PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 2, May 1971

ON MAXIMAL GROUPS OF ISOMETRIES

LUDVIK JANOS

ABSTRACT. The purpose of this note is to introduce the concept of "Optimal Metrization" for metrizable topological spaces. Let X be such a space, ρ a metric on X and $K(\rho)$ the group of all those homeomorphisms of X onto itself which preserve ρ. The metric ρ is said to be "optimal" provided there is no ρ^* with $K(\rho^*)$ properly containing $K(\rho)$. A space having at least one optimal metric is called "optimally metrizable." Examples of spaces which are and which are not optimally metrizable are given; it is shown that the real line R is, and that the usual metric is optimal.

1. **Introduction and notation.** Let X be a metrizable topological space. We denote by $G(X)$ the group of all homeomorphisms of X onto itself and by $M(X)$ the set of all metrics on X compatible with the topology of X. We observe that with each $\rho \in M(X)$ there is associated the subgroup $K(\rho) \subseteq G(X)$ (group of all isometries for ρ) defined by $K(\rho) = \{h \mid h \in G(X)$ and $\rho(x, y) = \rho(h(x), h(y))$ for all $x, y \in X\}$. The basic idea motivating our investigations is the classification of a metric $\rho \in M(X)$ according to the size of the corresponding group $K(\rho)$.

CONVENTION. In this paper the set-theoretical inclusion is denoted by $\supseteq$, reserving $\supset$ for the *proper* inclusion.

DEFINITION 1.1. A metric $\rho \in M(X)$ is said to be *optimal* iff there is no $\rho^* \in M(X)$ with $K(\rho^*) \supset K(\rho)$. A space X is said to be *optimally metrizable* iff there is at least one optimal metric in $M(X)$.

Denoting by $L(X)$ the lattice of all subgroups of $G(X)$ we have the mapping $K: M(X) \to L(X)$ defined by $K(\rho) \in L(X)$ for $\rho \in M(X)$.

DEFINITION 1.2. The image of $M(X)$ under K is the subset $P(X) \subseteq L(X)$ partially ordered by inclusion. Its elements are groups of isometries and its maximal element (if it exists) is called a *maximal group of isometry*.

It is obvious that X is optimally metrizable if and only if $P(X)$ has a maximal element.

If $A \in L(X)$ and $h \in G(X)$ we denote by (A, h) the subgroup generated by A and h.

REMARK 1.1. It is obvious that a space X is not optimally metrizable iff to each $A \in P(X)$ there exists $h \in G(X)$ such that $h \notin A$ and $(A, h) \in P(X)$.

Received by the editors May 3, 1970.
AMS 1969 *subject classifications*. Primary 5480; Secondary 2240.
Key words and phrases. Group of isometries, optimal metric, optimally metrizable.

2. **General properties of the set** $P(X)$. We observe that if $\rho \in M(X)$ then for each $g \in G(X)$ the function $g\rho$ defined by $g\rho(x, y) = \rho(g(x), g(y))$ for all $x, y \in X$ is again a metric $\in M(X)$; thus $G(X)$ acts on $M(X)$ in this natural way.

THEOREM 2.1. *For each* $\rho \in M(X)$ *and* $g \in G(X)$ *we have* $K(g\rho) = g^{-1}K(\rho)g$. *Thus* $P(X)$ *contains with each* $A \in P(X)$ *all its conjugates* $g^{-1}Ag \in P(X)$.

PROOF. By straightforward verification.

COROLLARY. *If* $\rho \in M(X)$ *is optimal then* $g\rho$ *is also optimal for every* $g \in G(X)$.

In the case X is compact we topologize $G(X)$ by the uniform convergence topology and it is a well-known fact (see for example [**1**] and [**2**]) that any compact subgroup K of $G(X)$ lies inside $K(\rho)$ for some ρ. Hence we obtain this obvious statement.

THEOREM 2.2. *If* X *is compact then* X *is optimally metrizable iff* $G(X)$ *has a maximal compact subgroup.*

3. **Optimal metrization property of some well-known spaces.** We first observe that any set X with the discrete topology is optimally metrizable and the optimal metric ρ is the most trivial one defined by $\rho(x, y) = 1$ for $x \neq y$. In this case we have $K(\rho) = G(X)$ and $P(X) = L(X)$. On the other hand we now show that the one-point compactification N^* of the set of positive integers N has *not* this property. Under N^* we understand the set $N \cup \{\infty\}$ metrized for example by: $\rho(n, m) = |1/n - 1/m|$ for $n, m \in N$ and $\rho(n, \infty) = 1/n$ for $n \in N$.

THEOREM 3.1. *The space* N^* *is not optimally metrizable.*

PROOF. Suppose that K were a maximal compact subgroup of $G(N^*)$. Any orbit $K(n) = \{g(n) | g \in K\}$ is compact; so if it were infinite, then it would include ∞. But no member of $G(N^*)$ moves ∞, so $K(n)$ is finite. Thus there are $n, m \in N$ with disjoint orbits $K(n)$, $K(m)$. Taking for h the simple transposition of m and n, we observe that the action of (K, h) differs from that of K only on the finite set $K(n) \cup K(m)$, and (K, h) is therefore again compact, which is impossible. Hence, N^* is not optimally metrizable.

THEOREM 3.2. *The compact interval* $[a, b]$ *is optimally metrizable and the usual metric* $|y - x|$ *is optimal.*

PROOF. We prove this showing that the group $K = \{e, r\}$ consisting of the identity e and the reflexion r ($r(x) = a + b - x$ for $x \in [a, b]$) is maximal compact in $G([a, b])$. If this were not the case then there would exist a larger compact group $K' \supset K$. If $h \in K' \setminus K$ we may assume h is increasing since $r \cdot h$ also belongs to $K' \setminus K$. Since $h \neq e$ there is $p \in [a, b]$ such that $h(p) \neq p$, and we know that there is an interval $[c, d] \subseteq [a, b]$ with p in the interior and the only points fixed by h are c and d. Now $\{h^n \mid n \geqq 1\}$ has a limit element g in K', so $g(c)$, $g(p)$, $g(d)$ are limit points of $\{h^n(c)\}$, $\{h^n(p)\}$, $\{h^n(d)\}$ respectively. Thus $g(c) = c$, $g(d) = d$ and $g(p) = c$ or d which is impossible. Hence $K' = K = \{e, r\} = K(\rho)$ where $\rho(x, y) = |y - x|$ showing that this metric is optimal.

THEOREM 3.3. *The circle S_1 is optimally metrizable.*

PROOF. Representing the circle S_1 in the form $S_1 = \{e^{i\pi x} \mid x \in [-1, 1]\}$ we shall prove that the group $G \subseteq G(S_1)$ consisting of all rotations $e^{i\pi x} \to e^{i\pi(x+a)}$ and the reflexion $e^{i\pi x} \to e^{-i\pi x}$ is a maximal compact subgroup. To this end we observe that the set $G(S_1; -1) \subseteq G(S_1)$ of all those elements of $G(S_1)$ which leave invariant the point $-1 = e^{-i\pi}$ is a *closed* subgroup of $G(S_1)$ which is homeomorphic and isomorphic to $G[-1, 1]$. If G^* were a compact group containing G, then according to Theorem 3.2 $G^* \cap G(S_1; -1) = G \cap G(S_1; -1)$. Now if g belongs to G^*, then we can find a rotation f such that $f(e^{-i\pi}) = g(e^{-i\pi})$. Thus $h = f^{-1}g$ belongs to $G^* \cap G(S_1; -1)$, h belongs to G, $g = fh$ belong to G and $G^* = G$ which proves our assertion.

THEOREM 3.4. *The real line R is optimally metrizable and the usual metric $|y - x|$ is optimal.*

PROOF. The metric $|y - x|$ is preserved by the group of all translations and reflexions. Denoting this group by K, assume that there is $\rho \in M(R)$ with $K(\rho) \supset K$. Let $f \in K(\rho) \setminus K$. Without loss of generality we may assume f increasing and having at least one fixed point since otherwise we would apply on f suitable operations in K. Let F be the set of all fixed points of f. From $F \neq \emptyset$ we know that $R \setminus F$ has a connected component C that is an interval with at least one endpoint a. If we choose b in C, then $\{f^n(b)\}$ or $\{f^{-n}(b)\}$ will approach a, but $\rho(f^{n+1}(b), f^n(b)) = \rho(f(b), b) = \rho(f^{-n-1}(b), f^{-n}(b))$ will not approach 0, which is impossible. Thus K is the maximal group of isometry corresponding to the metric $|y - x|$ which completes our proof.

REFERENCES

1. S. Eilenberg, *Sur les groupes compacts d'homéomorphies*, Fund. Math. **28** (1937), 75–80.

2. M. T. Wechsler, *Homeomorphism groups of certain topological spaces*, Ann. of Math. (2) **62** (1955), 360–373. MR **17**, 287.

UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32601