha + 1$. Let A be a $\sqrt[6]{1}$–matrix. The matrix B is obtained from A by a *proper scaling* of A if B is obtained by multiplying some row or column of A by a member of the set $\{\pm 1, \pm r, \pm(r-1)\}$. The following proposition is a routine consequence of the fact that the sixth roots of unity form a subgroup of the multiplicative group of the complex numbers.

Proposition 3.1. *Let A be a $\sqrt[6]{1}$–matrix, and B be a matrix over $\mathbf{C}$.*

1. *If B is obtained from A by a sequence of proper scalings, then B is a $\sqrt[6]{1}$–matrix.*
2. *If B is obtained from A by a sequence of pivots, then B is a $\sqrt[6]{1}$–matrix.*

A more or less immediate consequence of Proposition 3.1 is

Proposition 3.2. *The class of $\sqrt[6]{1}$–matroids is minor closed and is closed under duality.*

A straightforward argument proves

Proposition 3.3. *Direct sums and 2–sums of $\sqrt[6]{1}$–matroids are $\sqrt[6]{1}$–matroids.*

We now consider how $\sqrt[6]{1}$–matroids relate to other classes of matroids under consideration. It will eventually be shown that the class of near-regular matroids is the intersection of the classes of $\sqrt[6]{1}$–matroids and dyadic matroids. We first prove

Proposition 3.4. *Near-regular matroids are $\sqrt[6]{1}$–matroids.*

Proof. Say M is near-regular. Then M has a near-unimodular representation $[I|A]$. Regard $[I|A]$ as a matrix over $\mathbf{C}$ and let $[I|A](r)$ be the matrix obtained by making the substitution $\alpha = r$ in $[I|A]$. By 2.12, $M[I|A] = M[[I|A](r)]$. It follows from this that M is a $\sqrt[6]{1}$–matroid if $[I|A](r)$ is a $\sqrt[6]{1}$–matrix. We now show that this is indeed the case.

Say D is a submatrix of $[I|A]$, and $D(r)$ is the corresponding submatrix of $[I|A](r)$. If $|D| = 0$, then certainly $|D(r)| = 0$, while if $|D| = \pm\alpha^i(\alpha-1)^j$, then by [18, Lemma 5.6], $|D(r)| = \pm r^i(r-1)^j$. But $\{\pm1, \pm r, \pm(r-1)\}$ forms a group under multiplication, so $\pm r^i(r-1)^j \in \{\pm1, \pm r, \pm(r-1)\}$. It follows that $[I|A](r)$ is indeed a $\sqrt[6]{1}$–matrix. □

Of course the converse of Proposition 3.4 does not hold; it is shown in the proof of Theorem 5.1 that $AG(2,3)$ is a $\sqrt[6]{1}$–matroid (see also [1, Exercise 24.14], and [12, Exercise 6.4.9]). But $AG(2,3)$ is certainly not near-regular. We now consider the representability of $\sqrt[6]{1}$–matroids. For a $\sqrt[6]{1}$–matrix A, and a root ω of $\alpha^2-\alpha+1$ over a field $\mathbf{F}$ let $A(\omega,\mathbf{F})$ denote the matrix over $\mathbf{F}$ obtained by making the substitution $r = \omega$ in A.

Proposition 3.5. *Let M be a $\sqrt[6]{1}$–matroid represented by the $\sqrt[6]{1}$–matrix A, and let $\mathbf{F}$ be a field for which $\alpha^2-\alpha+1$ is not irreducible. If ω is a root of $\alpha^2-\alpha+1$ over $\mathbf{F}$, then $A(\omega,\mathbf{F})$ is an $\mathbf{F}$–representation of M.*

Proof. Let D be a square submatrix of A with corresponding submatrix $D(\omega,\mathbf{F})$ of $A(\omega,\mathbf{F})$. By (2.5) it suffices to show that $|D| = 0$ if and only if $|D(\omega,\mathbf{F})| = 0$. Consider the subset $\mathbf{I}_r$ of $\mathbf{C}$ defined by $\mathbf{I}_r = \{a + br : a, b \in \mathbf{Z}\}$. Evidently $\mathbf{I}_r$ is an integral domain. Define the function $\varphi : \mathbf{I}_r \to \mathbf{F}$ by $\varphi(a + br) = a + b\omega$. One routinely checks that φ is a homomorphism. It is clear that $A(\omega,\mathbf{F}) = \varphi(A)$. By (2.6), $|D(\omega,\mathbf{F})| = |\varphi(D)| = \varphi(|D|)$. But $|D| \in \{0, \pm1, \pm r, \pm(r-1)\}$. If $|D| = 0$, then it is immediate that $|D(\omega,\mathbf{F})| = 0$, so assume that $|D| \neq 0$. Then $|D(\omega,\mathbf{F})| \in \{\pm1, \pm\omega, \pm(\omega-1)\}$. But all members of this set are non-zero elements of $\mathbf{F}$, so $|D(\omega,\mathbf{F})| \neq 0$. □

4. A Matrix Lemma

In this section we prove a lemma for a certain type of near-unimodular matrix. The result is needed for the proofs of Theorem 5.1 and Lemma 5.2. An $n \times n$ matrix is *semi-cyclic* if for $1 \le i, j \le n$ we have $a_{ij} = 1$, if $i = j$ or $i = j + 1$, and $a_{ij} = 0$ if $i > j + 1$ or $i < j$ and $j \neq n$. In other words, A has the form

$$A = \begin{bmatrix} 1 & 0 & 0 & & 0 & 0 & a_1 \\ 1 & 1 & 0 & \cdots & 0 & 0 & a_2 \\ 0 & 1 & 1 & & 0 & 0 & a_3 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 1 & 0 & a_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & 1 & a_{n-1} \\ 0 & 0 & 0 & & 0 & 1 & 1 \end{bmatrix},$$

where, in general, no restriction is placed on the values of $\{a_1, \ldots, a_{n-1}\}$.

Lemma 4.1. *Let A be a near-unimodular, semi-cyclic matrix. Then all members of $\{a_1, \ldots, a_{n-1}\}$ belong to the set*

$$G = \{0, \pm1, \pm\alpha, \pm(\alpha-1), \pm1/\alpha, \pm1/(\alpha-1), \pm\alpha/(\alpha-1), \pm(\alpha-1)/\alpha\}.$$

We first note a lemma. Let $\mathcal{A}$ denote the set

$$\mathcal{A} = \{0\} \cup \{\pm\alpha^i(\alpha-1)^j : i, j \in \mathbf{Z}\}.$$

Lemma 4.2. *If a, b, $b-1$, $a-b$ and $a-b+1$ are all in $\mathcal{A}$, then a, b, $b-1$, $a-b$ and $a-b+1$ are all in G.*

Proof. It was noted in the proof of [18, Proposition 5.4] that a routine check shows that if both b and $b-1$ are in $\mathcal{A}$, then $b \in S$, where

$$S = \{0, 1, \alpha, -(\alpha-1), 1/\alpha, -1/(\alpha-1), \alpha/(\alpha-1), (\alpha-1)/\alpha\},$$

and both b and $b-1$ are in G. Now $a-b$ and $a-b+1$ are in $\mathcal{A}$, so $-(a-b)$ and $-(a-b)-1$ are in $\mathcal{A}$. Therefore $-(a-b)$ and $-(a-b)-1$ are in G, and it follows that $a-b$ and $a-b+1$ are in G. Moreover, $a-b \in -S$, where $-S = \{-s : s \in S\}$. But $a = b + (a-b)$, so a can be obtained by adding a member of S to a member of $-S$. A further routine check shows that the only members of $\mathcal{A}$ that can be obtained by adding a member of S to a member of $-S$ are in G. Since $a \in \mathcal{A}$, it follows that $a \in G$, and the lemma is proved. □

We now complete the proof of Lemma 4.1.

Proof. Say A is near-unimodular and semi-cyclic. The result is clear if A is 1×1 or 2×2, so assume that

$$A = \begin{bmatrix} 1 & 0 & 0 & & 0 & 0 & a_1 \\ 1 & 1 & 0 & \cdots & 0 & 0 & a_2 \\ 0 & 1 & 1 & & 0 & 0 & a_3 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 1 & 0 & a_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & 1 & a_{n-1} \\ 0 & 0 & 0 & & 0 & 1 & 1 \end{bmatrix}$$

is $n \times n$ where $n \geq 3$. Note that

$$\pm|A| = a_1 - a_2 + a_3 - \cdots \pm a_{n-1} \mp 1.$$

Now set $b = a_2 - a_3 + \cdots \pm a_{n-1}$. Then $\pm|A| = a_1 - b + (-1)^{n-1}$. We now consider some other subdeterminants of A. Certainly a_1 is a subdeterminant of A. By deleting the last row and second-to-last column of A we obtain a matrix with a determinant equal to $\pm(a_1 - b)$. Delete the first and last row, and the first and second-to-last column of A to obtain a submatrix with determinant equal to $\mp b$. Delete the first row and the first column of A to obtain a submatrix with determinant equal to $\mp(b + (-1)^n)$.

Since A is near-unimodular, all of the above subdeterminants are in $\mathcal{A}$. Clearly $x \in \mathcal{A}$ if and only if $-x \in \mathcal{A}$. Hence a_1, $a_1 - b$, $a_1 - b + (-1)^{n-1}$, b and $b + (-1)^n$ are all in $\mathcal{A}$. If n is odd, it follows immediately from Lemma 4.2 that $a_1 \in G$. If n is even, then $-a_1$, $-a_1 - (-b)$, $-a_1 - (-b) + 1$, $-b$ and $-b-1$ are all in $\mathcal{A}$, and again it follows by Lemma 4.2 that $-a_1 \in G$, that is, $a_1 \in G$.

Say $1 < i \leq n-1$. To show that $a_i \in G$, apply the above argument to the matrix obtained by deleting the first $i-1$ rows and first $i-1$ columns from A. □

It is not hard to show that all subdeterminants of a near-unimodular, semi-cyclic matrix belong to G, but we do not need this fact. Indeed the only fact needed is that $a_1 \in G$. A similar result that is even easier to show is

Lemma 4.3. *If A is a semi-cyclic dyadic matrix, then all entries of A are in* $\{0, \pm 1, \pm 1/2, \pm 2\}$.

5. The Main Theorem

The key theorem of this paper is

Theorem 5.1. *Let $\mathbf{F}$ be a field whose characteristic is not 3, and let M be a 3–connected matroid that is representable over both $GF(3)$ and $\mathbf{F}$. Then M is either a dyadic matroid or a $\sqrt[6]{1}$–matroid.*

Much of the work is involved in proving Lemma 5.2 below. This lemma is a generalisation of [18, Theorem 6.6].

Lemma 5.2. *Let $\mathbf{F}$ be a field whose characteristic is not 3, and let M be a 3–connected matroid that is representable over both $GF(3)$ and $\mathbf{F}$. Assume that M is not near-regular, but that all 3–connected minors of M are near-regular. Then M is either a dyadic matroid or a $\sqrt[6]{1}$–matroid but not both. If M is a dyadic matroid then M is uniquely representable over $\mathbf{F}$, and if M is a $\sqrt[6]{1}$–matroid, then M has at most two inequivalent representations over $\mathbf{F}$.*

Proof. Certainly $r(M) > 2$. Assume that $r(M) = 3$. Then the only rank–3 ternary matroids satisfying the hypotheses of the lemma are the non-Fano matroid F_7^- and the matroid $AG(2,3)\backslash p$ obtained by deleting a point from the ternary affine plane. This fact follows easily from analyses of 3–connected, rank–3, ternary matroids given in [4, 6, 10]. Now F_7^- is uniquely represented over any field whose characteristic is not 2 by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

As a matrix over $\mathbf{Q}$ (or $\mathbf{C}$) this is clearly a dyadic matrix, and certainly not a $\sqrt[6]{1}$–matrix. It follows that F_7^- is a dyadic matroid and, since the above representation is unique, F_7^- is not a $\sqrt[6]{1}$–matroid. Consider $AG(2,3)\backslash p$. For a field $\mathbf{F}$ with a root ω of $\alpha^2 - \alpha + 1$, this matroid is represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1-\omega & 1 \\ 0 & 0 & 1 & 0 & 1 & -\omega & -\omega & 1-\omega \end{bmatrix}.$$

Again, it is easily checked that when $\mathbf{F} = C$, this matrix is a $\sqrt[6]{1}$–matrix, and that the representation is unique up to the choice of ω, so that $AG(2,3)\backslash p$ has at most two inequivalent representations over any field. Finally, $AG(2,3)\backslash p$ is not representable over $\mathbf{Q}$, so it is certainly not a dyadic matroid.

Assume that M has rank r where $r > 3$. Certainly $r(M^*) > 2$. If $r(M^*) = 3$, then the conclusion of the theorem holds for M^*, and consequently for M. Therefore we may assume that both M and M^* have rank at least 4. Since whirls are near-regular ([18, Proposition 5.4]) M is not a whirl. By 2.2, there exists an element x in the ground set E of M with the property that either $M\backslash x$ or M/x is 3–connected and non-binary. It is routinely seen that under the current assumptions no generality is lost in assuming that $M\backslash x$ is 3–connected and non-binary. By 2.1, there exists an independent triple $\{a, b, c\}$ of distinct elements of $E - \{x\}$ with the property that $\mathrm{si}(M\backslash x/a)$, $\mathrm{si}(M\backslash x/b)$, $\mathrm{si}(M\backslash x/c)$, $\mathrm{si}(M\backslash x/a, b)$, and $\mathrm{si}(M\backslash x/a, c)$

are all 3–connected and non-binary. If $\{x,a,c\}$ is collinear, then clearly $\{x,a,b\}$ is not collinear. Assume without loss of generality that $\{x,a,b\}$ is not collinear.

We now focus on a particular representation of $M\backslash x$. Since $M\backslash x$ is a 3–connected minor of M, it is near-regular and therefore has a near-unimodular representation. Since one can scale and pivot on a near-unimodular matrix it follows that $M\backslash x$ can be represented by a near-unimodular matrix $[I|A]$ where the last two columns of I represent a and b respectively. Following standard practice we say that $M\backslash x$ is represented by A, the identity matrix being implicit. Let A_a, A_b, and $A_{a,b}$ denote the matrices obtained by deleting the second-to-last, the last, and the last two rows of A respectively. Under the current convention, A_a, A_b, and $A_{a,b}$ represent $M\backslash x/a$, $M\backslash x/b$, and $M\backslash x/a,b$ respectively. Say $s \in \{a,b,\{a,b\}\}$. Certainly x is not a loop of M/s. Since $\mathrm{si}(M/s)$ is 3–connected, $\mathrm{si}(M/s)$ is near-regular. Evidently, M/s is also near-regular. We now show that a near-unimodular representation of $M\backslash x/s$ extends uniquely to a near-unimodular representation of M/s where the vector representing x is chosen to have leading non-zero entry 1. If x is in a non-trivial parallel class of M/s this is clear, so assume that x is not in such a parallel class. Then $\mathrm{si}(M/s)$ is a 3–connected extension of $\mathrm{si}(M\backslash x/s)$. By 2.11, any near-unimodular representation of $\mathrm{si}(M\backslash x/s)$ extends uniquely to a near-unimodular representation of $\mathrm{si}(M/s)$. The fact that a near-unimodular representation of $M\backslash x/s$ extends uniquely to a near-unimodular representation of M/s now follows routinely. It follows that a unique column can be added to each of A_a, A_b and $A_{a,b}$ to obtain representations of M/a, M/b and $M/a,b$ respectively. Clearly the first $r-2$ entries of these column vectors agree.

Let $\mathbf{x} = (x_1,x_2,\dots,x_r)$ be defined as follows: $(x_1,x_2,\dots,x_{r-2})$ is the vector that can be added to $A_{a,b}$ to represent $M/a,b$, while x_{r-1} and x_r are the last entries of the vectors that can be added to A_b and A_a to represent M/b and M/a respectively. Let M' be the matroid on $E(M)$ that is represented by the matrix $[A|\mathbf{x}]$, where, of course, $\mathbf{x}$ represents x. It now follows that M' is a $\mathbf{Q}(\alpha)$–representable matroid on $E(M)$ with the property that $M'\backslash x = M\backslash x$, $M'/a = M/a$, and $M'/b = M/b$. Certainly $M \neq M'$, for otherwise, by 2.10, M would be near-regular. We now show that, for some $f \in \mathbf{F}$, the matrix obtained by evaluating each entry of $[A|\mathbf{x}]$ at f represents M.

By 2.12, every $\mathbf{F}$–representation of $M\backslash x$ is obtained from A by evaluating its entries at an appropriate member of $\mathbf{F}$. But some $\mathbf{F}$–representation of $M\backslash x$ extends to an $\mathbf{F}$–representation of M. Therefore there is an element $f \in \mathbf{F}$ with the property that the representation $A(f)$ of $M\backslash x$ obtained by evaluating the entries of A at f extends to a representation of M. Say $[A(f)|\mathbf{f}]$ represents M, where $\mathbf{f} = (f_1,f_2,\dots,f_r)$, and the leading non-zero entry of $\mathbf{f}$ is 1. Evidently,

$$(f_1,f_2,\dots,f_{r-1}) = (x_1(f),x_2(f),\dots,x_{r-1}(f)),$$

and

$$(f_1,f_2,\dots,f_{r-2},f_r) = (x_1(f),x_2(f),\dots,x_{r-2}(f),x_r(f)).$$

It follows that $\mathbf{f} = x(f)$. In other words, the matrix $[A|\mathbf{x}](f)$ obtained by evaluating each entry of $[A|\mathbf{x}]$ at f represents M. Certainly $f \notin \{0,1\}$.

Since M is ternary, the above argument holds when $\mathbf{F}$ is $GF(3)$. For this field we must have $f = -1$, that is, $[A|\mathbf{x}](-1)$ represents M over $GF(3)$. We now focus on a particular subdeterminant of $[A|\mathbf{x}]$.

Now $M[A|\mathbf{x}] \neq M$, so by 2.5 there exists a submatrix D of $[A|\mathbf{x}]$ with the property that, for the corresponding submatrices $D(f)$ and $D(-1)$ of $[A|\mathbf{x}](f)$ and $[A|\mathbf{x}](-1)$ respectively, it is either the case that $|D| = 0$ and both $|D(f)| \neq 0$ and $|D(-1)| \neq 0$, or the case that $|D| \neq 0$ and both $|D(f)| = 0$ and $|D(-1)| = 0$. Consider such a submatrix D. It is evident that D meets the rows indexed by a and b, so D is at least 2×2. We now prove

5.3. *It may be assumed without loss of generality that D is 2×2.*

Proof. The argument of this proof is probably unnecessarily pedantic but paranoia got the better of me. Assume that D is $n \times n$ where $n > 2$. Consider the entries of D that are in neither of the rows indexed by a or b, nor in the column $\mathbf{x}$. We first show that at least one of these entries is non-zero. Certainly if $n > 3$, then all these entries cannot be 0, for otherwise we would have $|D| = |D(f)| = 0$, contradicting the choice of D. Assume that D is 3×3. Say $D = [d_{ij}]$ where the second and third rows of D are indexed by a and b, and the third column of D contains entries from $\mathbf{x}$. We have $d_{11} = d_{12} = 0$. Set

$$D' = \begin{bmatrix} d_{21} & d_{22} \\ d_{31} & d_{32} \end{bmatrix}.$$

It is easily seen that $|D| = 0$ if and only if $|D'| = 0$ and that $|D(f)| = 0$ if and only if $|D'(f)| = 0$. It now follows by the choice of D that $|D'| = 0$ if and only if $|D'(f)| \neq 0$. But D and D' are submatrices of A and $A(f)$ respectively, and both of these matrices represent $M\backslash x$. Hence, by 2.5, $|D'| = 0$ if and only if $|D'(f)| = 0$; a contradiction.

Therefore there exists a non-zero entry of D that is in neither of the rows indexed by a and b nor in the column $\mathbf{x}$. Assume without loss of generality that d_{11} is such an entry. Then $d_{11}(f)$ is the corresponding entry of $D(f)$. We now do operations on D and $D(f)$ respectively that amount, up to a scalar multiple, to pivots on D and $D(f)$ respectively. Let D'' be the $n \times n$ matrix defined as follows: $d''_{ij} = d_{ij}$ if $i = 1$, and otherwise,

$$d''_{ij} = \begin{vmatrix} d_{11} & d_{1j} \\ d_{i1} & d_{ij} \end{vmatrix}.$$

Let $D(f)''$ be obtained from $D(f)$ in precisely the same way that D'' is obtained from D. By elementary matrix theory $|D| = |D''|$ and $|D(f)| = |D(f)''|$. We now show that $D(f)'' = D''(f)$. Consider corresponding entries of D'' and $D(f)''$. It is easily seen that these are either corresponding entries of D and $D(f)$ or subdeterminants of corresponding 2×2 near-unimodular submatrices of D and $D(f)$. In the first case it is clear, and in the second case it follows from 2.12, that an entry of $D(f)''$ is the evaluation at f of the corresponding entry of D'', so that indeed $D(f)'' = D''(f)$. Let D''_{11} denote the submatrix obtained by deleting the first row and column of D''. Evidently, $|D''_{11}| = 0$ if and only if $|D''| = 0$. Similarly $|D(f)''_{11}| = 0$ if and only if $|D(f)''| = 0$ so that $|D''(f)_{11}| = 0$ if and only if $|D''(f)| = 0$. We conclude that $|D''_{11}| = 0$ if and only if $|D''(f)_{11}| \neq 0$. It now follows routinely that after pivoting on d_{11} in $[A|\mathbf{x}]$ we obtain a matrix that has all the desired properties of $[A|\mathbf{x}]$ and has an $(n-1) \times (n-1)$ submatrix with the desired properties of D. Obvious comments complete the proof. □

Assume then, that D is 2×2. One of the columns of D meets the column $\mathbf{x}$ of A. Say $\mathbf{y}$ is the other column of A that meets D, and say $\mathbf{y}$ represents the element $y \in E$. We now prove

5.4. *It may be assumed without loss of generality that the entries of D are all in $\{0, \pm 1, \pm\alpha, \pm(\alpha - 1)\}$.*

Proof. We first show that $[A|\mathbf{x}]$ can be scaled to obtain a matrix A' with the property that the submatrix D' of A' corresponding to D has all of its entries in $\{0, \pm 1, \pm\alpha, \pm(\alpha - 1)\}$. This scaling is proper in the sense that rows and columns are multiplied only by entries of the form $\pm\alpha^i(\alpha - 1)^j$.

Recall the bipartite graph $B(H)$ associated with a matrix H. The vertices of $B(H)$ are the index sets of the rows and columns of H. An edge joins row i to column j if and only if the (i, j)–th entry of the matrix is non-zero. It is known [2, Proposition 2.4] that a matroid $M[I|H]$ is connected if and only if $B(H)$ is connected. Consider $B([A|\mathbf{x}])$. We let a, b, x, and y denote the vertices of $B([A|\mathbf{x}])$ corresponding to the rows and columns of $[A|\mathbf{x}]$ that are represented by a, b, x, and y respectively. In what follows we use a technique of finding shortest paths in certain subgraphs of $B([A|\mathbf{x}])$. This technique is familiar from Gerards' proof [3] of Tutte's characterisation of totally-unimodular matrices.

Since M is 3–connected, $\{a, b, y\}$ is either independent or is a circuit of M. We consider two cases. For the first case, assume that $\{a, b, y\}$ is independent.

Extending previous notation, let $[A|\mathbf{x}]_{a,b}$ denote the matrix obtained by deleting the last two rows from $[A|\mathbf{x}]$. Now $[A|\mathbf{x}]_{a,b}$ represents $M/a, b$. Since $\mathrm{si}(M\backslash x/a, b)$ is 3–connected, it follows that, apart from loops, $M/a, b$ is connected. Loops of $M/a, b$ are represented by columns of zeros of $[A|\mathbf{x}]_{a,b}$, and these correspond to isolated vertices of $B([A|\mathbf{x}]_{a,b})$. Therefore, apart from possible isolated vertices, $B([A|\mathbf{x}]_{a,b})$ is a connected graph. Now $\{a, b, x\}$ and $\{a, b, y\}$ are both independent, so neither x nor y is a loop of $M/a, b$. Therefore, neither x nor y is an isolated vertex of $B([A|\mathbf{x}]_{a,b})$. It follows that there exists a path joining x and y. In particular there exists a shortest path joining x and y. It is easily seen that the vertex induced subgraph of a shortest path connecting two vertices of a simple graph (as $B([A|\mathbf{x}]_{a,b})$ certainly is) contains just the path. It follows routinely that, after an appropriate permutation of its rows and columns, $[A|\mathbf{x}]_{a,b}$ has a submatrix of the form

$$\begin{bmatrix} * & * & 0 & & 0 & 0 & 0 \\ 0 & * & * & \cdots & 0 & 0 & 0 \\ 0 & 0 & * & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & * & 0 & 0 \\ 0 & 0 & 0 & \cdots & * & * & 0 \\ 0 & 0 & 0 & & 0 & * & * \end{bmatrix}.$$

In this matrix the elements labelled by $*$ are non-zero and the first and last columns correspond to the vertices x and y respectively. Now say

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}.$$

Then, up to a permutation of its columns, $[A|\mathbf{x}]$ has a submatrix equal to

$$\begin{bmatrix} * & * & 0 & & 0 & 0 & 0 \\ 0 & * & * & \cdots & 0 & 0 & 0 \\ 0 & 0 & * & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & * & 0 & 0 \\ 0 & 0 & 0 & \cdots & * & * & 0 \\ 0 & 0 & 0 & & 0 & * & * \\ d_{11} & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} & d_{12} \\ d_{21} & b_2 & b_3 & \cdots & b_{n-2} & b_{n-1} & d_{22} \end{bmatrix}.$$

It is routinely seen that $[A|\mathbf{x}]$ can be properly scaled so that the above matrix is transformed into a matrix of the form

$$\begin{bmatrix} 1 & 1 & 0 & & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & & 0 & 1 & 1 \\ d_1 & a'_2 & a'_3 & \cdots & a'_{n-2} & a'_{n-1} & 1 \\ d_2 & b'_2 & b'_3 & \cdots & b'_{n-2} & b'_{n-1} & 1 \end{bmatrix}.$$

Denote the matrices obtained from the above matrix by deleting the second-to-last and last rows by S_a and S_b respectively. Then S_a and S_b are submatrices of matrices obtained from $[A|\mathbf{x}]_b$ and $[A|\mathbf{x}]_a$ by a proper scaling. It follows that both S_a and S_b are near-unimodular. Moreover, S_a and S_b are transposes of semi-cyclic matrices. It now follows by Lemma 4.1 that both d_1 and d_2 are in

$$\{0, \pm 1, \pm\alpha, \pm(\alpha-1), \pm 1/\alpha, \pm 1/(\alpha-1), \pm\alpha/(\alpha-1), \pm(\alpha-1)/\alpha\}.$$

Perform a final scaling on $[A|\mathbf{x}]$ by multiplying the entries in the second-to-last and last row by the denominators of d_1 and d_2 respectively. Let A' denote the matrix that results from this final scaling, and D' denote the submatrix of A' corresponding to D. It is clear that all the entries of D' are in $\{0, \pm 1, \pm\alpha, \pm(\alpha-1)\}$. Clearly none of the scalings that have been performed have affected the desired properties of D and it follows that in this case we can indeed assume that 5.4 holds.

Consider the second case. Assume that $\{a, b, y\}$ is a circuit of M. We first show that $M\backslash x, y$ is 3–connected and non-binary. Since y is a loop of the non-binary matroid $M\backslash x/a, b$ it follows easily that $M\backslash x, y$ is non-binary. Assume that $M\backslash x, y$ is not 3–connected. Then $M\backslash x, y$ has a 2–separation $\{J, K\}$. If both a and b are in J, then it is evident that $\{J \cup \{y\}, K\}$ is a 2–separation of the 3–connected matroid $M\backslash x$. It follows from this contradiction that we may assume without loss of generality that $a \in J$ and $b \in K$. Since $M\backslash x$ is 3–connected, $M\backslash x, y$ is connected and it follows that this 2–separation is exact. Also, $M\backslash x, y$ has no parallel classes. Therefore, $r_{M\backslash x,y}(J) \geq 2$ and $r_{M\backslash x,y}(K) \geq 2$. Since $r(M\backslash x, y) \geq 4$, either $r_{M\backslash x,y}(J) > 2$ or $r_{M\backslash x,y}(K) > 2$. Assume without loss of generality that $r_{M\backslash x,y}(J) > 2$. Now, $a \notin \mathrm{cl}_{M\backslash x,y}(K)$, for otherwise we would again contradict the fact that $M\backslash x$ is 3–connected. It now follows from elementary facts on rank functions of contractions that $r_{M\backslash x,y/a}(J - \{a\}) = r_{M\backslash x,y}(J) - 1$, and $r_{M\backslash x,y/a}(K) = r_{M\backslash x,y}(K)$. We deduce that $\{J - \{a\}, K\}$ is a 2–separation of $M\backslash x, y/a$ with $r_{M\backslash x,y/a}(J - \{a\}) \geq 2$ and $r_{M\backslash x,y/a}(K) \geq 2$. It follows routinely from this fact that $\mathrm{si}(M\backslash x, y/a)$ is not 3–connected. But $\{y, a, b\}$ is a circuit of M so $\{b, y\}$ is a parallel pair in $M\backslash x/a$. Hence $\mathrm{si}(M\backslash x, y/a) \cong \mathrm{si}(M\backslash x/a)$ so that $\mathrm{si}(M\backslash x/a)$ is not 3–connected. But we know that this matroid is indeed 3–connected. This contradiction establishes that $M\backslash x, y$ is 3–connected.

We now know that $M\backslash x$ and $M\backslash x, y$ are 3–connected and non-binary. Of course, $M\backslash y$ is also 3–connected and non-binary. Let A_y denote the matrix obtained by deleting $\mathbf{y}$ from A. We now show that $[A_y|\mathbf{x}]$ is a near-unimodular matrix.

It was noted above that $\mathrm{si}(M\backslash x, y/a) \cong \mathrm{si}(M\backslash x/a)$. Similarly $\mathrm{si}(M\backslash x, y/b) \cong \mathrm{si}(M\backslash x/b)$, so that $\mathrm{si}(M\backslash x, y/a)$ and $\mathrm{si}(M\backslash x, y/b)$ are both 3–connected and non-binary. Therefore $\mathbf{Q}(\alpha)$–representations of $\mathrm{si}(M\backslash x, y/a)$ and $\mathrm{si}(M\backslash x, y/b)$ extend uniquely to representations of $\mathrm{si}(M\backslash y/a)$ and $\mathrm{si}(M\backslash y/b)$ respectively, where the vector representing x is chosen to have leading non-zero coordinate 1. Since $M\backslash y$ is near-regular, and $M\backslash x, y$ is 3–connected and non-binary, by 2.11 there is, up to scaling, a unique extension of a $\mathbf{Q}(\alpha)$–representation of $M\backslash x, y$ to a representation of $M\backslash y$. It now follows that the unique vector with leading non-zero coordinate 1 that extends A_y to a representation of $M\backslash y$ is $\mathbf{x}$. Hence $[A_y|\mathbf{x}]$ is a near-unimodular matrix.

We now use another argument based on shortest paths, but this time we remove some columns. Since $M\backslash x, y$ is connected there is a shortest path of $B(A_y)$ joining a and b. In this case it follows that after an appropriate permutation of its rows and columns, A_y has a submatrix of the form

$$\begin{bmatrix} * & 0 & 0 & & 0 & 0 & 0 \\ * & * & 0 & \cdots & 0 & 0 & 0 \\ 0 & * & * & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & * & * & 0 \\ 0 & 0 & 0 & \cdots & 0 & * & * \\ 0 & 0 & 0 & & 0 & 0 & * \end{bmatrix},$$

where the first and last rows are indexed by a and b respectively and again the elements labelled by a $*$ are non-zero. Hence $[A|\mathbf{x}]$ has a submatrix equal to

$$\begin{bmatrix} * & 0 & 0 & & 0 & 0 & 0 & d_{11} & d_{12} \\ * & * & 0 & \cdots & 0 & 0 & 0 & y_2 & x_2 \\ 0 & * & * & & 0 & 0 & 0 & y_3 & x_3 \\ & \vdots & & \ddots & & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & & * & * & 0 & y_{r-2} & x_{r-2} \\ 0 & 0 & 0 & \cdots & 0 & * & * & y_{r-1} & x_{r-1} \\ 0 & 0 & 0 & & 0 & 0 & * & d_{21} & d_{22} \end{bmatrix}.$$

Again it is clear that $[A|\mathbf{x}]$ can be properly scaled so that the above matrix is transformed into a matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 & & 0 & 0 & 0 & d_1 & d_2 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 & y'_2 & x'_2 \\ 0 & 1 & 1 & & 0 & 0 & 0 & y'_3 & x'_3 \\ & \vdots & & \ddots & & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & & 1 & 1 & 0 & y'_{r-2} & x'_{r-2} \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & y'_{r-1} & x'_{r-1} \\ 0 & 0 & 0 & & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The matrices S_x and S_y obtained from the above matrix by deleting the second-to-last and last column are semi-cyclic. But A and $[A_y|\mathbf{x}]$ are both near-unimodular so S_x and S_y are near-unimodular. Again it follows by Lemma 4.1 that both d_1 and d_2 are in

$$\{0, \pm 1, \pm\alpha, \pm(\alpha-1), \pm 1/\alpha, \pm 1/(\alpha-1), \pm\alpha/(\alpha-1), \pm(\alpha-1)/\alpha\}.$$

As before, perform a final scaling by multiplying the entries in the second-to-last and last column of the above matrix by the denominators of d_1 and d_2 respectively. Let A' denote the matrix that results from this final scaling, and D' denote the submatrix of A' corresponding to D. It is clear that all the entries of D' are in $\{0, \pm 1, \pm\alpha, \pm(\alpha-1)\}$.

It now follows that in both possible cases we can assume without loss of generality that the entries of D are in $\{0, \pm 1, \pm\alpha, \pm(\alpha-1)\}$. □

Assume then that D is 2×2 and that the entries of D are all in $\{0, \pm 1, \pm\alpha, \pm(\alpha-1)\}$. It is clear that if $|D| = 0$, then both $|D(f)| = 0$ and $|D(-1)| = 0$. Therefore we may assume that $|D| \neq 0$ and both $|D(f)| = 0$ and $|D(-1)| = 0$. Consider possible values for $|D|$. Certainly $|D|$ is not of the form $\pm\alpha^i(\alpha-1)^j$, for then neither $|D(f)|$ nor $|D(-1)|$ would be zero. If $|D|$ were of the form $\pm 2\alpha^i(\alpha-1)^j$, then $|D(-1)| \neq 0$, so this case does not occur. Assume that $|D|$ is in

$$\{\alpha^2+1, \alpha^2-\alpha-1, \alpha^2-2\alpha+2, \alpha^2-3\alpha+1, \alpha^2+\alpha-1, 2\alpha^2-2\alpha+1\}.$$

It is easily checked that -1 is a root of none of these polynomials over $GF(3)$, so that in any of these cases, $|D(-1)| \neq 0$, and it follows that this case does not occur. One routinely checks that the only remaining cases are that $|D| = \alpha^2 - \alpha + 1$, or that, apart from a possible factor of α or $\alpha - 1$, $|D|$ belongs to $\{\alpha+1, \alpha-2, 2\alpha-1\}$. It is clear that these last two cases can occur. We complete the proof of the lemma by showing that in these cases M is either a dyadic matroid or a $\sqrt[6]{1}$–matroid.

5.5. *If $|D| \in \{\alpha+1, \alpha-2, 2\alpha-1\}$, then M is a dyadic matroid and is uniquely representable over* $\mathbf{F}$.

Proof. In this case it is clear that $\mathbf{F}$ does not have characteristic 2, and that $f \in \{-1, 1/2, 2\}$. Certainly the choice of f is unique, so that M is uniquely representable over $\mathbf{F}$. We now sort out some notation. First simplify things slightly by setting $B = [A|\mathbf{x}]$. To make things completely unambiguous, let $B(e, \mathbf{E})$ denote the matrix obtained over a field $\mathbf{E}$ by making the substitution $\alpha = e$ in the entries of B. There are four matrices that we are interested in. We are already familiar with B, $B(f, \mathbf{F})$, and $B(-1, GF(3))$. Of course, $M[B(f, \mathbf{F})] = M[B(-1, GF(3))] = M$. Consider also the matrix, $B(f, \mathbf{Q})$. We complete the proof by showing that $M = M[B(f, \mathbf{Q})]$, and that $B(f, \mathbf{Q})$ is a dyadic matrix.

We first show that $M[B(f,\mathbf{Q})] = M$. Assume not. Then there is a submatrix D' of B with the property that $|D'(f,\mathbf{Q})| \neq 0$ if and only if both $|D'(f,\mathbf{F})| = 0$ and $|D'(-1,GF(3))| = 0$. It is clear that D' meets both of the rows indexed by a and b, and also meets the column $\mathbf{x}$. Moreover, arguing just as in 5.3 and 5.4, we may assume without loss of generality that D' is 2×2, and that the entries of D' are all in $\{0,\pm 1,\pm\alpha,\pm(\alpha-1)\}$. We now consider possible values for $|D'|$.

If $|D'| = 0$, then $|D'(f,\mathbf{Q})| = |D'(f,\mathbf{F})| = 0$, so $|D'| \neq 0$. If $|D'|$ is of the form $\pm\alpha^i(\alpha-1)^j$ or the form $\pm 2\alpha^i(\alpha-1)^j$, then both $|D'(f,\mathbf{Q})|$ and $|D'(f,\mathbf{F})|$ are non-zero, so this case does not occur. Say $D' = \alpha^2-\alpha+1$. Then, since the characteristic of $\mathbf{F}$ is not equal to 3, neither -1, 2, nor $1/2$ are roots of $\alpha^2-\alpha+1$ over $\mathbf{F}$. Hence $|D'(f,\mathbf{F})| \neq 0$. But $|D'(-1,GF(3))| = 0$; a contradiction. Therefore $|D'| \neq \alpha^2-\alpha+1$. Assume that $|D'|$ is in

$$\{\alpha^2+1, \alpha^2-\alpha-1, \alpha^2-2\alpha+2, \alpha^2-3\alpha+1, \alpha^2+\alpha-1, 2\alpha^2-2\alpha+1\}.$$

None of these polynomials has a root over $GF(3)$ or over $\mathbf{Q}$. Therefore $|D'(f,\mathbf{Q})| \neq 0$ and $|D'(-1,GF(3))| \neq 0$; a contradiction, so this case does not occur. Finally assume that $|D'| \in \{\alpha+1, \alpha-2, 2\alpha-1\}$. Say $|D'| = \alpha-2$. Then $|D'(-1,GF(3))| = 0$. But we have assumed that $|D'(f,\mathbf{Q})| \neq 0$ if and only if $|D'(-1,GF(3))| = 0$. Hence $|D'(f,\mathbf{Q})| \neq 0$. Therefore $f \neq 2$. But then $|D'(f,\mathbf{F})| \neq 0$. This is again a contradiction, so $|D'| \neq \alpha-2$. The arguments for $|D| = \alpha+1$ and $|D| = 2\alpha-1$ are similar. All possible cases have been covered, and each leads to a contradiction, so no submatrix D' of B exists with the property that $|D'(f,\mathbf{Q})| \neq 0$ if and only if both $|D'(f,\mathbf{F})| = 0$ and $|D'(-1,GF(3))| = 0$. It follows that $M[B(f,\mathbf{Q})] = M$.

It remains to show that $B(f,\mathbf{Q})$ is a dyadic matrix. Certainly every non-zero entry of $B(f,\mathbf{Q})$ is an integral power of 2. By scaling if necessary we may assume that every entry of $B(f,\mathbf{Q})$ is a non-negative power of 2. Assume that $B(f,\mathbf{Q})$ is not a dyadic matrix. Then there exists a subdeterminant that has an odd prime p as a factor. Consider the matrix $B(f,GF(p))$. This is just the matrix over $GF(p)$ obtained by interpreting the entries of $B(f,\mathbf{Q})$ as integers mod p. Then $B(f,\mathbf{Q})$ has a submatrix with a non-zero determinant that has the property that the corresponding submatrix of $B(f,GF(p))$ has a zero determinant. It follows that $M[B(f,\mathbf{Q})] \neq M[B(f,GF(p))]$. By 2.4, $M[B(f,GF(p))]$ is a weak-map image of $M[B(f,\mathbf{Q})]$, and by 2.5 this weak-map must be proper. But $M[B(f,GF(p))]\backslash x$ is obtained by making the substitution $f=\alpha$ in a near-unimodular matrix for some f in $\mathbf{F}-\{0,1\}$. It follows that $M[B(f,GF(p))]\backslash x = M[B]\backslash x = M\backslash x$. Now $M\backslash x$ is 3–connected and non-binary. Hence $M[B(f,GF(p))]$ is a single-element extension of a 3–connected, non-binary matroid. Moreover it is easy to see that x is neither a loop nor a coloop of $M[B(f,GF(p))]$, so this matroid is connected. Finally we note that since $M[B(f,GF(p))]$ is a rank-preserving, weak-map image of a ternary matroid, it has neither $U_{2,5}$ nor $U_{3,5}$ as a minor, and since it is representable over $GF(p)$ and p is odd, it has neither F_7 nor F_7^* as a minor. Hence $M[B(f,GF(p))]$ is ternary. We conclude that $M[B(f,GF(p))]$ is a connected, ternary, non-binary matroid that is a proper, rank-preserving, weak-map image of the 3–connected ternary matroid M. By 2.3, this cannot happen. This contradiction shows that $B(f,\mathbf{Q})$ has no subdeterminant having p as a factor. We conclude that $B(f,\mathbf{Q})$ is a dyadic matrix, and it follows that M is a dyadic matroid. □

Note that, regarded as a matrix over $\mathbf{C}$, $B(f,\mathbf{Q})$ is certainly not a $\sqrt[6]{1}$–matrix. But $B(f,\mathbf{Q})$ is the unique representation of M over $\mathbf{C}$, and it follows that M is not a $\sqrt[6]{1}$–matroid.

5.6. *If $|D| = \alpha^2 - \alpha + 1$, then M is a $\sqrt[6]{1}$–matroid and M has at most two inequivalent representations over $\mathbf{F}$.*

Proof. Since $B(f,\mathbf{F})$ represents M over $\mathbf{F}$, and $|D(f,\mathbf{F})| = 0$, it must be the case that f is a root of $\alpha^2 - \alpha + 1$. It follows that M has at most two inequivalent representations over $\mathbf{F}$. (Note that for some fields (e.g. GF(4) or $\mathbf{C}$) a field automorphism takes one root of $\alpha^2 - \alpha + 1$ to the other, so that in these cases M is uniquely representable. However for other fields (e.g. GF(7)) no such automorphism exists.)

As in 5.5 we have the matrices B, $B(f,\mathbf{F})$, and $B(-1, GF(3))$. Consider also the matrix $B(r,\mathbf{C})$ over $\mathbf{C}$ obtained by making the substitution $\alpha = r$ in B where r is a complex root of $\alpha^2-\alpha+1$. We complete the proof by showing that $M = M[B(r,\mathbf{C})]$ and that $B(r,\mathbf{C})$ is a $\sqrt[6]{1}$–matrix.

Assume that $M[B(r,\mathbf{C})] \neq M$. Then arguing just as in the proof of 5.5 we may assume that there exists a 2×2 submatrix D' of B with entries in $\{0, \pm 1, \pm\alpha, \pm(\alpha-1)\}$, and the property that $|D'(r,\mathbf{C})| \neq 0$ if and only if both $|D'(f,\mathbf{F})| = 0$ and $|D'(-1, GF(3))| = 0$. As in 5.5 we show that this situation does not occur for any of the possible values for $|D'|$. Evidently, the case $|D'| = 0$ causes no difficulties. For all other cases, except the case $|D'| = \alpha^2 - \alpha + 1$, it is easily checked that $|D'(r,\mathbf{C})| \neq 0$ and that $|D'(-1, GF(3))| \neq 0$, so these cases do not occur. Finally, if $|D'| = \alpha^2 - \alpha + 1$, then of course $|D'(r,\mathbf{C})| = |D'(f,\mathbf{F})| = |D'(-1, GF(3))| = 0$. We conclude that no submatrix D' of B exists with the property that $|D'(r,\mathbf{C})| \neq 0$ if and only if both $|D'(f,\mathbf{F})| = 0$ and $|D'(-1, GF(3))| = 0$. It follows that $M[B(r,\mathbf{C})] = M$.

It remains to show that $B(r,\mathbf{C})$ is a $\sqrt[6]{1}$–matrix. Since $r^2 = r - 1$, every subdeterminant of $B(r,\mathbf{C})$ can be expressed in the form $ar + b$, where a and b are integers. We begin by showing that if a and b are both even, then $a = b = 0$.

Consider the field $GF(4)$. Say the elements of $GF(4)$ are $\{0, 1, \omega, \omega+1\}$. Then ω is a root of $\alpha^2 - \alpha + 1$ over $GF(4)$. Let $B(\omega, GF(4))$ denote the matrix over $GF(4)$ obtained by making the substitution $\omega = r$ in $B(r,\mathbf{C})$ (or equivalently, the substitution $\omega = \alpha$ in B.) We now show that $M[B(\omega, GF(4))] = M$. Consider the integral domain $\mathbf{I}_r = \{ar + b : a, b \in \mathbf{Z}\}$. Define the function $\varphi : \mathbf{I}_r \to GF(4)$ by $\varphi(ar+b) = (a \bmod 2)r + b \bmod 2$. One routinely checks that φ is a homomorphism. (This is just a case of the homomorphism defined in the proof of Proposition 3.5.) It now follows by 2.7 that $M[B(\omega, GF(4))]$ is a weak-map image of $M = M[B(r,\mathbf{C})]$. This weak map is clearly rank preserving. We need to show that it cannot be proper. We first show that $M[B(\omega, GF(4))]$ is ternary. Assume not. Then, since no rank-preserving weak-map image of a ternary matroid has $U_{2,5}$ or $U_{3,5}$ as a minor, $M[B(\omega, GF(4))]$ has either F_7 or F_7^* as a minor. Assume that $M[B(\omega, GF(4))]$ has an F_7–minor. Then there exists an independent set I and a coindependent set J of $M[B(\omega, GF(4))]$ such that $M[B(\omega, GF(4))]/I\backslash J \cong F_7$. It is easily seen that in this case $M[B(\omega, GF(4))]/I\backslash J$ is a rank-preserving, weak-map image of $M/I\backslash J$, a ternary matroid. A straightforward argument shows that the only ternary matroid that has F_7 as a rank-preserving, weak-map image is the non-Fano matroid F_7^-. But F_7^- is 3–connected and is not near-regular, so F_7^- cannot be a proper minor of M. Hence $M = F_7^-$. This argument may be dualised so that we can conclude that either M or M^* is isomorphic to F_7^-. It follows that either M or M^* has rank less than or equal to 3. This contradicts the assumption that both these matroids have rank at least 4. We conclude that $M[B(\omega, GF(4))]$ has neither F_7 nor F_7^* as a minor and hence that $M[B(\omega, GF(4))]$ is ternary. We now know that $M[B(\omega, GF(4))]$ is a ternary, rank-preserving, weak-map image of M. Moreover, arguing just as in 5.5, we deduce that $M[B(\omega, GF(4))]$ is connected and non-binary. Therefore, $M[B(\omega, GF(4))]$ is a connected, ternary, non-binary matroid that is a rank-preserving, weak-map image of the 3–connected ternary matroid M. Hence, by 2.3, $M = M[B(\omega, GF(4))]$.

Now assume that some non-zero subdeterminant of $B(r, \mathbf{C})$ is of the form $ar + b$ where both a and b are even. Then, by 2.6, the corresponding subdeterminant of $B(\omega, GF(4))$ is equal to $\varphi(ar + b)$, and $\varphi(ar + b) = 0$. But this implies that $M[B(\omega, GF(4))] \neq M$; a contradiction. Hence all subdeterminants of $B(r, \mathbf{C})$ are of the form $ar + b$ where at least one of a and b is odd.

To show that $B(r, \mathbf{C})$ is a $\sqrt[6]{1}$–matrix we need to show that all subdeterminants are in $\{0, \pm 1, \pm r, \pm(r-1)\}$. Assume that $B(r, \mathbf{C})$ has a submatrix that is not in this set. By a familiar argument we may assume without loss of generality that there exists a 2×2 submatrix D' of B whose entries are in $\{0, \pm 1, \pm\alpha, \pm(\alpha - 1)\}$ having the property that $|D'(r, \mathbf{C})| \notin \{0, \pm 1, \pm r, \pm(r-1)\}$. Once again we consider possible values for $|D'|$. Clearly $|D'| \neq 0$ and $|D'|$ is not of the form $\pm\alpha^i(\alpha-1)^j$. If $|D'|$ is of the form $\pm 2\alpha^i(\alpha-1)^j$, then $|D'(r, \mathbf{C})|$ is non-zero and is equal to $ar + b$ for some even integers a and b. We have shown above that this case does not occur. Say $|D'|$ is in

$$\{\alpha^2 + 1, \alpha^2 - \alpha - 1, \alpha^2 - 2\alpha + 2, \alpha^2 - 3\alpha + 1, \alpha^2 + \alpha - 1, 2\alpha^2 - 2\alpha + 1\}.$$

Then $|D'(r, \mathbf{C})|$ is in

$$\{r, -2, -(r-1), -2r, 2(r-1), -1\}.$$

Members of this set are either in $\{0, \pm 1, \pm r, \pm(r-1)\}$ or of the form $ar + b$ for even integers a and b, so this case does not occur. Clearly $|D'| \neq \alpha^2 - \alpha + 1$. The only other case is if $|D'|$ is, up to sign and a factor of α or $\alpha - 1$, in $\{\alpha - 2, \alpha + 1, 2\alpha - 1\}$. Now $B(-1, GF(3))$ represents M over $GF(3)$ so that in this case $|D'(-1, GF(3))| = 0$. But $|D'(r, \mathbf{C})| \in \{r - 2, r + 1, 2r - 1\}$ so that $|D'(r, \mathbf{C})| \neq 0$. It now follows from the fact that both $B(r, \mathbf{C})$ and $B(-1, GF(3))$ represent M that this last case cannot occur. No possible case is consistent with our assumption and it follows that the assumption is false. We conclude that $B(r, \mathbf{C})$ is indeed a $\sqrt[6]{1}$–matrix and 5.6 is proved. □

The lemma follows on combining 5.5 with 5.6, and noting that it is now clearly the case that M cannot be both a dyadic matroid and a $\sqrt[6]{1}$–matroid. □

With Lemma 5.2 in hand we are in a position to establish some more facts on dyadic matroids and $\sqrt[6]{1}$–matroids.

Lemma 5.7. *Let M be a 3–connected dyadic matroid that is not near-regular, and let $\mathbf{F}$ be a field.*

1. *If M is representable over $\mathbf{F}$, then M is uniquely representable over $\mathbf{F}$.*
2. *M is representable over $\mathbf{F}$ if and only if the characteristic of $\mathbf{F}$ is not 2.*
3. *M is not a $\sqrt[6]{1}$–matroid.*

Proof. Consider 1. Say M is representable over $\mathbf{F}$. Clearly M has a 3–connected minor N satisfying the hypotheses of Lemma 5.2. By this lemma, N is uniquely representable over $\mathbf{F}$. It is shown in [17] that an $\mathbf{F}$–representation of a 3–connected ternary matroid that extends to an $\mathbf{F}$–representation of a 3–connected ternary extension or coextension does so uniquely. That M is uniquely representable over $\mathbf{F}$ follows from this fact.

Consider 2. It follows from 2.16 that if the characteristic of $\mathbf{F}$ is not 2, then M is representable over $\mathbf{F}$. Assume that the characteristic of $\mathbf{F}$ is 2. It clearly suffices to consider dyadic matroids satisfying the hypotheses of Lemma 5.2. But, as noted in the proof of 5.5, such matroids are not representable over a field of characteristic 2.

Consider 3. This follows from the fact that M has a minor satisfying the hypotheses of Lemma 5.2 and the fact that the class of $\sqrt[6]{1}$–matroids is minor closed. □

The proof of Lemma 5.8 below is similar to that of Lemma 5.7 and is omitted.

Lemma 5.8. *Let M be a 3–connected $\sqrt[6]{1}$–matroid that is not near-regular, and let $\mathbf{F}$ be a field.*

1. *M is representable over $\mathbf{F}$ if and only if $\mathbf{F}$ has a root of $\alpha^2 - \alpha + 1$.*
2. *M is not a dyadic matroid.*
3. *Say M is representable over $\mathbf{F}$. Then M is uniquely representable over $\mathbf{F}$ if an automorphism of $\mathbf{F}$ takes one root of $\alpha^2 - \alpha + 1$ to the other. Otherwise M has exactly two inequivalent representations over $\mathbf{F}$.*

We now prove Theorem 5.1.

Proof. Clearly $r(M) > 2$. Assume that M has rank 3. A similar check to that for the rank–3 case of Lemma 5.2 shows that M is either a restriction of the rank–3 ternary Dowling geometry $Q_3(GF(3)^*)$, or a restriction of the ternary affine plane $AG(2,3)$. One easily checks that the following matrix is a dyadic matrix that represents $Q_3(GF(3)^*)$:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}.$$

Hence $Q_3(GF(3)^*)$ is a dyadic matroid. Also, if r is a complex root of $\alpha^2 - \alpha + 1$, then

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1-r & 1 & 1-r \\ 0 & 0 & 1 & 0 & 1 & -r & -r & 1-r & 1-r \end{bmatrix}$$

represents $AG(2,3)$. Again it is easily checked that this matrix is a $\sqrt[6]{1}$–matrix so that $AG(2,3)$ is a $\sqrt[6]{1}$–matroid. Since the classes of dyadic matroids and $\sqrt[6]{1}$–matroids are minor closed we conclude that the theorem holds when M has rank 3.

Assume that M has rank r where $r > 3$. If M is near-regular, then the theorem certainly holds, so assume that M is not near-regular. If $r(M^*) = 3$, then the conclusion of the theorem holds for M^*, and consequently for M. Hence we may assume that both M and M^* have rank at least 4. The proof is by induction on the cardinality of the matroids satisfying the hypotheses of the theorem. If M is near-regular, then by 2.17 and Proposition 3.4, M is both a dyadic matroid and a $\sqrt[6]{1}$–matroid. If all 3–connected minors of M are near-regular, then the conclusion follows by Lemma 5.2. This establishes the base case of the argument.

Assume that the theorem holds for all 3–connected matroids satisfying the hypotheses of the theorem whose ground sets have cardinality less than $|E(M)|$. Assume that M has a proper 3–connected minor that is not near-regular. It follows from a routine application of the splitter theorem [14] that there exists an element $x \in E(M)$ with the property that $M\backslash x$ or M/x is 3–connected and not near-regular. It is easily seen that under the current assumptions no generality is lost in assuming that $M\backslash x$ is 3–connected and not near-regular. Using 2.1, and arguing as in the proof of Lemma 5.2, we see that there exists a pair of elements in $E(M) - \{x\}$ with the property that $\{x, a, b\}$ is not collinear and the property that $\mathrm{si}(M\backslash x/a)$, $\mathrm{si}(M\backslash x/b)$, and $\mathrm{si}(M\backslash x/a, b)$ are all non-binary and 3–connected. Assume that M is represented over $\mathbf{F}$ by the matrix $[I|A|\mathbf{x}]$, where $\mathbf{x}$ represents x, the last two columns of I represent a and b respectively, and $\mathbf{x}$ is scaled so that its leading non-zero entry is 1. Again we typically suppress reference to I, and again we let A_a, A_b, and $A_{a,b}$ denote the matrices obtained by deleting the second-to-last, the last, and the last two rows of A respectively. We also let $\mathbf{x}_a$, $\mathbf{x}_b$, and $\mathbf{x}_{a,b}$ denote the vectors obtained by deleting the second-to-last, the last, and the last two coordinates of $\mathbf{x}$ respectively. We will extend this notational convention to other matrices and vectors in an obvious way.

For reasons that will become apparent we further manipulate the representation to ensure that A has a submatrix of a certain type. Since $M\backslash x/a, b$ is non-binary, it has a $U_{2,4}$–minor. A routine argument now shows that by further pivoting, scaling, and row and column permutations we may assume without loss of generality that

$$A_{12} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & u \end{bmatrix},$$

where $u \notin \{0, 1\}$. We first ensure that a potentially unpleasant situation does not arise. Say $s \in \{a, b, \{a, b\}\}$.

5.9. *If $M\backslash x$ is a dyadic matroid, then M/s is a dyadic matroid, and if $M\backslash x$ is a $\sqrt[6]{1}$–matroid, then M/s is a $\sqrt[6]{1}$–matroid.*

Proof. Say $M\backslash x$ is a dyadic matroid. Then there exists a dyadic matrix $[I|A']$ that represents $M\backslash x$ over $\mathbf{C}$. (While dyadic–matrices are defined to be matrices over $\mathbf{Q}$ we can of course regard them as matrices over $\mathbf{C}$.) Evidently we may assume that each column of $[I|A']$ represents the same element of $E(M\backslash x)$ as the corresponding column of $[I|A]$. Equivalently, each row and column of A' represents the same element of $E(M)$ as the corresponding row and column of A. For a matrix B over $\mathbf{C}$, let $B(\mathbf{F})$ denote the matrix over $\mathbf{F}$ obtained by interpreting the entries of B as entries of $\mathbf{F}$ when this is well defined. By Lemma 5.7.2, $\mathbf{F}$ does not have characteristic 2. It is easily seen that $A'(\mathbf{F})$ is well defined and represents $M\backslash x$ over $\mathbf{F}$. But by Lemma 5.7.1, M is uniquely representable over $\mathbf{F}$. It now follows that A and $A'(\mathbf{F})$ are equal up to a scaling of A'. Perform this scaling if necessary and assume without loss of generality that $A = A'(\mathbf{F})$.

Now consider M/s. Clearly M/s is a dyadic matroid if and only if $\mathrm{si}(M/s)$ is a dyadic matroid. Assume that $\mathrm{si}(M/s)$ is not a dyadic matroid. Then, by the induction assumption, $\mathrm{si}(M/s)$ is a $\sqrt[6]{1}$–matroid. Arguing as above, there exists a $\sqrt[6]{1}$–matrix over $\mathbf{C}$ that represents $\mathrm{si}(M/s)$ and has the property that the matrix over $\mathbf{F}$ obtained by evaluating its entries at one of the roots of $\alpha^2 - \alpha + 1$ is equal to the submatrix of $[A|\mathbf{x}]$ that represents $\mathrm{si}(M/s)$. Clearly we can extend this to M/s. That is, there exists a $\sqrt[6]{1}$–matrix A'' over $\mathbf{C}$ with the property that, for some roots ω and r of $\alpha^2 - \alpha + 1$ over $\mathbf{F}$ and $\mathbf{C}$ respectively, the matrix $A''(\mathbf{F})$ obtained by making the substitution $\omega = r$ in A'', has the property that $A''(\mathbf{F}) = [A_s|\mathbf{x}_s]$. Consider

$$A'_{12} = \begin{bmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{bmatrix}.$$

Clearly

$$A'_{12}(\mathbf{F}) = \begin{bmatrix} 1 & 1 \\ 1 & u \end{bmatrix}.$$

As we have set it up this does not guarantee that

$$A'_{12} = \begin{bmatrix} 1 & 1 \\ 1 & u' \end{bmatrix}$$

for some $u' \in \mathbf{C}$. However an easy argument shows that we may further properly scale A' to ensure that this is the case. In other words we may assume without loss of generality that

$$A'_{12} = \begin{bmatrix} 1 & 1 \\ 1 & u' \end{bmatrix}.$$

But A'_{12} is a dyadic matrix and $u' \notin \{0, 1\}$, so $u' \in \{-1, 1/2, 2\}$. It follows that $u \in \{-1, 1/2, 2\}$. Now consider A''_{12}. Arguing as for A'_{12} we may assume that

$$A''_{12} = \begin{bmatrix} 1 & 1 \\ 1 & u'' \end{bmatrix}.$$

Here A''_{12} is a $\sqrt[6]{1}$–matrix, and $u'' \notin \{0, 1\}$. It follows that $u'' \in \{r, -(r-1)\}$, and that $u \in \{\omega, -(\omega - 1)\}$. But $-(\omega - 1)$ is just the other root of $\alpha^2 - \alpha + 1$. We conclude that some member of $\{-1, 1/2, 2\}$ is a root of $\alpha^2 - \alpha + 1$ over $\mathbf{F}$. An easy argument shows that the only fields for which a member of $\{-1, 1/2, 2\}$ is a root of $\alpha^2 - \alpha + 1$ are fields of characteristic 3. This contradicts the assumption that $\mathbf{F}$ does not have characteristic 3. We conclude that M/s is not a $\sqrt[6]{1}$–matroid, and it follows that M/s is indeed a dyadic matroid.

The proof in the case that $M\backslash x$ is a $\sqrt[6]{1}$–matroid is similar and is omitted. □

We complete the proof of the theorem by proving

5.10. *If $M\backslash x$ is a dyadic matroid, then M is a dyadic matroid, and if $M\backslash x$ is a $\sqrt[6]{1}$–matroid, then M is a $\sqrt[6]{1}$–matroid.*

Proof. Say $Z \in \{dyadic, \sqrt[6]{1}\}$. Assume that $M\backslash x$ is a Z–matroid, and let A' be a Z–matrix over $\mathbf{C}$. As in 5.9 we may assume without loss of generality that $A'(\mathbf{F}) = A$. Say $s \in \{a, b, \{a, b\}\}$. We now show that there exists a unique vector $\mathbf{x}'_s$ over $\mathbf{C}$, with leading non-zero coefficient 1, having the property that $[A'_s|\mathbf{x}'_s]$ represents M/s over $\mathbf{C}$. By 5.9, M/s is a Z–matroid. We consider possible cases.

Assume that $M\backslash x/s$ and M/s are both near-regular. Then arguing just as in the proof of Lemma 5.2, it follows that there is a unique vector $\mathbf{x}'_s$ over $\mathbf{C}$ such that $[A'_s|\mathbf{x}'_s]$ represents M/s over $\mathbf{C}$. Assume that neither $M\backslash x/s$ nor M/s are near-regular. Then both $\mathrm{si}(M\backslash x/s)$ and $\mathrm{si}(M/s)$ are uniquely representable over $\mathbf{C}$. (This is true when $Z = \sqrt[6]{1}$, for there is an automorphism of $\mathbf{C}$ taking one root of $\alpha^2 - \alpha + 1$ to the other.) An easy argument now shows that there is a unique vector $\mathbf{x}'_s$ over $\mathbf{C}$ such that $[A'_s|\mathbf{x}'_s]$ represents M/s over $\mathbf{C}$. The final case requires the most effort. Assume that $M\backslash x/s$ is near-regular but M/s is not. In this case it is clear that some Z–matrix A''_s that represents $M\backslash x/s$ extends to a matrix $[A''_s|\mathbf{x}''_s]$ over $\mathbf{C}$ that represents M/s. Moreover, by 2.16 or Proposition 3.5, $[A''_s|\mathbf{x}''_s](\mathbf{F})$ represents M/s over $\mathbf{F}$.

Say $Z = dyadic$. Then $[A''_s|\mathbf{x}''_s](\mathbf{F})$ is the unique representation of M/s over $\mathbf{F}$. It follows that we may assume without loss of generality that $[A''_s|x''_s](\mathbf{F}) = [A_s|\mathbf{x}_s]$. Therefore the submatrix of $[A''_s|\mathbf{x}''_s](\mathbf{F})$ indexed by its first two rows and columns is equal to

$$U = \begin{bmatrix} 1 & 1 \\ 1 & u \end{bmatrix}.$$

Since $[A''_s|\mathbf{x}''_s]$ is a dyadic matrix, the submatrix of $[A''_s|\mathbf{x}''_s]$ indexed by the first two rows and columns is equal to U. But we already know that the submatrix of A'_s indexed by the first two rows and columns is equal to U. By 2.12 both $[A''_s|\mathbf{x}''_s]$ and A'_s are evaluations of a near–unimodular matrix. It is now routinely seen that since they agree on the above-mentioned submatrix A''_s and A'_s must, up to a proper scaling, be equal. We conclude that in this case there does exist a unique vector $\mathbf{x}_s$ with leading coefficient 1 such that $[A'|\mathbf{x}'_s]$ represents M/s over $\mathbf{C}$. If $Z = \sqrt[6]{1}$, then the same argument applies except that A''_s and A'_s may differ in that we may have $A''_s = \overline{A'_s}$. But conjugation is an automorphism of $\mathbf{C}$, so A''_s extends to a representation of M/s if and only if $\overline{A'_s}$ does. In all possible cases there exists a unique vector $\mathbf{x}'_s$ over $\mathbf{C}$, with leading non-zero coefficient 1, having the property that $[A'_s|\mathbf{x}'_s]$ represents M/s over $\mathbf{C}$.

Arguing just as in Lemma 5.2 it now follows that there exists a unique vector $\mathbf{x}'$, with leading coefficient 1, that has the properties that $[A'|\mathbf{x}'](\mathbf{F}) = [A|\mathbf{x}]$, $M[A'_a|\mathbf{x}'_a] = M/a$, and $M[A'_b|\mathbf{x}'_b] = M/b$. Our aim now is to show that $M[A'|\mathbf{x}'] = M$, and that $[A'|\mathbf{x}']$ is a Z–matrix. We first obtain another representation of M.

Consider the matrix $[A'|\mathbf{x}'](GF(3))$. Since A' is a dyadic matrix, it follows by 2.16 that $M[A'(GF(3))] = M\backslash x$. Since M is ternary, and $M\backslash x$ is connected, this representation of $M\backslash x$ extends to a representation of M over $GF(3)$ (there are other ways of seeing this too). This representation agrees with $[A'|\mathbf{x}'](GF(3))$ on $[A'_s|\mathbf{x}'_s](GF(3))$ for $s \in \{a, b, \{a, b\}\}$, since this is a Z–matrix representing M/s. By a now familiar argument it follows that the vector that extends $[A'](GF(3))$ to a representation of M is $\mathbf{x}'(GF(3))$, that is, $M[[A'|\mathbf{x}'](GF(3))] = M$.

We now show that $M[A'|\mathbf{x}'] = M$. The argument is similar to argument in Lemma 5.2 so some detail is omitted. Say $M \neq M[A'|\mathbf{x}']$. Then there exists a submatrix D' of $[A'|\mathbf{x}']$ with the property that $|D'| \neq 0$ if and only if both $|D'(\mathbf{F})| = 0$ and $|D'(GF(3))| = 0$. Arguing just as in 5.4 we may assume without loss of generality that D' is 2×2.

Say $Z = dyadic$. Then arguing just as in 5.4, but this time using Lemma 4.3, we may further assume that

$$D' = \begin{bmatrix} d_1 & d_2 \\ 1 & 1 \end{bmatrix}$$

where $d_1, d_2 \in \{\pm 1, \pm 2, \pm 1/2\}$. It follows that

$$|D'| \in \{0, \pm 1, \pm 2, \pm 1/2, \pm 3, \pm 4, \pm 1/2, \pm 3/2, \pm 5/2\}.$$

Evidently, if $|D'| = 0$, then $|D'(\mathbf{F})| = 0$ and $|D'(GF(3))| = 0$, so assume that $|D'| \neq 0$. Then $|D'(GF(3))| = 0$, so $|D'| \in \{\pm 3, \pm 3/2\}$. But $\mathbf{F}$ does not have characteristic 3, so in this case $|D'(\mathbf{F})| \neq 0$, contradicting the assumption that this determinant is equal to 0.

Say $Z = \sqrt[6]{1}$. Again it follows that we may assume without loss of generality that

$$D' = \begin{bmatrix} d_1 & d_2 \\ 1 & 1 \end{bmatrix},$$

in this case $d_1, d_2 \in \{\pm 1, \pm r, \pm(r-1)\}$, and hence $|D'|$ is in

$$\{0, \pm 1, \pm r, \pm(r-1), \pm 2, \pm 2r, \pm 2(r-1), \pm(r+1), \pm(2r-1), \pm(r-2)\}.$$

Again we may assume that $|D'| \neq 0$. Then by our assumption, $|D'(GF(3))| = 0$. But $D'(GF(3))$ is obtained by making the substitution $r = -1$. It follows that $|D'| \in \{1+r, 2r-1, r-2\}$. But $\mathbf{F}$ does not have characteristic 3, and it is routinely checked that for a root ω of $\alpha^2 - \alpha + 1$ over $\mathbf{F}$, $1+\omega$, $2\omega - 1$ and $\omega - 2$ are all non-zero. Again we have contradicted our assumption and we can conclude that in this case we also have $M = M[A'|\mathbf{x}']$.

The fact that $[A'|\mathbf{x}']$ is a Z–matrix now follows from arguments essentially identical to those in the latter parts of 5.5 and 5.6 in the cases $Z =$ *dyadic* and $Z = \sqrt[6]{1}$ respectively. It follows immediately that M is either a dyadic matroid or a $\sqrt[6]{1}$–matroid so that 5.10, is proved. □

Theorem 5.1 follows immediately. □

6. Summing Up

We now prove Theorems 1.1—1.5. Essentially these theorems represent little more than a packaging of the information we have in an attempt to maximise the readership of this paper. We first note a proposition.

Proposition 6.1. *A matroid M is both dyadic and a $\sqrt[6]{1}$–matroid if and only if it is near-regular.*

Proof. This follows from Lemma 5.7 and Lemma 5.8 in combination with 2.17 and Proposition 3.4. □

Proof of Theorem 1.1. By 2.16, if part 6 holds, then all other parts hold. Consider the converses. Using quadratic residues it is easily seen that if q is an odd prime power that is congruent to 2 (mod 3), then $\alpha^2 - \alpha + 1$ has no root over $GF(q)$. Let i be any one of parts 1—5, and let M be a 3–connected matroid representable over all the fields satisfying the conditions of part i. It is clear that at least one of the fields satisfying the conditions of part i has no root of $\alpha^2 - \alpha + 1$. By Lemma 5.8.1, if M is a $\sqrt[6]{1}$–matroid, then M is near-regular, and hence, by Proposition 6.1, a dyadic matroid. On the other hand, if M is not a $\sqrt[6]{1}$–matroid, then by Theorem 5.1, M is a dyadic matroid. By 2.15, dyadic matroids are closed under direct sums and 2–sums, and the theorem follows. □

Proof of Theorem 1.2. Part 1 is just a case of part 2. We show that parts 2 and 3 are equivalent. Since $GF(2^k)$ has characteristic 2, by Lemma 5.7.2, any dyadic matroid that is representable over this field is near-regular and is therefore a $\sqrt[6]{1}$–matroid. It now follows by Theorem 5.1, that any 3–connected matroid representable over $GF(3)$ and $GF(2^k)$ is a $\sqrt[6]{1}$–matroid. By Proposition 3.3, $\sqrt[6]{1}$–matroids are closed under direct sums and 2–sums. Hence any matroid that is representable over $GF(3)$ and $GF(2^k)$ is a $\sqrt[6]{1}$–matroid. On the other hand, since k is even, $GF(2^k)$ does have a root of $\alpha^2 - \alpha + 1$, and hence, by Proposition 3.5, any $\sqrt[6]{1}$–matroid is representable over $GF(2^k)$. □

Proof of Theorem 1.3. Each of the fields of conditions 1, 2, and 3 has a root of $\alpha^2 - \alpha + 1$, and none has characteristic 2. Hence, by 2.16 and Proposition 3.5, if M is either a dyadic matroid or a $\sqrt[6]{1}$–matroid, then M is representable over any of these fields. Moreover any matroid that can be obtained by taking 2–sums and direct sums of dyadic matroids and $\sqrt[6]{1}$–matroids is representable over each of these fields. On the other hand, by Theorem 5.1, any 3–connected matroid that is representable over each of the fields of condition 1, 2, or 3 is either a dyadic matroid or a $\sqrt[6]{1}$–matroid. It follows that any matroid that is representable over each of these fields can be constructed by taking 2–sums and direct sums of dyadic matroids and $\sqrt[6]{1}$-matroids. □

Proof of Theorem 1.4. At least one of the fields of conditions 1, 2, 3 or 4 has characteristic 2, and at least one does not have a root of $\alpha^2 - \alpha + 1$. It now follows by Lemmas 5.7 and 5.8 that if M is representable over each of the fields satisfying conditions 1, 2, 3 or 4, then M must be near-regular. On the other hand, none of the fields is $GF(2)$, so by 2.13, any near-regular matroid is representable over each of these fields. □

Proof of Theorem 1.5. Say $\mathcal{M}$ is the class of matroids representable over $GF(3)$ and $GF(q)$ where $q \in \{2, 3, 4, 5, 7, 8\}$. If $q = 2$, then $\mathcal{M}$ is the class of regular matroids; if $q = 3$, then $\mathcal{M}$ is the class of ternary matroids; if $q = 4$, then $\mathcal{M}$ is the class of $\sqrt[6]{1}$–matroids; if $q = 5$, then $\mathcal{M}$ is the class of dyadic matroids; if $q = 7$, then $\mathcal{M}$ is the class of matroids obtained by taking 2–sums and direct sums of $\sqrt[6]{1}$–matroids and dyadic matroids; and if $q = 8$, then $\mathcal{M}$ is the class of near-regular matroids. In the light of results in this paper it is easily seen that if $\mathcal{F}$ is a set of fields containing $GF(3)$, then the class of matroids representable over all fields in $\mathcal{F}$ must belong to one of the above classes. □

Finally we note that we can specify exactly when a 3–connected ternary matroid is uniquely representable over a field whose characteristic is not 3. The proof is a straightforward consequences of results in this paper and is omitted.

Theorem 6.2. *Let* $\mathbf{F}$ *be a field whose characteristic is not* 3*, and let* M *be a* 3*–connected, ternary matroid representable over* $\mathbf{F}$*.* M *is not uniquely representable over* $\mathbf{F}$ *if and only if* M *is either a non-binary, near-regular matroid, or* M *is a* $\sqrt[6]{1}$*–matroid that is not near-regular and no automorphism of* $\mathbf{F}$ *takes one root of* $\alpha^2 - \alpha + 1$ *to the other. In the latter case,* M *has exactly two inequivalent* $\mathbf{F}$*–representations.*

References

1. Brylawski, T. H., and Kelly, D., *Matroids and combinatorial geometries.* Department of Mathematics, University of North Carolina, Chapel Hill, 1980.
2. Brylawski, T. H., and Lucas, D., Uniquely representable combinatorial geometries, in *Teorie Combinatorie (Proc. 1973 Internat. Colloq.),* pp. 83–104, Accademia Nazionale del Lincei, Rome, 1976.
3. Gerards, A. M. H., A short proof of Tutte's characterisation of totally unimodular matrices, *Linear Algebra Appl.* **114/115** (1989), 207–212. MR **90b:**05033
4. Kung, J. P. S., Combinatorial geometries representable over $GF(3)$ and $GF(q)$. I. The number of points, *Discrete Comput. Geom.* **5** (1990), 83–95. MR **90i:**05028
5. Kung, J. P. S. and Nguyen, H. Q., Weak Maps, in *Theory of Matroids* (ed. N. White), pp. 254–271, Cambridge University Press, Cambridge, 1986. CMP 18:15
6. Kung, J. P. S. and Oxley, J. G., Combinatorial geometries representable over $GF(3)$ and $GF(q)$. II. Dowling geometries, *Graphs Combin.* **4** (1988), 323–332. MR **90i:**05029

7. Lee, J., Subspaces with well-scaled frames, *Linear Algebra and its Applications* **114/115** (1989), 21–56. MR **90k:**90111
8. Lee, J., The incidence structure of subspaces with well-scaled frames, *J. Combin. Theory Ser. B* **50** (1990), 265–287. MR **92d:**05041
9. Lucas, D., Weak maps of combinatorial geometries, *Trans. Amer. Math. Soc.* **206** (1975), 247–279. MR **51:**7911
10. Oxley, J. G., A characterization of the ternary matroids with no $M(K_4)$–minor, *J. Combin. Theory Ser. B* **42** (1987), 212–249. MR **88d:**05039
11. Oxley, J. G., A characterization of certain excluded-minor classes of matroids, *Europ. J. Combin.* **10** (1989), 275–279. MR **91a:**05025
12. Oxley, J. G., *Matroid Theory,* Oxford University Press, New York, 1992. MR **94d:**05033
13. Oxley, J. G. and Whittle, G. P., On weak maps of ternary matroids, *Europ. J. Combin.* (to appear).
14. Seymour, P. D., Decomposition of regular matroids, *J. Combin. Theory Ser. B* **28** (1980), 305–359. MR **82j:**05046
15. Tutte, W. T., A homotopy theorem for matroids, I, II, *Trans. Amer. Math. Soc.* **88** (1958), 144–174. MR **21:**336
16. Tutte, W. T., Lectures on matroids, *J. Res. Nat. Bur. Standards Sect. B* **69B** (1965), 1–47. MR **31:**4023
17. Whittle, G. P., Inequivalent representations of ternary matroids, *Discrete Math.* **149** (1996), 233–238.
18. Whittle, G. P., A characterisation of the matroids representable over $GF(3)$ and the rationals, *J. Combin. Theory Ser. B.* **65** (1995), 222–261.
19. Zaslavsky, T. H., Signed Graphs, *Discrete Appl. Math.* **4** (1982), 47–74. MR **84e:**05095a

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