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ON MAXIMAL FAMILIES OF COMPACT COMPLEX
SUBMANIFOLDS OF COMPLEX MANIFOLDS

MAKOTO MAMBA

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Introduction. Let W be a complex manifold. Let V be a compact
complex submanifold of W. Let F be the sheaf over V of germs of

holomorphic sections of the normal bundle F of V in W. Let H1(V, F)
be the first cohomology group of F. In 1962, Kodaira proved the follow-
ing theorem:

THEOREM (K. Kodaira [4]). If H1(V, F) = 0, then there exists a maxi-
mal family {Vs}s•¸S of compact complex submanif olds of W such that
V0= V for a point o e S where the parameter space S is a complex
manifold.

The main purpose of this paper is to drop the assumption H1(V, F) =0.
We get:

THEOREM 1. There exists a maximal family {Vs}s•¸S of compact com-
plex submanif olds of W such that V0 = V for a point o e S where the
parameter space S is an analytic space.
The idea of the proof is due to Kuranishi's proof of his theorem on
the existence of the local moduli spaces of complex structures [6]. (See
also [7].)
Fixing W, we can easily patch these maximal families together and
get the following theorem.
THEOREM 2. Let W be a complex manifold. Then the set of all
compact complex submanif olds of W forms a (not necessarily connected)
analytic space S (W) in a natural way.

Our space 5(W) is naturally identified with an open subspace of the
Douady space [1].
For each point s •¸S (W), we denote VS the corresponding compact
complex submanif old of W. Using our concrete construction of maximal
families, we get:

THEOREM 3. Let W be a complex manifold. Let S(W) be the analytic
space in Theorem 2. Then,

582 M. NAMBA

{(s, t)•¸S(W)xS(W)|Vs •¸ Vt}

is a closed subvariety of S(W) x S(W).

COROLLARY. Let V be a compact complex submanif old of a complex
manifold W. Then

{s•¸S(W)|Vs •½ V}

is a closed subvariety of S(W).

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gratitude to Professsor Masataka Kuranishi, the thesis advisor, for his
instruction, guidance and many thoughtful comments.

1. Preliminaries. Let W be a (r + d)-dimensional (connected) com-

plex manifold. Let V be a d-dimensional (connected) compact complex
submanifold of W. We may assume that V is covered by a finite num-
ber of open subsets {Wi}i •¸ I of W, each of which has a local coordinate
system:

(wi, zi)=(wi1, ... , wri, z1i, ... , zdi)

such that V is defined in Wi by the equation wi = 0. We put Ui =
Wi•¿ V. Let

wi = fik(wk, zk),

zi = gik(wk, zk)

be the coordinate transformations in Wi •¿ Wk, where fik, and gik are vector-
valued holomorphic functions of (wk, zk) •¸ Wi •¿ Wk. We define matrix-
valued holomorphic functions Fik(zk) by

Fik(zk) (•Ýfik/•Ýwk) (o,zk) for zk •¸ Ui •¿ Uk .

Then we get the following identities:

Fij(zj)Fjk(zk) = Fik(zk) for zk •¸ Ui •¿ Uj •¿ Uk and zj = gik(a, zk)

Thus the system {Fik} defines a holomorphic vector bundle F on V. We
call this bundle the normal bundle of V in W. We denote F the sheaf
of germs of holomorphic sections of F.

Now we consider another compact complex submanifold V' of W
covered by {Wi}i•¸I. We assume that V' is defined in Wi by the equation:

wi = ƒÓi(zi)

where ƒÓi is a vector-valued holomorphic function of zi•¸Ui. These ƒÓi

ON MAXIMAL FAMILIES OF COMPACT COMPLEX 5$3

must satisfy the following compatibility conditions:

for

We want to consider families of such V'.

DEFINITION 1.1. Let X and S be analytic spaces* and let ƒÎ: X S
be a proper surjective holomorphic map. The triple (X,ƒÎ, S) is called
a family of compact complex manifolds if and only if there are an open
covering {Xƒ¿} of X, open subsets ƒ¶ƒ¿ of Cn, and holomorphic isomorphisms

where Sƒ¿ = ƒÎ(Xƒ¿) is open in S, such that the diagram

commutes for each ƒ¿. S is called the parameter space of the family
(X, ƒÎ, 5).

DEFINITION 1.2. Let W be a complex manifold. A family (X, ƒÎ, S)
of compact complex manifolds is called a family of compact complex sub-
manifolds of W if and only if X is an analytic subvariety of W x S and
ƒÎ is the the restriction to X of the projection map: W x S S.
For each point s •¸ S of a family (X, ƒÎ, S) of compact complex sub-
manifolds of W, the fiber ƒÎ-1(s) can be written as ƒÎ-1(s) = Vs x s where
Vs is a compact complex submanifold of W. We identify ƒÎ-1(s) with Vs
and write the family as {Vs}s•¸S to simplify the notations.

DEFINITION 1.3. A family {Vs}s•¸S of compact complex submanifolds
of a complex manifold W is said to be maximal at s0 •¸ S if and only if
for any family {Vt}S•¸T of compact complex submanifolds of W with a
point t0 •¸ T such that Vt0 = Vs0, there exist a neighbourhood U of t0 in
T and a holomorphic map f of U into S such that f(t0) = s0 and such
that

Vf(t) = Vt for all t •¸ U.

A family of compact complex submanifolds of W is called a maximal
family if and only if it is maximal at every point of the parameter
space.

* By an analytic space we mean a reduced, connected, Hausdorff complex analytic space.

584 M. NAMBA

Now, let W, V, Wi and (wi, zi) be as above. In order to prove
Theorem 1, we may take the parameter spaces of families as small as
we want. Thus, by the implicit mapping theorem, we may restrict our
attention to families {Vs}s•¸S of compact complex submanifolds of W such
that V0 = V for a point o •¸ S and such that, for each point s •¸ S, Vs is
defined in Wi by the equation:

wi= ƒÓi(zi, s)

where ƒÓi is a vector-valued holomorphic function of (zi, s) •¸ Ui x S.

2. Some lemmas. Let W, V, {Wi}i•¸I, (wi, zi) and F be as above. We
may assume that, for each i•¸I, Wi (the closure of Wi in W) is compact,
the local coordinate system (wi, zi) is extended to an open set Wi•½ Wi
and that Wi = {(wi, zi)•¸ Wi||wi|< 1 and |zi| < 1} where

and

We also assume that V is defined in Wi by the equation wi = 0. W e
put Ui= V •¿ Wi. Then

We may assume that, for each positive integer n and for each n-tuple
(i1,• • • , in) of indices, Ui1•¿... •¿ Uin and Ui1 •¿ ... •¿ Uin are connected and
Stein, unless they are empty.
Let Cp = Cp(V, F, {Ui}) be the (not necessarily skew symmetric) p-th
cochain group of F on the nerve of the covering {Ui}. We introduce a
norm || || in Cp. For each ƒÌ = {ƒÌi0...ip} •¸ C, we define ||ƒÌ|| by

||ƒÌ|| = sup {| ƒÌƒÉi0 ...1p(z)|:ƒÉ= 1, ..., r, z •¸ Ui
0 •¿ ... •¿ U1, (i0, ..., 2p)•¸Ip+1}

where ƒÌƒÉi0 ...ip is the representation of the component ƒÌi0 ...2 p of ƒÌ with respect
to the coordinate (wi0, zi0). We put

It is easy to see that Cp(|| ||) is a Banach space and the coboundary map o maps G (|| ||) continuously into Cp+l(|| ||). We put

ON MAXIMAL FAMILIES OF COMPACT COMPLEX 5$5

LEMMA 2.1. Hp and Hp(|| ||) are canonically isomorphic to Hp(V, F) (the p-th cohomology group of F).
PROOF. Hp is canonically isomorphic to Hp(V, F) by Leray's theorem. Since Zp(|| ||) is a subgroup of Zp and

we have the canonical injection:
Hp(|| ||)Hp.

Let Cp be the (not necessarily skew symmetric) p-th cochain group of F on the nerve of the covering {Ui}i•¸I. We put

Then it is canonically isomorphic to Hp. Since the restriction maps
map into into and into we have
a homomorphism:

HpHp(|| ||).
It is clear that the diagram

is commutative. Hence Hp(|| ||) = Hp. q.e.d. Let e be a small positive number such that the open sets

again cover V. We put Uei = Wei •¿ V = {(o, zI) e Wi||zi| < 1-e}.
Besides Cp, we must consider additive groups Cpe. An elementƒÌ =
{ƒÌi0..,ip} •¸ Cpe is a function which associates to each (p + 1)-ple (i0, • • •, ip)
of indices in I a holomorphic section ƒÌi0...ip on Uei0 •¿ ... •¿Ueip-1 •¿ Uip. In particular, C0e = C0. We define the coboundary map
ƒÂ e : Cpe Cp+1e
by

586 M. NAMBA

for

We introduce a norm J Je in Cpe. For each ƒÌ= {ƒÌi0. .,i p} E Cpe, we define J e
by

where ƒÌƒÉi0...i p is the representation of the component ƒÌi0...ip of e with respect
to the coordinate (wi0, zi0). In particular, we define

|ƒÌ|e=||ƒÌ|| for ƒÌ•¸C0e=C0.
We put

It is easy to see that Cpe((| |e) is a Banach space and the coboundary map
ƒÂ e maps Cpe(| |e) continuously into Cp+1e (| |e)• We put

LEMMA 2.2. There is a canonical identification: Z1e(| |e) = Z'(|| ||) and
the norms J Je and || || are equivalent in them.

PROOF. Each element ƒÌ = {ƒÌik} •¸ Z1(|| ||) corresponds to the element
ƒÌ '= {ƒÌ'
ik}•¸ Z1e (e) with ƒÌ'ik=ƒÌik | Uei •¿ Uk• It is clear that |ƒÌ'|e•…||ƒÌ||
Conversely, let ƒÌ' = {ƒÌ'ik}•¸ Z1e(||e). We take a point z •¸ Ui •¿ Uk. Since
{ Uz } is a covering of V, there is an index j such that z •¸ Uej;. We define
an element ƒÌik(z) of the fiber Fz of F by

(1)

We show that ƒÌik(z) does not depend on the choice of the index j. Let
us take another index 1 such that z•¸Ule. Since ƒÌ'•¸ Z1e(||e)'

ON MAXIMAL FAMILIES OF COMPACT COMPLEX 587

Thus ƒÌik(z) is a well defined holomorphic section of F on Ui •¿ Uk, we
set ƒÌ= {ƒÌik}. We express (1) in the coordinate in Ui:

where z = (0, zi) and zj= gji(0, zi). Then we get

where Hence

we show that e is a cocycle. Let us take a point z •¸ Ui •¿ Uj •¿ Uk.
There is an index l such that z •¸ Uel. Thus

Now, ƒÌ'•¸Z1e(| |e) corresponds to ƒÌ•¸Z1(||||). q.e.d.
The following lemma is a slight modification of Kuranishi's Proposi-
tion 2.5', [5].

LEMMA 2.3. There is a continuous linear map

such that ƒÂeE = the identity map on B2e(||e).

PROOF. First of all, we define additive groups Cpe (q, q'), p, q, q'
0, 1, 2, ... An element ƒÌ•¸ Cpe (q, q') is a function which associates to each
(p+1)-pie (i0, • • •,,ip) of indices a C•‡-differential (q, q')-form ƒÌi0...ip on
Uei0 •¿• • •,•¿ Ueip-1 •¿ Uip with coefficients in F. We define a norm |ƒÌ|e by

where is the coordinate expression in Ui0 of the component

We also define a map

by

It is clear that Cpe is a subgroup of Cpe (0, 0) and

588 M. NAMBA

ƒÂ

e: Cpe Cp+1e
is the restriction map of

ƒÂe: Cpe(0, 0) Cp+1(0, 0)

defined above.
Let {qi}i•¸I be a partition of unity subordinate to the covering { Uie}i •¸I .
Given ƒÌ•¸B2e(| |e) we define an element ƒÅ= {ƒÅil}•¸Cle (0, 0) by ƒÅjl=
1Ji•¸I qiƒÌiJL. Then

where cl is a constant. We claim that ƒÂeƒÅ=ƒÌ.

Let •ÝƒÅ={•ÝƒÅjk} Then •ÝƒÅ is an element of Cle(O, 1) and satisfies ƒÂe•ÝƒÅ= 0,
for •Ý(ƒÅjk) -•Ý (ƒÅjk) + •Ý(ƒÅjk) =•Ý(ƒÌilk) = 0. Let ƒÉj = Then

Since ƒÉj= a qi•ÝƒÅj, is) `= L ,k giƒÌ ki•Ýgk, we can find a constant

c2 such that

(1)
where •ÝƒÀ1= •Ý/•ÝzƒÀ1 etc. and p = 0, 1, ..., d. We now denote by K; the
Newlander-Nirenberg operator on Uj([8] or p. 186 [9]), and use its properties;
for a (vector valued) (0, 1)-form ƒÉj on Uj,

(N1) with c3 a constant ,

(N2)
From (1) and (N1) above, it follows that

with c4 a constant .

Now we get on Hence defines a global
form w with coefficients in
Let 0 < a < 1 be a constant and be the Kodaira-Nirenberg-
Spencer norm [3]. Then by estimating|ƒÖ|d+1 on Uej we have

with c5 a constant .

We introduce a Hermitian metric on V and let •Ý* and G be the ad joint
operator of a and the Green operator respectively. Let's ƒÌ'ijl be the restric-
tion of ƒÌijl on Uie •¿ Uej •¿ Uel . Since ƒÌ •¸ B2e (| |e),ƒÌ' = {ƒÌ'ijl} is a coboundary

ON MAXIMAL FAMILIES OF COMPACT COMPLEX 589

of F on the nerve of the covering {Uei}, ft is clear that w corresponds
to ƒÌ' by Dolbeault's isomorphism. Since ƒÌ' is a coboundary,
w = •Ý•Ý*GƒÖ .

We put ƒÎ=•Ý*GƒÖ. Then there is a constant c6 such that

Let us denote ƒÎi the restriction of ƒÎ on Ui. Then we have

where c7 is a constant, by (N1) above, we get

where c8 is a constant.

We put ƒÉ'i = ƒÉi - ƒÎi. Then we have

Hence we have

by (N2) above.
Now, we define ƒÀ= {ƒÀij} by ƒÀij=ƒÅij-kjƒÉ'j+kiƒÉ'i. Then it is an

element of Ce and there is a constant c such that

We define E:ƒÌ ƒÀ. We claim ƒÂƒÀ =ƒÌ.

Using the map E in Lemma 2.3, we define a map

by A = 1- EƒÂe. Then A is a projection map.
Since the proof of the following lemma is similar to (and simpler than)
that of Lemma 2.3, we omit it.

LEMMA 2.4. There is a continuous linear map

590 M. MAMBA

such that ƒÂeE0 = the identity on B1e (| |e). Finally, we prove the following

lemma.

LEMMA 2.5.

(1) There is a canonical identification: B1e(||||)= B1e(| |e)
(2) H1e (| |e) is canonically isomorphic to H1(V, F),
(3) ƒÂeC0e(||e) = B1e(| |e),
(4) B1e (| |e) is closed in Z1e (| |e).
PROOF. First of all, we show (1). Each element ƒÌ = {ƒÌik} e B1(||||)
corresponds to the element ƒÌ' = {ƒÌ'ik} E B1e (| |e) with ƒÌ'ik =ƒÌik | Uze •¿ Uk. It is
clear that |ƒÌ'|e •…||ƒÌ|| . Conversely, we take an element e B1e (| |e) .By
Lemma 2.4, E0ƒÌ is an element of C0(||) and each component of = ƒÂeE0ƒÌ
is the restriction of the corresponding component of a(E0ƒÌ) •¸ C1(||||). Since
U2 fl Uk is connected for each pair (i, k), the extension ƒÂ(E0ƒÌ) is uniquely
determined by . We associate ƒÂ(E0eƒÌ) toƒÌ . Thus we get (1). (2) follows
from (1), Lemma 2.1 and Lemma 2.2. ƒÂe(E0ƒÌ) = shows (3).
To prove (4), let {ƒÌ(n)} be a sequence in B1e(| |e) which converges to
•¸ Z1e(| |e)• We put ƒÅ(n) = E0 ƒÌ(n) •¸C0e(| |e), n = 1, 2, ... Then

where c and M are constants. Thus, for all point zi •¸ Uz,

where =
By Montel's theorem, there is a subsequence

n1,i2, ... •‡

such that ƒÅi(nv)(zi) converges absolutely and uniformly on each compact

subset of Uz for each i e I.

We put ƒÅi(zi) = limv ƒÅi(nv) (zi). Then ƒÅi is holomorphic on Uz. We put
= {rci} and regard ? as an element of C0. For each fixed zi •¸Ui, we
have

Thus |ƒÅ|e•… M so that ƒÅ•¸ C0e (| |e). Now, for each fixed zi •¸ Uei •¿ Uk, we

have

where

Letting v •‡,we have

ON MAXIMAL FAMILIES OF COMPACT COMPLEX 591

Hence ƒÂeƒÅ = ƒÌ. q.e.d.

It is well known that H1(V,F) is of finite dimensional. Hence, by
(2) and (4) of Lemma 2.5, there is a subspace He(| |e)* of Z1e(| |e) isomorphic to H1(V,F) such that Z1e(| |e) splits into the direct sum of B1e(| |e) and H1e (| |e)

Let

and

be the projection maps corresponding to the splitting.

3. Proof of Theorem 1. Let W, V, {Wi}i•¸I , {WI}i•¸i, {Wei}i•¸I,(wi, zi)
and F be as above. We assume that a compact complex submanifold V'
covered by { Wi}i•¸I is defined in Wi by the equation:

Then, for such V', we associate an element

ƒÓ must satisfy the compatibility conditions:

Conversely, an element ƒÓ = {ƒÓI} •¸ C0(||||) which satisfies || ƒÓ || < 1 and the
above compatibiliy conditions defines a complex submanifold VƒÓ of W
by the equation:

We show that there is a small number } 0 such that VƒÓ is compact if
||ƒÓ|| < ƒÃ. For this purpose, we need the following lemma. The proof
will be given at the end of this section.

LEMMA 3.1. There is a small positive number such that if || ƒÓ || <
and if ƒÓ defines a submanif old VƒÓ, then VƒÓ is covered by {WEK}k•¸I.

Now we show that VƒÓ is compact if ||ƒÓ || <ƒÃ where satisfies Lemma
3.1. Let {Pv}y=1,2,... be an arbitrary sequence of points of VƒÓ. By Lemma
3.1, {Pv}v_1,2,... •¼U Wei . We want to choose a subsequence of {Py}v=1,2,...
* we use the same notation for the convenience.

592 M. NAMBA

converging to point of VƒÓ. Since the number #(I) of indices is finite, we
may assume that Pv belongs to a fixed Wei for all v. We write Pv =
(wvi, zvi) in the local coordinate (wi, zi). Then

For each Pv, we associate a point Qv in V defined by

Since V is compact, we may assume that {Qv}v=1,2,... itself converges to a point

Now, we put

Then P •¸ VƒÓ and

Hence {Pv}v=1,2, .. converges to P. This shows that VƒÓ is compact.
Now, we need the following two lemmas. The proofs will be given
at the end of this section.

LEMMA 3.2. There is a small positive number such that if |wk| < ƒÃ,
then (wk, zk) •¸ Wi •¿ Wk for all zk •¸ Uei •¿ Uk.

LEMMA 3.3. Let e' be a small positive number greater than e such
that the open sets

again cover V. Then there is a small positive numbers such that if
| wk| < ƒÃ and if (wk, zk) e We'i •¿ We'i, then zk •¸ Uei •¿ Uk.

Now, let B(ƒÃ) be the open s-ball of C0(||||) = C0e(| |e) with the center
0, where satisfies Lemmas 3.1, 3.2 and 3.3. We define a map

by

where zk = gki(0, zi). Since (ƒÓk(zk), zk) •¸ Wi •¿ Wk by Lemma 3.2, K maps
B(s) into C1e. It is clear that |KƒÓ|e < 1 +ƒÃ so that K maps B(ƒÃ) into
C1e (| |e)•
We assume that c e B(E) satisfies KƒÃ = 0. If zk •¸ Uk satisfies

ON MAXIMAL FAMILIES OF COMPACT COMPLEX 593

then zk •¸ Uei •¿ Uk by Lemma 3.3 so that

Thus the equations: wi= ƒÓi(zi) define a compact complex submanifold VƒÓ.
Conversely, we assume that ƒÓ •¸ B(ƒÃ) defines a compact complex sub-
manifold VƒÓ defined by the equations: wi = ƒÓi(zi), then

for

Hence KƒÓ = 0 by Lemma 3.2.
Thus the problem is reduced to analyze the set

LEMMA 3.4. There is a small positive number ƒÃ' <ƒÃ such that

is an analytic map and K'(0) = ƒÂe.

PROOF. We want to show that there is a small positive number
ƒÃ' <ƒÃ such that for any affine line L in C0(||||), K is analytic map of
L •¿ B(ƒÃ') into C1e(| |e). This implies that K: B(ƒÃ') Ce(| |e} is analytic.
(See e.g., Proposition 2, [1]).
We take a point ƒÓ0 •¸ L •¿ B(ƒÃ'). Then L can be written as

where s e C and ƒÓ1 •¸ C0(||||). We may assume that ƒÓ1 •¸ B(ƒÃ') and L(s) •¸ B(ƒÃ')
for all s •¸ ‡™ where 4 is the unit disc in C. Now, putting zk = gkz(0, zi),
we have

we put

we show that B{s} is an analytic map of 4 into C1e (||e). Similar argu-ments show that A(s) and C(s) are analytic. We put

594 M. MAMBA

We also put wk = ƒÓ0k(zk), and x = sƒÓ1k(zk). Then

If |wk + x |< ƒÃ, then |gik(wk + x, zk)| < 1 for all zk •¸ UeI •¿ Uk by Lemma
3.2. Let ƒÃ' be a positive number smaller than . Then by Cauchy's
estimate,

for |wk| < ƒÃ'

where is extended over all non-negative integers v1,...,vr with
vi + ... + vr •† 1 and « means that the absolute values of the coefficients
of y in the formal power series in x1,• • •., xr are less than the absolute
values of the corresponding coefficients of D(x). Hence

E(s) converges for s •¸‡™ and is equal to

provided ƒÃ' < ƒÃ/2.
Taking ƒÃ' sufficiently small, we may assume that

for all s •¸‡™.
Thus,

(1) for all s •¸‡™.

Next, if |wk| < then by Cauchy's estimate,

Thus, if IwkI < ƒÃ', then

Taking ƒÃ' sufficiently small, we may assume that

thus

(2)
Now,

where

ON MAXIMAL FAMILIES OF COMPACT COMPLEX 595

(2) shows that z'i < 1 - e/2 when ||ƒÓ0|| <ƒÃ'. Thus, again by
Cauchy's estimate, we have

Thus

Hence

F() converges for |y| < e/2 so that F (E (s)) converges for all s e 4 by
(1). This shows that B (s) is analytic.
Similar arguments show that A(s) and C (s) are analytic provided ƒÃ' is
sufficiently small. Since ƒÓ0 is an arbitrary point of L •¿ B(ƒÃ'), K is analytic
on L •¿ B(ƒÃ').
Finally, we show that K'(0) = ƒÂe. Since K(0) = 0, KƒÓ - K0 = KƒÓ.

Now

where zk = gki(0, zi) and o(ƒÓ) is some function of ƒÓ (and of zk) such that

as

There is a constant M1 such that

for

On the other hand, there is a constant M2 such that

Thus,

Hence

so that

q.e.d.
Let ƒÃ' be a small positive number such that Lemma 3.4 holds. We

596 M. NAMBA

define an analytic map L: B(ƒÃ') C0(|| ||) by

where E0, B and A are continuous linear maps defined in •˜ 2. Then we
get

By the inverse mapping theorem, L is an analytic isomorphism of an open neighbourhood ƒ¶ of 0 in B(ƒÃ') onto an open neighbourhood ƒ¶' of 0
in C0(|| ||). We put

LEMMA 3.5.

PROOF. If
We put and put

q. e. d.

Then it is clear that S = L(M) and ƒÓ(S) = M. Let s •¸ H0(V, F) •¿ ƒ¶'.
We put ƒÓ = ƒÓs. Then

Let H be the projection map defined at the end of •˜ 2. Then

In other words, KƒÓ(s) has no coboundary part.

LEMMA 3.6. Taking ƒ¶' snfficiently small, we have

PROOF. Let e' be a small positive number greater than e such that
the open sets
cover V.

We put Ue'i = We'i •¿ V. We introduce a norm I in Ce as follows: for
each ƒÌ= {ƒÌijk} •¸ C2e, we define e |ƒÌ|e' by

where ƒÌƒÉijk is the representation of the component ƒÌijk of ƒÌ with respect
to the coordinate (wi, zi). Then it is clear that

ON MAXIMAL FAMILIES OF COMPACT COMPLEX 597

(1)

We show that there is a constant c0> 0 such that

(2)

Let zi•¸ e Uei •¿ Uej •¿ Uk. Since { Ue'i} covers V, there is an index 1 such that
zi •¸ Ue'i. Since ƒÌ•¸Z2e (| |e), we have

where zl = gli(0,zi). Thus

where || F|| = sup {| FƒÉikv (zk)| : ƒÉ, v =1, ..., r, i, k •¸ I, zk •¸ Ui •¿ Uk}.
Now, let zi •¸ Uei •¿ Uej •¿ Uk. Taking ||ƒÓ|| sufficiently small, we may
assume that (ƒÓk(zk), zk)•¸ We/2i •¿ We/2j •¿ Wk where zk = gki(0, zi) and

This follows from Lemma 3.2 by replacing Wz to W1/2i (and Wj to We/2j).
We putj = gjk(ƒÓk(zk), zk). Then

Again, taking sufficiently small, we may assume that

where

We put

R'(KƒÓ,ƒÓ) is an element of C2e(| |e). Then it is easy to see that there is

a constant cl such that

(3)

provided |KƒÓ|e and ||ƒÓ|| are sufficiently small.
This follows from the mean value theorem applied on the real and
the imaginary parts of the functions f a (wj, zj), ƒÉ = 1, ..., r.
In a similar way, if we put

598 M. NAMBA

then, we can show that there is a constant c2 such that

(4)
Now, we put

we assume that zi belongs to Ue'i •¿ Uej •¿ Uk. Then, taking ||ƒÓ|| sufficiently
small, we may assume that Lj = gjk(ƒÓk(zk), zk) •¸ Uei ƒÓ U 1/2j. Thus

where
Applying the mean value theorem on the real and the imaginary parts
of the functions (KƒÓ)ƒÉij(zi), ƒÉ= 1, ..., r, we have

with c3 constant. Hence

(5)

Now, we assume that zi belongs to Uei •¿ Uej •¿ Uk. Then

Hence we have

By (1), (3), (4) and (5), we have

with c4 a constant. Thus, by (2),

Thus there is a constant c such that

ON MAXIMAL FAMILIES OF COMPACT COMPLEX 599

Now, let s •¸ H0(V, F) •¿ ƒ¶'. Taking ƒ¶' sufficiently small, we may as-
sume that

Now, We assume that Then

If a contradiction. Hence

q.e.d.
Now, for each s •¸S, ƒÓ(s) = {ƒÓi(zi, s)} defines a compact complex sub-
manifold Vs and ƒÓi(zi, s) is a vector valued holomorphic function of

This is easily seen, because for each fixed zi •¸ Ui,

is a continuous linear map, so that

is an analytic map. Thus { Vs}s•¸S forms a family (X, ƒÎ, S) of compact
complex submanifolds of W.
We show that (X, ƒÎ, S) is a maximal family. Let s0 •¸ S. Let (Y, ƒÊ, T)
be a family of compact complex submanifolds of W with a point p e T
such that ƒÊ-1(p) = Vs0. Let w'i = wi - ƒÓi(zi, s0). Then (shrinking T if
necessary) we may assume that there are vector valued holomorphic
functions ƒÆ'i(zi, t) on Ui x T such that ƒÆ'i(zi, p) = 0 and that the equation
w'i = ƒÆ'i(zi, t) defines the submanifold e ƒÊ-1(t). We put

and

Then it is easy to see that

is an analytic map. We may assume that ƒÆ(T) •¼ ƒ¶. We have K(ƒÆ(t)) = 0.
Let f (t) = L(ƒÆ(t)). Then f is a holomorphic map of T into S with
f (p) = s0. We have ƒÆ(f(t)) = a(t). Hence we get ƒÊ-1(t) = ƒÎ-1(f (t}}.
This completes the proof of Theorem 1.

REMARK 3.1. It is clear that the map f, with the property: ƒÊ-1(t) =
ƒÎ-1 (f(t)) for t •¸T, is uniquely determined.

600 M. NAMBA

REMARK 3.2. If H1(V,F) = 0, then S = H0(V,F) •¿ ƒ¶' by Lemma 3.6.
This is Kodaira's case (see the Introduction).

PROOF OF LEMMA 3.1. Let ƒÎk: WK,Uk be the projection map defined
by ƒÎk(wk, zk) = zk. For each positive integer n, we set

and

(the closure in W) .

Since Ak(n) is compact,

is also compact. It is clear that A(n) contains V. We show that

Let b •¸ •¿•‡ n=1, A(n). Then there are an index k •¸ I and a subsequence

such that for

Then

Hence wk(b) = 0 so that b •¸ V. If we find an integer n such that

then ƒÃ=1/n satisfies Lemma 3.1. Thus the proof of Lemma 3.1 reduces
to the following lemma. The proof is straightforward.

LEMMA 3.7. Let A be a compact subset of a Hausdorf space X. Let
A(n) be compact subsets of X such that

(1)

(2)

Let U be an open neighbourhood of A. Then there exists an integer n
such that U•½A(n).

PROOF OF LEMMA 3.2. Let Wk(n) be as in the proof of Lemma 3.1.
We put

Here, the closure is taken in W.

ON MAXIMAL FAMILIES OF COMPACT COMPLEX 601

First of all, we show that A(n) is a closed subset of W. Let {bV}v=i,2,...
be a sequence of points of A(n) converging to a point b of W. b belongs
to Wk(n). Since ƒÎk(bV) •¸ Uei •¿ Uk converges to ƒÎk(b), b also belongs to
ƒÎ-1k (Uei •¿ Uk). This shows that A(n) is closed. Since A(n) •¼ Wk(n) and
the later is compact, A(n) is also compact. We put A = Uei •¿Uk. In
order to apply lemma 3.7 to X = W, U = U, it is enough to see that

and

Thus wk(b) = 0 so that b •¸ Uk. Since
Hence b •¸ A.
Thus there is an integer n such that A(n) •¼ Wi. Hence

On the other hand, is contained in Wk. Hence

Now, ƒÃ=1/n satisfies the requirement. q.e.d.

PROOF OF LEMMA 3.3. Let Wk(n) be as above. In order to prove the lemma, it is enough to find an integer n such that

We put

and

A(n) and A are compact.
We claim that
Hence

Then
implies that
On the other hand,

Hence b •¸ Ue'i •¿ Ue'k = A. Now, we apply Lemma 3.7 to the case X = W,
U = ƒÎk-1(Uei •¿ Uk). Thus there is an integer n such that

Hence

602 M. NAMBA

q. e. d.

4. Proof of Theorem 2. Let W be a complex manifold. Let S(W)
be the set of all compact complex submanifolds of W. Let V be an
element of S(W). Let (X, ƒÎ, S) be the maximal family with the center
V constructed in •˜ 3. Two different points s t in S have different
fibers ƒÎ-1(s) •‚ƒÎ-1(t). Thus, there is a unique injective map

S S(W)

defined by s ƒÎ-1(s). We want to take this map as a local chart around
V •¸ S(W). Using the maximality of (X, ƒÎ, S) and Remark 3.1, these local
charts patch up to give a (locally finite dimensional, not necessarily con-
nected) analytic space structure on S(W).
We prove that the underlying topological space of S(W) is a :Haus-
dorff space. For this purpose we need the following two lemmas.

LEMMA 4.1. Let W be a metric space with metric d. Let C(W) be
the set of all compact subsets of W. For any two elements A and B in
C (W), we associate a number d'(A, B) defined y
d'(A, B) = sup {d(x, B) x •¸A} + sup {d(A, y) y •¸ B}. Then d' is a
metric on C (W).

PROOF. It is easy to check that d' satisfies the three axioms for
metric. q.e.d.

LEMMA 4.2. Let (X, ƒÎ, S) be a family of compact complex submani-
folds of W. With an Hermitian metric on W, we regard W as a metric
space (W, d). Let o •¸ S. Then d' (ƒÎ-1(s), ƒÎ-1(o)) is a continuous function
of s •¸ S, where d' is the metric in C (W) introduced in Lemma 4.1.

PROOF. It suffices to prove that

It is known [7] that there is an open neighbourhood S' of o in S and a continuous retraction

such that R| ƒÎ-1(s) is a C•‡-diffeomorphism of ƒÎ-1(s) onto ƒÎ-1(o) for each
s •¸ S'. We fix on point s •¸ S'. We take a point P •¸ƒÎ-1(s). Then

Hence

ON MAXIMAL FAMILIES OF COMPACT COMPLEX 003

The right hand side is finite, for R is continuous and ƒÎ-1(s) is compact.
In a similar way, we get

Hence

We show that

We assume the converse. Then there are a positive number ƒÃ > 0, a
sequence {sn}n=1,2,... of points of S' converging to o and a sequence {Pn}n=1,2,..
of points of ƒÎ-1(S') with Pn •¸ ƒÎ-1(s), n = 1, 2, ... such that

Since each fiber ƒÎ-1(s) is compact, there is a subsequence n1 < n2 < ...
such that converges to a point P •¸ ƒÎ-1(o). Then

a contradiction. q.e.d.

Now, it is easy to prove that S(W) is a Hausdorif space. Using an Hermitian metric on W, we regard W as a metric space (W, d). Let d' be the metric on the set C(W) of all compact subsets of W defined in Lemma 4.1. Lemma 4.2 asserts that the identity map

I: S(W) (S(W), d'}

is a continuos map where (S(W), d') is a metric space with the metric d'. Since (S(W), d') is a Hausdor space, S(W) is also a Hau sdorff space. This completes the proof of Theorem 2.

Henceforth, for each point t e S(W), we denote Vt the corresponding compact complex submanifold in W. Let

We first show that X(W) is closed in W x S(W). Let (Pn, tn), n =1, 2,...
be a sequence in X(W) which converges to a point (P, t) •¸ Wx 5(W).
We claim that P •¸ Vt.

by Lemma 4.2.

Hence d(P, Vt) = 0 so that P •¸ Vt.

604 M. MAMBA

Let (F, t) •¸ X (W). There is a maximal family (X, ƒÎ, S) where S is an
open neighbourhood of t in S(W) such that

X is a subvariety of W x S containing (F, t). Since

X is an open subset of X (W) containing (F, t). Hence X (W) is a sub-
variety of W x S(W). Let ƒÎ: W x S(W) S(W) be the projection map.
Then (X, ƒÎ, S) is the restriction of (X(W),ƒÎ , S(W)) to the open subset
S of S(W). Thus we conclude that (X(W),ƒÎ , S(W)) is a maximal family
of compact complex submanif olds of W.
It is clear that the family (X (W), ƒÎ, S(W)) has the following uni-
versal property:
For any family (X, ƒÎ, S) of compact complex submanifolds of W,
there is a unique holomorphic map

f: SS(W)

such that ƒÎ-1(s) = ƒÎ-1(f (s)) for all s •¸ S.
In the rest of this section, we show that S (W) is identified with an
open subset of the Douady space D = D(W). Douady [1] constructed a
fiat family (Z, ƒÅ, D) of all (not necessarily reduced) compact complex
subvarieties of W. We consider the reduced analytic space Dred associated
to D. Let

By Theorem 3.1 and Corollary 3.3 in VI, [2], we see that S'(W) is
an open subset of Dred and that the triple (ƒÅ-1red (S'(W)),ƒÅred, S'(W)) forms
a family of compact complex submanifods of W in our sense where red:
Zrec Dred is the holomorphic map associated to ƒÅ.
We note that ƒÅ-1red(t) = ƒÅ-1(t)red = ƒÅ-1(t) for t •¸S'(W). Let (X(W),ƒÎ ,
S(W)) be the family which we constructed above. There is a unique
holomorphic map

such that ƒÅ-1red(t) =ƒÎ-1(f(t)) for all t •¸ S'(W). On the other hand, by the
universal property of the family (Z, ƒÅ, D), there is a unique morphism

g: S(W) D

such that ƒÎ-1(s) = ƒÅ-1(g(s)) for all s •¸ S(W).
Since S(W) is reduced, there is a unique holomorphic map

ON MAXIMAL FAMILIES OF COMPACT COMPLEX 605

g0. S(W) Dred

such that the diagram

is commutative, where Dred D is the canonical morphism. Now it is
clear that g0(S(W)) = S'(W) and that

g0f = IS'(w) ,
fg0 = IS(w)

where IS' (w) and Is(W) are the identity maps of A'(W) and S(W) respectively.
Hence f is a holomorphic isomorphism. Thus we conclude that 2(W) is
identified with an open subspace of Dred•

5. Proof of Theorem 3. Let S(W) be the analytic space constructed
in •˜4. We put

We first show that A is closed in S(W) x S(W). Let (sn, tn), n =1, 2,
be a sequence of points in A converging to a point (s, t) •¸ 2(W) x 2(W).
We assume that Vs •¼ Vt. Then there is a point P •¸ Vs - Vt. We put
ƒÃ = d(P, Vt) > 0 where d is an hermitian metric on W. Let d' be the
metric on the set of all compact subsets in W defined in Lemma 4.1.
By Lemma 4.2, we can choose ii so large that

(1) d' (Vs, Vsn) < ƒÃ/ 2

(2) d'(Vt, Vtn) < ƒÃ/2.

By (1), there is a point Pn •¸ V,, such that d (P, Pn) < ƒÃ/2. We note that
Pn •¸ Vsn•¸ Vtn by the assumption. Hence

This contradicts to (2). Thus Vs •¼ Vt. Hence A is closed in 2(W) x S(W).

Now, let X and V be two compact complex submanifolds in W such
that V •¼ X. Let {Wi}i•¸I and { Wi}i •¸I be finite coverings of V by open

606 M. NAMBA

subsets Wi and Wi in W such that Wi•½ Wi (the closure in W) for all
i •¸ I. Moreover, we assume that there is on each Wi a local coordinate
system

such that

and such that X and V are defined in W by the equations

X:ui=O,
V:ui=wi=0.

We assume that { Wi •¿ V } and { Wi •¿ V } satisfy the similar conditions to
those in •˜ 2.

Let {WƒÁ}ƒÁ •¸ „C and{WƒÁ}ƒÁ•¸ „C be finite collections of open subsets of W
having the following properties:
(a) WƒÁ•½ WƒÁ (the closure in W) for all ƒÁ•¸ „C,
(b) each WƒÁ does not intersect with V,
(c) { Wi}i •¸I •¾{WƒÁ}ƒÁ•¸ „C is a covering of X,
(d) there is on each WƒÁ a local coordinate system

such that

and such that X is defined in WƒÁ by the equation: uƒÁ = 0.

N

(e) { Wi•¿ X }•¾ { WƒÁ •¿ X } and { Wi •¿ X } •¾{WƒÁ •¿ X } satisfy the similar
conditions to those in •˜ 2.

Let F be the normal bundle of X in W and let G be the restriction
of F on V. Let H and N be the normal bundles of V in W and in X
respectively. Then we have an exact sequence

(3) 0NH G 0.

Let C 0(V, H) = C0(V, H, { Wi •¿ V }i •¸ I), etc., be the 0-th cochain groups
of the sheaf H, etc., over V of germs of holomorphic sections of H, etc.,
on the nerve of the covering { Wi •¿ V }i •¸ I. Then, by (3), we have a
canonical isomorphism

(4)

Let

ON MAXIMAL FAMILIES OF COMPACT COMPLEX 607

be the projection maps with respect to (4). Let H H be norms in C0(V, H),
etc., defined as in § 2. Let C0(V, H,||||), etc., be the Banach spaces of
elements in C0(V, H), etc., with finite norms. Then it is clear that a
and ,3 are continuous linear maps of C °(V, H,||||) into C 0(V, N, ||||) and
C0(V, G,||||) respectively.
We put Y = X •¿ (Ui•¸I WI). Then Y is an open subset of X contain
ing V. Let C 0(X, F) (resp. C0(Y, F|Y)) be the 0-th cochain group of

the sheaf F over X (resp. F | Y over Y) of germs of holomorphic sections
of F (resp. F | Y) on the nerve of the covering { Wi •¿ X}i•¸ I •¾ { WƒÁ•¿X}ƒÁ•¸„C
(resp. { Wi X}1)i•¸I). Then we have a canonical linear map

We introduce a norm H H in C0(X, F) (resp. C0(Y, F | Y)) and define a
Banach space C 0(X, F,||||) (resp. C 0(Y, F|Y, ||||)) as in •˜ 2. Then l
maps C0 (X, F,||||) continuously into C0(Y, F | Y, ||||).
Now, let X' and V' be compact complex submanifolds of W near
from X and V respectively, defined in Wi by the equations:

Then V' •¼ X' if and only if

for

We assume that X' is defined in WƒÁ, by the equation:
uƒÁ =ƒÉƒÁ(vƒÁ} .

We may consider

Then l (ƒÉ) = {ƒÉi}i•¸I •¸ C0(Y, F|Y, ||||). We may also consider

Now, we define a map

by

608 M. NAMBA

where B(1) is the 1-ball of C 0(V, H,||||) with the center 0. Then it is
easy to see that Q is analytic in an open neighbourhood ƒ¶1 x ƒ¶2 of (0, 0).
On the other hand, (taking ƒ¶1 and ƒ¶2 sufficiently small), i) and X satisfy
the equations:

K1ƒÅ=0 and K2ƒÉ = 0
where

are analytic maps defined as in •˜3. (C1e(V, H,| |e) and C1e(X, F, | |e) are
defined as in •˜2.) By the arguments in •˜3, we know that the sets

are (finite dimensional) analytic spaces which are taken as local charts of S(W) in neighbourhoods of V and X respectively. Now,

is a finite dimensional subvariety of ƒ¶1 x ƒ¶2.
It is clear that, using A(V, X) as a local chart of the set A in a
neighbourhood of the point (V, X), we can give an analytic space structure
in A. It is also clear that the analytic space A thus defined is a closed
subvariety of S(W) x S(W). This proves Theorem 3.
Let o be a point of S(W). Then

This proves the corollary of Theorem 3.

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MATHEMATICAL INSTITUTE
TOHOKU UNIVERSITY
SENDAI, JAPAN