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ON MEANS OF SUBHARMONIC FUNCTIONS

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1. INTRODUCTION

Let P, Q denote points in R^n $(n \geqq 2)$, let PQ be the euclidean distance of P from Q, and write

$$B(P, a) = \{Q \in R^n : PQ < a\} \quad (0 < a \leqq +\infty),$$

$$S(P, a) = \{Q \in R^n : PQ = a\} \quad (0 < a < +\infty).$$

Let O denote the origin of axes in R^n. For brevity, we put $B(a) = B(0, a)$, $S(a) = S(0, a)$. We denote the element of Lebesgue surface area on a sphere by $d\sigma$ and the element of Lebesgue volume by dv. For a function f, defined in $B(a)$, and integrable over every $S(r)$ for $0 < r < a$, the spherical mean $\mathcal{M}(f, \cdot) : (0, a) \to R$ is given by

$$\mathcal{M}(r, f) = \frac{1}{s_n r^{n-1}} \int_{S(r)} f \, d\sigma,$$

where s_n denotes the surface area of $S(1)$. If further f is locally integrable in $B(a)$ the volume mean $\mathcal{A}(f, \cdot) : (0, a) \to R$ is given by

$$\mathcal{A}(f, r) = \frac{1}{v_n r^n} \int_{B(r)} f \, dv,$$

where v_n denotes the volume of $B(1)$. Provided that $\mathcal{M}(f, \cdot)$ is Cauchy-Riemann integrable on every subinterval $(0, r]$ of $(0, a)$, the two means are related by the equation

$$\mathcal{A}(f, r) = \frac{n}{r^n} \int_0^r t^{n-1} \mathcal{M}(f, t) \, dt. \tag{1}$$

When f is subharmonic in $B(a)$, certain properties of the means $\mathcal{M}(f, r)$ and $\mathcal{A}(f, r)$ are well-known. For example, both means are continuous, increasing*)

*) The terms 'increasing' and 'decreasing' are used in the wide sense.

functions of r, and convex functions of $\log r$ (when $n = 2$) and r^{2-n} (when $n \geqq 3$).

In this paper we examine the behaviour of the quotient

$$\mathcal{Q}(f, r) = \mathcal{A}(f, r)/\mathcal{M}(f, r) \quad (\mathcal{M}(f, r) \neq 0),$$

in particular indicating conditions on f which guarantee that $\mathcal{Q}(f, r)$ is a decreasing function of r on $(0, a)$. Our first result concerns positive powers of harmonic functions.

Theorem 1. *If h is harmonic and not identically zero in $B(a)$ then $\mathcal{Q}(h^2, \cdot)$ is decreasing on $(0, a)$. If $p > 0$, $p \neq 2$, then there exists a harmonic function H in R^n such that $\mathcal{Q}(|H|^p, \cdot)$ is not decreasing on any non-empty interval $(0, \alpha)$.*

For a sufficiently differentiable function f denote by $\Delta^j f$ the j-th iterated laplacian of f (i.e. $\Delta^0 f = f$, $\Delta^1 f = \Delta f$, $\Delta^j f = \Delta(\Delta^{j-1} f)$, $j = 1, 2, \ldots$). The positive part of Theorem 1 will follow from the more general

Theorem 2. *Let $f : B(a) \to R$ be analytic and suppose that $\Delta^j f(O) \geqq 0$ for each non-negative integer j.*

(i) *If $\Delta^k f(O) > 0$ for at least one non-negative integer k, then $\mathcal{Q}(f, \cdot)$ is decreasing on $(0, a)$.*

(ii) *If $\Delta^j f(O) = 0$ for each non-negative integer j, then $\mathcal{M}(f, \cdot) \equiv 0$ on $(0, a)$.*

We give an example in § 6 to show that the condition $\Delta^j f(O) \geqq 0$ for all j cannot be relaxed. Initially we derive Theorem 2 from the following theorem which, especially in its application to harmonic functions, seems to be of some independent interest.

Theorem 3. *If $f : B(a) \to R$ is analytic, $\Delta^j f(O) \geqq 0$ for each non-negative integer j and $\Delta^k f(O) > 0$ for at least one non-negative k, then $\log \mathcal{M}(f, r)$ is a convex function of $\log r$ for $0 < r < a$.*

Corollary. *If h is harmonic and not identically zero in $B(a)$, then $\log \mathcal{M}(h^2, r)$ is a convex function of $\log r$ for $0 < r < a$.*

The counterexamples proving the negative part of Theorem 1 are given in § 4. They also serve to show that, in Theorem 2, f cannot be replaced by $|f|^p$ for any $p > 0$, $p \neq 1$. Further, they show indirectly that Theorem 3 and its corollary become false if f (respectively h^2) are replaced by $|f|^p$ (respectively $|h|^{2p}$) with $p > 0$, $p \neq 1$.

It will be noticed that the counterexamples satisfy $H(O) = 0$, and we may ask whether, if the extra condition $h(O) \neq 0$ is inserted in Theorem 1, any positive result for $\mathcal{Q}(|h|^p, \cdot)$ with $p > 0$, $p \neq 2$ can be obtained (e.g. with $p = 1$ we have, trivially, $\mathcal{Q}(|h|, \cdot)$ constant on some interval $(0, \alpha)$). More generally, we shall consider $\mathcal{Q}(s, \cdot)$ for a subharmonic function s. We have the following result concerning the behaviour of $\mathcal{Q}(s, r)$ for small values of r.

Theorem 4. *Let s be subharmonic and analytic in $B(a)$.*

(i) *If $s(O) > 0$ then there exists $\alpha \in (0, a]$ such that $\mathscr{Q}(s, \cdot)$ is decreasing on $(0, \alpha)$.*

(ii) *If $s(O) < 0$ then there exists $\alpha \in (0, a]$ such that $\mathscr{Q}(s, \cdot)$ is increasing on $(0, \alpha)$.*

(iii) *If $s(O) = 0$ and $\mathscr{M}(s, r) > 0$ for each $r \in (0, a)$ then there exists $\alpha \in (0, a]$ such that $\mathscr{Q}(s, \cdot)$ is monotonic on $(0, \alpha)$. The possibilities $\mathscr{Q}(s, \cdot)$ strictly increasing, strictly decreasing, and constant can all occur.*

(iv) *There exists an infinitely differentiable subharmonic function u in R^n such that $u(O) > 0$ and $\mathscr{Q}(u, \cdot)$ is not monotonic on any non-empty interval $(0, \alpha)$, and there exists an infinitely differentiable, non-negative subharmonic function v in R^n such that $v(O) = 0$ and the limit*

$$\lim_{r \to 0+} \mathscr{Q}(v, r)$$

does not exist.

Corollary. *Let h be harmonic in $B(a)$ and suppose that $h(O) \neq 0$.*

(i) *If $p \geqq 1$ then there exists $\alpha \in (0, a]$ such that $\mathscr{Q}(|h|^p, \cdot)$ is decreasing on $(0, \alpha)$.*

(ii) *If $0 < p \leqq 1$ then there exists $\alpha \in (0, a]$ such that $\mathscr{Q}(|h|^p, \cdot)$ is increasing on $(0, \alpha)$.*

The counterexamples proving the negative part of Theorem 1 show that, if the condition $h(O) \neq 0$ is dropped from this corollary, part (i) becomes false except for $p = 2$. We shall give an example in § 6 to show that, without the condition $h(O) \neq 0$, part (ii) also becomes false. We shall show also that in general $\alpha < a$ (§ 6).

The key result in the proof of Theorem 4 is

Theorem 5. *Let j, k be integers such that $0 < j < k$ and let $f : B(a) \to R$ be $2k + 2$ times continuously differentiable with $\Delta^i f(O) = 0$ $(0 \leqq i < k,\ i \neq j)$, $\Delta^j f(O) \neq 0$, $\Delta^k f(O) \neq 0$. If $\Delta^j f(O)$, $\Delta^k f(O)$ have the same (respectively opposite) signs then there exists $\alpha \in (0, a]$ such that $\mathscr{Q}(f, \cdot)$ is decreasing (respectively increasing) on $(0, \alpha)$. If $f : B(a) \to R$ is not identically zero and is analytic, and $\Delta^i f(O) \neq 0$ for only one value of i then $\mathscr{Q}(f, \cdot)$ is constant on $(0, a)$.*

Finally we give some results for large values of *r*.

Theorem 6. *Let h be harmonic in R^n and let $p \geqq 1$. Then h is a polynomial of degree m if and only if*

$$\lim_{r \to \infty} \mathscr{Q}(|h|^p, r) = \frac{n}{n + mp},$$

and h is not a polynomial if and only if

$$\lim_{r \to \infty} \mathscr{Q}(|h|^p, r) = 0 .$$

The question whether $\mathcal{Q}(h^p, r)$, when p is an even integer, is ultimately increasing or decreasing, shows a difference in behaviour between the cases $n = 2$, $n \geqq 3$ for harmonic polynomials.

Theorem 7. (i) *If h is a harmonic polynomial in R^2, then $\mathcal{Q}(h^{2q}, r)$ $(q = 1, 2, \ldots)$ decreases for sufficiently large r.*

(ii) *When $n \geqq 3$ there exists a harmonic polynomial h in R^n such that $\mathcal{Q}(h^{2q}, r)$ $(q = 2, 3, \ldots)$ increases strictly for sufficiently large r.*

(iii) *There exists h harmonic in R^2 such that $\mathcal{Q}(h^4, \cdot)$ is not monotonic on any interval $(\varrho, +\infty)$.*

2. PROOF OF THEOREM 3

First we prove

Lemma 1. *If $f : B(a) \to R$ is analytic, then $\mathcal{M}(f, \cdot)$ is analytic on $(0, a)$.*

Suppose that $r_0 \in (0, a)$ and that $P \in S(r_0)$. Choose polar coordinates $r, \theta_1, \theta_2, \ldots$ $\ldots, \theta_{n-1}$ centred at O such that $P = (r_0, \pi/2, \pi/2, \ldots, \pi/2)$. Since f is an analytic function of $x_1, x_2, \ldots, x_n$, which are in turn analytic functions of $r, \theta_1, \ldots, \theta_{n-1}$ in a neighbourhood of P, f is an analytic function of $r, \theta_1, \ldots, \theta_{n-1}$ in a neighbourhood of P (see e.g. H. Cartan [2, § IV.2.2]). Hence there is a positive number δ_P such that, in $B(P, \delta_P)$, f has an absolutely convergent, uniformly convergent series representation of the form

$$f(r, \theta_1, \ldots, \theta_{n-1}) = \sum_{m=0}^{\infty} (r - r_0)^m f_m(\theta_1, \ldots, \theta_{n-1}) .$$

If now $N(P)$ is a measurable subset of $S(r_0) \cap B(P, \frac{1}{2}\delta_P)$ and

$$N(P, r) = \{(r, \theta_1, \ldots, \theta_{n-1}) \in R^n : (r_0, \theta_1, \ldots, \theta_{n-1}) \in N(P)\} ,$$

then provided that $|r - r_0| < \frac{1}{2}\delta_P$, $N(P, r) \subset B(P, \delta)$ and

$$\int_{N(P,r)} f \, d\sigma = \sum_{m=0}^{\infty} (r - r_0)^m \int_{N(P,r)} f_m \, d\sigma ,$$

whence it follows that the function

$$r \to \int_{N(P,r)} f \, d\sigma$$

is analytic on $(r_0 - \frac{1}{2}\delta_P, r_0 + \frac{1}{2}\delta_P)$. The set $\{B(P, \frac{1}{2}\delta);\ P \in S(r_0)\}$ is an open cover of $S(r_0)$ and therefore has a finite subcover

$$\{B(P_1, \tfrac{1}{2}\delta_{P_1}), B(P_2, \tfrac{1}{2}\delta_{P_2}), \ldots, B(P_q, \tfrac{1}{2}\delta_{P_q})\} ,$$

say. Let

$$N(P_1) = B(P_1, \tfrac{1}{2}\delta_{P_1}) \cap S(r_0),$$

$$N(P_j) = (B(P_j, \tfrac{1}{2}\delta_{P_j}) \cap S(r_0)) \setminus \bigcup_{k=1}^{j-1} B(P_k, \delta_{P_k}) \quad (j = 2, 3, \ldots, q).$$

Then for any positive number r, $\{N(P_j, r) : j = 1, 2, \ldots, q\}$ is a disjoint measurable cover of $S(r)$. Hence, if $|r - r_0| < \frac{1}{2} \min \{\delta_{P_1}, \ldots, \delta_{P_q}\} = \delta$, say, then

$$\mathscr{M}(f, r) = (1/s_n r^{n-1}) \sum_{j=1}^{q} \int_{N(P_j, r)} f \, d\sigma .$$

Since each term in this sum is an analytic function of r on $(r_0 - \delta, r_0 + \delta)$, it follows that $\mathscr{M}(f, r)$ is analytic on this interval and therefore, since r_0 is arbitrary, on $(0, a)$.

Now suppose that f satisfies the hypotheses of Theorem 3. Since f is analytic there exists a positive number b such that

$$f(P) = \sum_{m=0}^{\infty} F_m(P) \quad (P \in B(b)),$$

where F_m is a homogeneous polynomial of degree m in the coordinates $(x_1, \ldots, x_n)$ of P and the series converges uniformly in $B(b)$. Hence, if $r \in (0, b)$,

$$\mathscr{M}(f, r) = \sum_{m=0}^{\infty} \mathscr{M}(F_m, r) = \sum_{m=0}^{\infty} r^{2m} \mathscr{M}(F_{2m}, 1) = \sum_{m=0}^{\infty} a_m r^{2m}, \tag{2}$$

say, the odd values of m making no contribution to the right-hand side since, when m is odd, each term of F_m is an odd function of at least one of the coordinates $x_1, \ldots, x_n$, so its integral over $S(r)$ is zero. By comparison with Pizzetti's formula (see e.g. duPlessis [3; p. 30]) or by direct computation, we see that a_m in (2) is given by

$$a_m = (2^m m! \, n(n + 2) \ldots (n + 2m - 2))^{-1} \Delta^m f(0), \tag{3}$$

which is non-negative by hypothesis.

Next we show that the series on the right-hand side of (2) converges to $\mathscr{M}(f, r)$ for $r \in (0, a)$. Let c be the radius of convergence of the series. Since the series converges to $\mathscr{M}(f, r)$ for $r \in (0, b)$ and, by Lemma 1, $\mathscr{M}(f, r)$ is analytic on $(0, a)$ by the principle of analytic continuation the series converges to $\mathscr{M}(f, r)$ for $r \in (0, \min \{a, c\})$. Hence it is enough to prove that $c \geqq a$. Since $a_m \geqq 0$ for each m, the sum function of the series in (2) has no analytic continuation to any neighbourhood of c. (See e.g. Titchmarsh [4; § 7.21] for a proof of the corresponding result for complex series. The proof for real series is the same). However, if $c < a$, then $\mathscr{M}(f, .)$ would provide such a continuation, so we conclude that $c \geqq a$, and the required result follows.

Now define a function g on the open disc with radius a and centre the origin in the complex plane by

$$g(z) = \sum_{m=0}^{\infty} a_m z^{2m}.$$

Then, by the result of the last paragraph and the fact that $a_m \geqq 0$ for each m,

$$\mathscr{M}(f, r) = g(r) = \sup_{0 \leqq \theta \leqq 2\pi} g(re^{i\theta}) \quad (0 < r < a) .$$

Hence, by applying the Hadamard three circles theorem to g, we obtain the convexity of $\log \mathscr{M}(f, r)$ as a function of $\log r$ for $r \in (0, a)$.

To prove the Corollary to Theorem 3, we note that a harmonic function h in $B(a)$ is analytic (see e.g. BRELOT [1; Appendix § 15]), and therefore h^2 is analytic. If h is not identically zero, then $\mathscr{M}(h^2, \cdot)$ is not identically zero and, by (2), (3), h^2 has at least one iterated laplacian which does not vanish at the origin. It suffices to show therefore that, for each $j \geqq 0$, $\Delta^j h^2 \geqq 0$, and this is straightforward. In fact if ∇ denotes the gradient operator in R^n

$$\Delta^0 h^2 = h^2 \geqq 0 , \quad \Delta^1 h^2 = 2|\nabla h|^2 \geqq 0 ,$$

$$\Delta^2 h^2 = 4 \sum_{i=1}^{n} \left|\nabla \frac{\partial h}{\partial x_i}\right|^2 = 2 \sum_{i=1}^{n} \Delta \left(\frac{\partial h}{\partial x_i}\right)^2 \geqq 0 ,$$

but for each $i = 1, 2, \ldots, \partial h / \partial x_i$ is itself a harmonic function, and the result may be proved by induction in an obvious way.

3. PROOF OF THEOREM 2

Theorem 2 (ii) is immediate from (2), (3). Suppose that the hypotheses of Theorem 2 (i) hold. If we again write $\mu(r) = \mathscr{M}(f, r)$, the condition that $\log \mu(r)$ is a twice continuously differentiable function of $\log r$ on $(0, a)$ is equivalent to the condition that $r\,\mu'(r)/\mu(r)$ is a continuously differentiable increasing function on $(0, a)$. Now

$$\mathscr{Q}'(f, r) = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{n}{r^n \mu(r)} \int_0^r t^{n-1} \mu(t)\,\mathrm{d}t\right) =$$

$$= \frac{-n^2}{r^{n+1}\mu(r)} \int_0^r t^{n-1} \mu(t)\,\mathrm{d}t - \frac{n\,\mu'(r)}{r^n(\mu(r))^2} \int_0^r t^{n-1} \mu(t)\,\mathrm{d}t + \frac{n}{r} =$$

$$= \frac{n}{r^{n+1}\mu(r)} \int_0^r t^n \mu'(t)\,\mathrm{d}t - \frac{n\,\mu'(r)}{r^n(\mu(r))^2} \int_0^r t^{n-1} \mu(t)\,\mathrm{d}t =$$

$$= \frac{n}{r^{n+1}(\mu(r))^2} \int_0^r t^{n-1}(t\,\mu'(t)\,\mu(r) - r\,\mu'(r)\,\mu(t))\,\mathrm{d}t \leqq 0 ,$$

since $r\,\mu'(r)/\mu(r)$ increases.

Theorem 2 (i) may also be proved directly, that is, without using Theorem 3, by using equations (1) and (2) to establish the equation

$$\mathscr{A}(f, r) = \sum_{m=0}^{\infty} \frac{n}{2m + n} a_m r^{2m}, \tag{4}$$

and then computing $\mathscr{Q}'(f, r)$ when $\mathscr{Q}(f, \cdot)$ is expressed as the quotient of the power series in (4) and (2).

4. PROOF OF THEOREM 1

The result for $\mathscr{Q}(h^2, \cdot)$ when h is harmonic and not identically zero follows from Theorem 2 (i), by noting that $\Delta^j h^2 > 0$ for each non-negative j and that h^2 has at least one iterated laplacian which does not vanish at O. It remains to give the counter-examples to show that when $p > 0$, $p \neq 2$, there exists a harmonic function H in R^n such that $\mathscr{Q}(|H|^p, \cdot)$ does not decrease on $(0, \alpha)$ for any positive α. When $0 < p < 2$ such an H is given by

$$H(x_1, x_2, \ldots, x_n) = x_1(1 + (n-1)x_1^2 - 3(x_2^2 + \ldots + x_n^2)).$$

Clearly H is harmonic in R^n, and with polar coordinates $(r, \theta_1, \ldots, \theta_{n-1})$ such that $x_1 = r \sin \theta_1$

$$|H(r, \theta_1, \ldots, \theta_{n-1})|^p = r^p |\sin \theta_1|^p \, |1 - r^2\{3 - (n+2)\sin^2 \theta_1\}|^p =$$
$$= r^p |\sin \theta_1|^p \, [1 - pr^2\{3 - (n+2)\sin^2 \theta_1\}] + O(r^{p+4})$$

for small r. Hence

$$\mathscr{M}(|H|^p, r) = ar^p - br^{p+2} + O(r^{p+4}), \tag{5}$$

where $a > 0$ and depends only on n, p and (see e.g. [3; p. 18])

$$b = s_n^{-1} p \int_{-\pi/2}^{\pi/2} \ldots \int_{-\pi/2}^{\pi/2} \int_{-\pi}^{\pi} \{3 - (n+2)\sin^2 \theta_1\} \, |\sin \theta_1|^p$$
$$\cos^{n-2} \theta_1 \cos^{n-3} \theta_2 \ldots \cos \theta_{n-2} \, d\theta_{n-1} \, d\theta_{n-2} \ldots d\theta_1 =$$
$$= s_n^{-1} s_{n-1} p \int_{-\pi/2}^{\pi/2} \cos^{n-2} \theta_1 \{3|\sin \theta_1|^p - (n+2)\,|\sin \theta_1|^{p+2}\} \, d\theta_1 > 0.$$

To obtain the last inequality, we used the equations

$$\int_{-\pi/2}^{\pi/2} \cos^{n-2} \theta_1 |\sin \theta_1|^{p+2} \, d\theta_1 = (p+1)(p+n)^{-1} \int_{-\pi/2}^{\pi/2} \cos^{n-2} \theta_1 |\sin \theta_1|^p \, d\theta_1,$$
$$3 - (p+1)(n+2)(p+n)^{-1} = (2-p)(n-1)(p+n)^{-1}.$$

From equations (1) and (5) we easily obtain

$$(6) \qquad \mathscr{A}(|H|^p, r) = an(p + n)^{-1} r^p - bn(p + n + 2)^{-1} r^{p+2} + O(r^{p+4}).$$

Since $a > 0$ and $b > 0$ it follows from (5) and (6) that, for sufficiently small r,

$$\mathscr{Q}(|H|^p, r) > \frac{n}{p + n},$$

and

$$\lim_{r \to 0+} \mathscr{Q}(|H|^p, r) = \frac{n}{p + n},$$

so that $\mathscr{Q}(|H|^p, \cdot)$ is not decreasing on $(0, \alpha)$ for any positive α.

When $p > 2$ and H is given by

$$H(x_1, x_2, \ldots, x_n) = x_1(1 - (n - 1) x_1^2 + 3(x_2^2 + \ldots + x_n^2)),$$

then $\mathscr{Q}(|H|^p, \cdot)$ is not decreasing on $(0, \alpha)$ for any positive α, the details of the proof being similar to those in the previous case.

5. PROOF OF THEOREM 5

Since f is $2k + 2$ times continuously differentiable Pizzetti's formula [3, p. 30] holds, and, under the hypotheses of Theorem 5 reduces to

$$(7) \qquad \mathscr{M}(f, r) = (2^j j!\, n(n + 2) \ldots (n + 2j - 2))^{-1} \Delta^j f(0)\, r^{2j} + (2^k k!\, n(n + 2) \ldots$$
$$\ldots (n + 2k - 2))^{-1} \Delta^k f(0)\, r^{2k} + O(r^{2k+2}) = cr^{2j} + dr^{2k} + O(r^{2k+2}),$$

say, for small r.

Using (1) we obtain

$$\mathscr{A}(f, r) = \frac{cn}{2j + n} r^{2j} + \frac{dn}{2k + n} r^{2k} + O(r^{2k+2}),$$

whence, using (7),

$$\mathscr{Q}(f, r) = \left(\frac{cn}{2j + n} + \frac{dn}{2k + n} r^{2(k-j)} + O(r^{2(k-j)+2})\right) \times (c + dr^{2(k-j)} +$$
$$+ O(r^{2(k-j)+2}))^{-1} = \frac{n}{2j + n} - \frac{2d(k - j)}{c(2k + n)} r^{2(k-j)} + O(r^{2(k-j)+2}),$$

and so $\mathscr{Q}(f, \cdot)$ decreases for small r if c and d have the same sign, and increases if c and d have opposite signs, which is the first result of the theorem. If f is not identically zero and analytic in $B(a)$, and $\Delta^i f(O) \neq 0$ for only one value of i then the Pizzetti representation

$$\mathcal{M}(f, r) = (2^i i!\, n(n+2)\dots(n+2i-2))^{-1}\, \Delta^i f(0)\, r^{2i} \quad (0 < r < a)$$

is exact, and clearly $\mathcal{Q}(f, \cdot)$ is constant on $(0, a)$.

6. PROOF OF THEOREM 4

To prove parts (i) and (ii) we first note that if $\mathcal{M}(s, \cdot)$ is constant on $(0, a)$ then these results are trivial. Otherwise there exists a smallest positive integer j such that $\Delta^j s(O) \neq 0$ for, if not, Pizzetti's formula gives that $\mathcal{M}(s, r) - s(O) = O(r^{2k+2})$ for all positive integers k whence $\mathcal{M}(s, r) - s(O)$, being an analytic function of r for small r, is zero on $(0, a)$ and $\mathcal{M}(s, \cdot)$ is constant. Further $\Delta^j s(O) > 0$ since otherwise, again by Pizzetti's formula,

$$\mathcal{M}(s, r) = s(0) - cr^{2j} + O(r^{2j+2}),$$

with $c > 0$, and $\mathcal{M}(s, \cdot)$ would decrease for small r. Parts (i) and (ii) now follow from Theorem 5.

To prove part (iii) we note that $\mathcal{M}(s, \cdot)$ is not constant on $(0, a)$ and, as in the proof of parts (i) and (ii), the first non-vanishing iterated Laplacian $\Delta^j s(O)$ is positive. If $\Delta^i s(O) = 0$ for all $i > j$ then $\mathcal{M}(s, r) = cr^{2j}$ $(c > 0,\ 0 < r < a)$ and $\mathcal{Q}(s, \cdot)$ is constant on $(0, a)$. An example of this case (with $j = 1$) is

$$s(x_1, x_2, \dots, x_n) = x_1^2 .$$

Otherwise there exists a smallest $i > j$ such that $\Delta^i s(O) \neq 0$ and, by Theorem 5 $\mathcal{Q}(s, \cdot)$ is either decreasing or increasing on $(0, \alpha)$ for some $\alpha > 0$, according to whether $\Delta^i s(O)$ is positive or negative. Examples of these cases are given respectively (with $a = +\infty$) by

$$s_2(x_1, x_2, \dots, x_n) = x_1^2 + x_1^4, \quad s_3(x_1, x_2, \dots, x_n) = x_1^2 - x_1^4 + x_1^6 .$$

In connection with the last example it is worth noting that, by a straightforward calculation,

$$\mathcal{M}(s_3, r) = \frac{\pi s_{n-1}}{4s_n}\left(r^2 - \frac{1}{4}r^4 + \frac{1}{16}r^6\right),$$

$$\mathcal{A}(s_3, r) = \frac{\pi s_{n-1}}{4s_n}\left(\frac{n}{n+2}r^2 - \frac{n}{n+4}r^4 + \frac{n}{n+6}r^6\right),$$

and

$$\operatorname{sign} \mathcal{Q}'(s_3, r) = \operatorname{sign}\left(\frac{1}{(4+n)(2+n)} - \frac{r^2}{(6+n)(2+n)} + \frac{r^4}{16(4+n)(6+n)}\right),$$

so that, for example, $\mathcal{Q}'(s_3, 2) < 0$ and therefore $\alpha < a$ in general. This example also serves to show that the result of Theorem 2 fails to hold if one of the iterated laplacians $\Delta^j f(O)$ is negative. In fact $\mathcal{Q}(s_3, r)$ is increasing both for small r and for large r.

The Corollary to Theorem 4 follows by applying the theorem to $|h|^p$ in the case $p \geqq 1$ and to $-|h|^p$ in the case $0 < p \leqq 1$ ($|h|^p$ is analytic in some neighbourhood of O since it is the composition of $P \to |h(P)|$ which is analytic in some neighbourhood of O, and $x \to x^p$, which is analytic in some neighbourhood of $|h(O)|$).

To show that part (ii) of the Corollary is false without the condition $h(O) \neq 0$, we again use the example, previously employed in § 4,

$$H(x_1, x_2, \ldots, x_n) = x_1(1 - (n-1)\, x_1^2 + 3(x_2^2 + \ldots + x_2^n)) .$$

When $0 < p < 2$, similar reasoning to that in § 4 yields that $\mathcal{Q}(|H|^p, \cdot)$ is not increasing on $(0, \alpha)$ for any positive α, and this includes the range $0 < p \leqq 1$ of part (ii) of the Corollary.

It remains therefore to give the example of a subharmonic function $u \in C^\infty(R^n)$ such that $u(O) > 0$ and $\mathcal{Q}(u, \cdot)$ is neither increasing nor decreasing on any non-empty interval $(0, \alpha)$. In order to reduce the length of the proof, we work only with $n = 3$. The generalization to higher dimensions is straightforward but involves lengthy calculations.

Define for each $j = 1, 2, \ldots, f_j : [0, +\infty) \to R$ by $f_j(x) = (2^j - x^{-1})^+$ $(x \geqq 0)$, $f_j(0) = 0$, and $u_j : R^3 \to R$ by

$$u_j(P) = f_j(OP) \quad (P \in R^3) .$$

Then u_j is subharmonic in R^3 and we have

Lemma 2. *There exists an infinitely differentiable subharmonic function u_j^* in R^3, depending only on OP, such that $u_j^*(P) = u_j(P)$ whenever $0 \leqq OP \leqq 2^{-j-1/12}$ or $OP \geqq 2^{-j+1/12}$, and*

$$(8) \qquad 0 \leqq u_j^*(P) - u_j(P) \leqq j^{-j} \quad (P \in R^3) .$$

In fact, using the infinitely differentiable mollifying function given by

$$\phi_j(P) = \alpha_j \exp\left((OP)^2 - \beta_j^2\right)^{-1} \quad (OP < \beta_j), \qquad \phi_j(P) = 0 \quad (OP \geqq \beta_j),$$

where $\beta_j > 0$ and α_j is chosen such that the integral of ϕ_j over R^3 is 1, we may take u_j^* to be the convolution $u_j * \phi_j$ given by

$$u_j * \phi_j(P) = \int_{R^3} \phi_j(Q)\, u_j(P - Q)\, \mathrm{d}v(Q) \quad (P \in R^3) .$$

It then follows from familiar theorems that $u_j^* \in C^\infty(R^3)$ and is subharmonic in R^3, and it is clear that u_j^*, like u_j, depends only on OP. Further, since u_j is harmonic in $R^3 \setminus S(2^{-j})$, it follows that $u_j^* = u_j$ when $OP \leqq 2^{-j} - \beta_j$ and when $OP \geqq 2^{-j} + \beta_j$ [1, Appendix § 4] and the invariance of harmonic functions under convolution with ϕ_j also gives, with $H_j(P) = 2^j - (OP)^{-1}$ $(P \in R^3 \setminus \{0\})$, that when $OP > \beta_j$

$$\begin{aligned} u_j^*(P) &= \int_{R^3} \phi_j(Q)\, H_j^+(P - Q)\, \mathrm{d}v(Q) \geqq \\ &\geqq \left\{ \int_{R^3} \phi_j(Q)\, H_j(P - Q)\, \mathrm{d}v(Q) \right\}^+ = \\ &= H_j^+(P) = u_j(P)\,. \end{aligned}$$

Taking $\beta_j < 2^{-j-1}$, we have that $u_j^*(P) = u_j(P) = 0$ when $OP \leqq \beta_j$, so that $u_j^* \geqq u_j$ in R^3, and the easily established inequality

$$|u_j^*(P) - u_j(P)| \leqq \sup_{OQ \leqq \beta j} |u_j(P - Q) - u_j(P)| \quad (P \in R^3)$$

together with the uniform continuity of u_j on R^3 shows that the inequalities (8) hold for suitably small β_j. This completes the proof of the lemma.

Define $f_j^* : [0, +\infty) \to R$ by

$$f_j^*(x) = u_j^*(x, 0, \ldots, 0)\,,$$

and write

$$a_j = \max_{0 \leqq i \leqq j} \sup_{x \in (0,1)} |f_j^{*(i)}(x)|\,, \quad b_j = (a_1 + a_2 + \ldots + a_j)^{-1}\,.$$

Now let $f : [0, +\infty) \to R$ be defined by

$$f(x) = \sum_{j=1}^{\infty} (2j)^{-j}\, b_j f_j^*(x)$$

and $u : R^3 \to R$ by $u(P) = f(OP) + 1$.

We shall show that

(i) u is subharmonic in R^3,

(ii) $u \in C^\infty(R^3)$,

(iii) $\varphi(u, \cdot)$ is not decreasing on any non-empty interval $(0, \alpha)$.

To establish (i) we note that, when $P \neq O$, u is the sum a finite numbei of subharmonic functions plus the limit of an increasing sequence of harmonic functions (since u_j^* is harmonic and non-negative when $OP \geqq 2^{-j+1/12}$), and u is bounded above in R^3, since

$$u \leqq 1 + b_1 \sum_{j=1}^{\infty} j^{-j} < +\infty\,.$$

Hence u is subharmonic in $R^3 \setminus \{O\}$. Since, as is proved below, $u \in C^\infty(R^3)$, it remains only to point out that the mean-value inequalities for u, for balls with centre O, are trivially satisfied since $u \geqq u(O) = 1$ in R^3.

We now turn to (ii) and prove that (a) $f \in C^\infty(0, +\infty)$ and (b) $f^{(i)}(x)/x \to 0$ as $x \to 0+$ for $i = 0, 1, 2, \ldots$, which is enough since u is a function of OP only.

(a) If $x_0 \geqq 2^{-1/2}$ then in the neighbourhood $(2^{-11/12}, +\infty)$ of x_0

$$f(x) = \sum_{j=1}^{\infty} (2j)^{-j} b_j (2^j - x^{-1}) = \sum_{j=1}^{\infty} j^{-j} b_j - x^{-1} \sum_{j=1}^{\infty} (2j)^{-j} b_j ,$$

so f is infinitely differentiable at x_0. If $0 < x_0 < 2^{-1/2}$, then there exists a unique positive integer m such that $2^{-m-1/2} < x_0 \leqq 2^{-m+1/2}$. Then, in some neighbourhood of x_0,

$$f(x) = (2m)^{-m} b_m f_m^*(x) + \sum_{j=m+1}^{\infty} (2j)^{-j} b_j f_j(x) =$$

$$= (2m)^{-m} b_m f_m^*(x) + \sum_{j=m+1}^{\infty} j^{-j} b_j - x^{-1} \sum_{j=m+1}^{\infty} (2j)^{-j} b_j ,$$

so that f is infinitely differentiable at x_0.

(b) If $x > 0$, then, in some neighbourhood of x,

$$f(y) = \sum_{2^{-j-1/12} \leqq x} (2j)^{-j} b_j f_j^*(y) ,$$

so that, for any non-negative integer i,

$$f^{(i)}(x) = \sum_{2^{-j-1/12} \leqq x} (2j)^{-j} b_j f_j^{*(i)}(x) ,$$

differentiation of the series for f yielding uniformly convergent series by the choice of b_j. If now $x < 2^{-i-1/12}$, then

$$|f^{(i)}(x)| \leqq \sum_{j+1/12 \geqq -\log x/\log 2} (2j)^{-j} = o(x) \quad (x \to 0+) .$$

In the last step we used

$$\sum_{j=p}^{\infty} (2j)^{-j} \leqq \sum_{j=p}^{\infty} j^{-j} = O(pe^{-p-\log p}) \quad (p \to \infty) .$$

Finally we establish (iii). To do this we show that for sufficiently large m,

$$\mathscr{Q}(u, 2^{-m-1/6}) < \mathscr{Q}(u, 2^{-m-1/12}) .$$

First

$$\mathscr{M}(u, 2^{-m-1/6}) = 1 + \sum_{j=m+1}^{\infty} (2j)^{-j} b_j f_j^*(2^{-m-1/6}) > \tag{9}$$

$$> 1 + (2m+2)^{-m-1} b_{m+1} f_{m+1}(2^{-m-1/6}) = 1 + b_{m+1}(m+1)^{-m-1}(1 - 2^{-5/6}) .$$

Next

$$(10)\quad \mathcal{M}(u, 2^{-m-1/12}) = 1 + \sum_{j=m+1}^{\infty} (2j)^{-j}\, b_j f_j^*(2^{-m-1/12}) \leqq$$

$$\leqq 1 + (2m+2)^{-m-1}\, b_{m+1}(2^{m+1} - 2^{m+1/12}) + b_{m+1} \sum_{j=m+2}^{\infty} j^{-j} =$$

$$= 1 + (m+1)^{-m-1}\, b_{m+1}(1 - 2^{-11/12} + o(1)),$$

as $m \to \infty$. Thirdly, using equation (1), we have

$$(11)\quad \mathcal{A}(u, 2^{-m-1/12}) \geqq 1 + \sum_{j=1}^{\infty} (2j)^{-j}\, b_j\, 3.2^{3(m+1/12)} \int_0^{2^{-m-1/12}} l^2 f_j(l)\, \mathrm{d}l >$$

$$> 1 + b_{m+1}\, 3\,.\,(2m+2)^{-m-1}\,.\,2^{3m+1/4} \int_{2^{-m-1}}^{2^{-m-1/12}} (2^{m+1} l^2 - l)\, \mathrm{d}l =$$

$$= 1 + b_{m+1}(m+1)^{-m-1}\left(1 + \tfrac{1}{8} 2^{-3/4} - \tfrac{3}{2} 2^{-11/12}\right).$$

Finally, using equation (1), inequality (8) and the fact that, by the subharmonicity of each u_j^*, $\mathcal{A}(u_j^*, 2^{-m-1/6}) \leq \mathcal{M}(u_j^*, 2^{-m-1/6})$,

$$(12)\quad \mathcal{A}(u, 2^{-m-1/6}) \leqq 1 + b_{m+1}(2m+2)^{-m-1}\, \mathcal{A}(u_{m+1}^*, 2^{-m-1/6}) +$$

$$+ \sum_{j=m+2}^{\infty} b_j (2j)^{-j}\, \mathcal{M}(u_j^*, 2^{-m-1/6}) \leqq$$

$$\leqq 1 + b_{m+1}(2m+2)^{-m-1}\, 3.2^{3(m+1/6)} \int_{2^{-m-1}}^{2^{-m-1/6}} l^2 (f_{m+1}(l) +$$

$$+ (m+1)^{-m-1})\, \mathrm{d}l + b_{m+1} \sum_{j=m+2}^{\infty} (2j)^{-j} (f_j(2^{-m-1/6}) + j^{-j}) <$$

$$< 1 + b_{m+1}(m+1)^{-m-1}\, 3.2^{2m-1/2}\,.$$

$$.\left(\int_{2^{-m-1}}^{2^{-m-1/6}} (l^2\, 2^{m+1} - l)\, \mathrm{d}l + O((m+1)^{-m-1})\right) + 2b_{m+1} \sum_{j=m+2}^{\infty} j^{-j} =$$

$$= 1 + b_{m+1}(m+1)^{-m-1}\left(1 + \tfrac{1}{8} 2^{-1/2} - \tfrac{3}{2} 2^{-5/6} + o(1)\right)$$

as $m \to \infty$. To prove that $\mathcal{Q}(u, 2^{-m-1/6}) < \mathcal{Q}(u, 2^{-m-1/12})$ for sufficiently large m, it is enough to prove, by inequalities (9), (10), (11) and (12) that

$$(1 + C(m)\, D_1)(1 + C(m)\, D_2) < (1 + C(m)\, D_3)(1 + C(m)\, D_4),$$

where

$$C(m) = b_{m+1}(m+1)^{-m-1}, \quad D_1 = 1 + \tfrac{1}{8} 2^{-1/2} - \tfrac{3}{2} 2^{-5/6},$$

$$D_2 = 1 - 2^{-11/12}, \quad D_3 = 1 + \tfrac{1}{8} 2^{-3/4} - \tfrac{3}{2} 2^{-11/12}, \quad D_4 = 1 - 2^{-5/6},$$

and to prove this equality for large m, it is enough to show that $D_1 + D_2 < D_3 + D_4$.

By rearrangement this condition may be reduced to $2^{1/4}(2^{5/3} - 1) > 2^{11/6} - 1$, which is easily verified.

By a similar construction we may also obtain an example of an infinitely differentiable, subharmonic function v in R^3 such that $v > 0$ in $R^3 \setminus \{O\}$, $v(O) = 0$ and $\lim \mathcal{Q}(v, r)$ does not exist as $r \to 0+$. In fact, with u defined as in the previous example, we take $v = u - 1$ in R^3. Using obvious modifications of inequalities (9) to (12), we obtain that

$$\mathcal{Q}(v, 2^{-m-1/6}) < D_1/D_4 + o(1), \quad \mathcal{Q}(v, 2^{-m-1/12}) > D_3/D_2 + o(1)$$

as $m \to \infty$. The non-existence of $\lim \mathcal{Q}(v, r)$ as $r \to 0+$ now follows from the inequality $D_1 D_2 < D_3 D_4$ which is easily verifiable by direct computation.

7. PROOF OF THEOREM 6

Given a function $f : R^n \to R$ and $r \in (0, +\infty)$, let $U(f, r)$ be the supremum of f over $S(r)$.

Suppose that h is harmonic in R^n and that α, l are numbers such that $\alpha > 1$, $l > 0$. Then $|h|$ is subharmonic in R^n and is therefore dominated by its Poisson integral $I_{|h|}$ in $B(\alpha l)$. Applying a Harnack inequality to $I_{|h|}$, we obtain

$$U(|h|, l) \leqq U(I_{|h|}, l) \leqq C(\alpha, n)\, I_{|h|}(O) = C(\alpha, n)\, \mathcal{M}(|h|, \alpha l),$$

where

$$C(\alpha, n) = \alpha^{n-2}(\alpha + 1)(\alpha - 1)^{1-n}.$$

By Hölder's inequality, if $p \geqq 1$, then

$$U(|h|^p, l) \leqq (C(\alpha, n))^p\, \mathcal{M}(|h|^p, \alpha l).$$

By applying this formula twice, we see that if $r, t > 0$,

$$\text{(13)} \qquad \frac{\mathcal{M}(|h|^p, t)}{\mathcal{M}(|h|^p, r)} \leqq \frac{(C(\alpha, n))^{-p}\,(U(|h|, t))^p}{(U(|h|, r/\alpha))^p} \leqq \left\{\frac{\mathcal{M}(h^2, \alpha t)}{\mathcal{M}(h^2, r/\alpha)}\right\}^{p/2}.$$

Now define $\gamma : (0, +\infty) \to R$ by

$$\gamma(r) = \log \mathcal{M}(h^2, r)/\log r.$$

If h is not a polynomial, then for each real k, $r^{-k}\, \mathcal{M}(h^2, r) \to \infty$ $(r \to \infty)$ (see e.g. [1; Appendix]), so that $\gamma(r) \to \infty$ $(r \to \infty)$. If $r > \alpha^2 > 1$, then

$$\text{(14)} \qquad \mathcal{M}(h^2, r/\alpha^2)\,(\mathcal{M}(h^2, r/\alpha))^{-1} =$$

$$= \exp\left(\gamma(r/\alpha^2) \log r/\alpha^2 - \gamma(r/\alpha) \log r/\alpha\right) \leqq \exp\left(-\gamma(r/\alpha) \log \alpha\right).$$

Since $\mathcal{M}(|h|^p, \cdot)$ is increasing on $(0, +\infty)$, we have, by using (1), (13), (14) and the fact that $\gamma(r) \to \infty$ $(r \to \infty)$,

$$\mathcal{Q}(|h|^p, r) = nr^{-n} \int_0^r t^{n-1} \mathcal{M}(|h|^p, t) (\mathcal{M}(|h|^p, r))^{-1} \, dt \leqq$$

$$\leqq nr^{-n} \left\{ \int_0^{r/\alpha^3} t^{n-1} \mathcal{M}(|h|^p, r/\alpha^3) (\mathcal{M}(|h|^p, r))^{-1} \, dt + \int_{r/\alpha^3}^r t^{n-1} \, dt \right\} \leqq$$

$$\leqq nr^{-n} \int_0^{r/\alpha^3} t^{n-1} \left\{ \frac{\mathcal{M}(h^2, r/\alpha^2)}{\mathcal{M}(h^2, r/\alpha)} \right\}^{p/2} dt + 1 - \alpha^{-3n} \leqq$$

$$\leqq nr^{-n} \int_0^{r/\alpha^3} t^{n-1} \exp(-\tfrac{1}{2} p \, \gamma(r/\alpha) \log \alpha) \, dt + 1 - \alpha^{-3n} \to$$

$$\to 1 - \alpha^{-3n} \quad (r \to \infty).$$

Since this holds for each $\alpha > 1$, $\mathcal{Q}(|h|^p, r) \to 0$ $(r \to \infty)$.

If P is a polynomial of degree m in R^n, then it is easy to see that

$$\mathcal{M}(|P|^p, r) = Cr^{mp} + O(r^{mp-1}) \quad (r \to \infty),$$

where $C > 0$, whence, by using (1) to estimate $\mathcal{A}(|P|^p, r)$, we find that $\mathcal{Q}(|P|^p, r) \to n/(n + mp)$ $(r \to \infty)$.

The various results of the theorem now follow.

8. PROOF OF THEOREM 7

To prove part (i) we note that, except in the trivial case where h is homogeneous and $\mathcal{Q}(h^{2q}, r)$ is constant, we may write (taking polar coordinates (r, θ) with origin O)

$$h(r, \theta) = ar^M \cos(M\theta + \delta_M) + br^N \cos(N\theta + \delta_N) + h_1(r, \theta),$$

where $a \neq 0$, $b \neq 0$, M, N are positive integers with $M > N$, δ_M and δ_N lie in the range $[0, 2\pi)$ and

$$h_1(r, \theta) = \sum_{m=0}^{N-1} c_m r^m \cos(m\theta + \delta_m),$$

with c_m constant and $\delta_m \in [0, 2\pi)$ for $m = 0, 1, 2, \ldots, N - 1$. We then have that

$$(h(r, \theta))^{2q} = (ar^M \cos(M\theta + \delta_M) + br^N \cos(N\theta + \delta_N))^{2q} +$$
$$+ 2q(ar^M \cos(M\theta + \delta_M) + br^N \cos(N\theta + \delta_N))^{2q-1} h_1(r, \theta) +$$
$$+ O(r^{M(2q-2)+2N-2}) =$$

$$= a^{2q}e^{2qM}(\cos(M\theta + \delta_M))^{2q} + 2qa^{2q-1}br^{M(2q-1)+N}$$
$$(\cos(M\theta + \delta_M))^{2q-1}\cos(N\theta + \delta_N) + q(2q-1)\,a^{2q-2}b^2r^{M(2q-2)+2N}$$
$$(\cos(M\theta + \delta_M))^{2q-2}(\cos(N\theta + \delta_N))^2 + O(r^{M(2q-3)+3N}) +$$
$$+ 2qa^{2q-1}r^{M(2q-1)}(\cos(M\theta + \delta_M))^{2q-1}\,h_1(r, \theta) + O(r^{M(2q-2)+2N-1}).$$

Since $M > N$

$$\int_0^{2\pi}(\cos(M\theta + \delta_M))^{2q-1}\cos(N\theta + \delta_N)\,\mathrm{d}\theta = 0,$$

and

$$\int_0^{2\pi}(\cos(M\theta + \delta_M))^{2q-1}\,h_1(r, \theta) = 0,$$

and so

$$\mathcal{M}(h^{2q}, r) = cr^{2qM} + dr^{M(2q-2)+2N} + O(r^{M(2q-2)+2N-1}),$$

where c and d are positive. The result now follows easily by a technique similar to that used in proving Theorem 5.

To demonstrate part (ii) we use the example

$$h(x_1, x_2, \ldots, x_n) = 1 - 2x_1^2 + x_2^2 + x_3^2 = 1 - h_1(x_1, x_2, \ldots, x_n),$$

say, which is harmonic in R^n for $n \geqq 3$. Since h^{2q} is a polynomial of degree $4q$, $\Delta^k h^{2q}$ is identically zero for $k > 2q$. We shall prove that $\Delta^{2q}h^{2q}(O) > 0$ and $\Delta^{2q-1}h^{2q}(O) < < 0$, and since

$$h^{2q} = h_1^{2q} - 2qh_1^{2q-1} + P,$$

where P is a polynomial of degree $4q - 4$, it is enough to prove that $\Delta^{2q}h_1^{2q}(O) > 0$ and $\Delta^{2q-1}h_1^{2q-1}(O) > 0$ or equivalently that $\Delta^m h_1^m(O) > 0$ for any integer $m > 2$. First we prove by induction on m that $\Delta^m(r^{2i}h_1^{m-i})(O) \geqq 0$ for $m = 2, 3, \ldots$, and $i = 0, 1, \ldots, m$, where $r^2 = x_1^2 + x_2^2 + x_3^2$. For $m = 2$ it is easy to verify that $\Delta^2 r^4(O) > 0$, $\Delta^2 r^2 h_1(O) = 0$, and $\Delta^2 h_1^2(O) > 0$. Suppose the indicated inequalities $\Delta^m(r^{2i}h_1^{m-i})(O) \geqq 0$, $i = 0, 1, \ldots, m$ hold for some $m \geqq 2$. Then for $j = 0, 1, \ldots$ $\ldots, m + 1$,

$$\Delta^{m+1}(r^{2j}h_1^{m+1-j}) = \Delta^m(2j(2j+1)\,r^{2j-2}h_1^{m+1-j} +$$
$$+ 2j(m+1-j)\,r^{2j-2}h_1^{m-j}(\nabla r^2 \,.\, \nabla h_1) + (m+1-j)(m-j)\,r^{2j}h_1^{m-1-j}|\nabla h_1|^2).$$

Now

$$\nabla r^2 \,.\, \nabla h_1 = 4h_1, \quad |\nabla h_1|^2 = 8r^2 + 4h_1.$$

Hence

$$\Delta^{m+1}(r^{2j}h_1^{m+1-j}) = \Delta^m(2j(4m - 2j + 5)\,r^{2j-2}h_1^{m+1-j} +$$
$$+ 4(m+1-j)(m-j)(2r^{2j+2}h_1^{m-1-j} + r^{2j}h_1^{m-j})).$$

Note that the first term on the right vanishes if $j = 0$ and the second term vanishes if $j = m$ or $j = m + 1$, so may we write

$$\Delta^{m+1}(r^{2j}h_1^{m+1-j}) = \Delta^m(\sum_{i=1}^{m} a_i r^{2i} h_1^{m-i}),$$

where $a_i \geqq 0$ for $i = 0, 1, \ldots, m$, whence $\Delta^{m+1}(r^{2j}h_1^{m+1-j})(O) \geqq 0$, and the induction is complete. To prove the strict positivity of $\Delta^m h_1^m(O)$ for $m \geqq 2$ we note that

$$\Delta^{m+1}h_1^{m+1} = 4m(m+1)\,\Delta^m(2r^2h_1^{m-1} + h_1^m),$$

whence

$$\Delta^{m+1}h_1^{m+1}(0) \geqq 4m(m+1)\,\Delta^m h_1^m(0),$$

and the result follows by induction on m, noting that $\Delta^2 h_1^2(0) > 0$. Hence $\Delta^{2q}h^{2q}(O) > > 0$, $\Delta^{2q-1}h^{2q}(O) < 0$, and Pizzetti's formula gives

$$\mathcal{M}(h^{2q}, r) = cr^{4q} - dr^{4q-2} + O(r^{4q-4}),$$

where $c > 0$, $d > 0$, whence $\mathcal{Q}(h^{2q}, r)$ increases strictly for sufficiently large r, by a technique similar to that used in proving Theorem 5.

We note that if we took $1 + h_1$ instead of $1 - h_1$ in this example then $\mathcal{Q}(h^{2q}, r)$ would decrease strictly for sufficiently large r. This exhausts the possibilities for the behaviour of $\mathcal{Q}(h^{2q}, r)$ for large r, when h is a polynomial, since $\mathcal{Q}(h^{2q}, r)$, being a rational function of r, must be ultimately monotonic.

To prove part (iii), we show first that there exists a sequence (h_m) of harmonic polynomials in R^2 and sequences (λ_m), (λ'_m), $(\varkappa_m)$ of positive numbers such that for each positive integer m

(α) $|h_m(r, \theta)| < 2^{-m}e^r$, where (r, θ) are polar coordinates centred at O,

(β) $\lambda_m < \lambda'_m < \frac{1}{2}\lambda_{m+1}$,

(γ) $\mathcal{Q}((\sum_{j=1}^{m} h_j)^4, \lambda'_l) - \mathcal{Q}((\sum_{j=1}^{m} h_j)^4, \lambda_l) > \varkappa_l \; (l = 1, 2, \ldots, m)$.

We have seen (§ 4) that there exists a harmonic polynomial h_1 in R^2 such that $\mathcal{Q}(h_1^4, \cdot)$ is not decreasing on $(0, \infty)$. Hence there exist positive numbers $\lambda_1, \lambda'_1, \varkappa_1$ such that $\lambda_1 < \lambda'_1$ and

$$\mathcal{Q}(h_1^4, \lambda'_1) - \mathcal{Q}(h_1^4, \lambda_1) > \varkappa_1 .$$

Now suppose that we have found $h_1, \ldots, h_m, \lambda_1, \ldots, \lambda_m, \lambda'_1, \ldots, \lambda'_m, \varkappa_1, \ldots, \varkappa_m$ satisfying (α), (β), (γ). Choose an integer k such that $k > 21$ and $k/3$ is larger than the degree of $\sum_{j=1}^{m} h_j$, and put

$$h_{m+1}(r, \theta) = \gamma(r^k \cos k\theta - \delta r^{3k} \cos 3k\theta),$$

where γ, δ are constants to be fixed later satisfying $0 < \gamma < (2^{m+1}(3k)!)^{-1}$ and

$0 < \delta < 1$. Then h_{m+1} is harmonic in R^2 and

$$|h_{m+1}(r, \theta)| \leqq \gamma(r^k + r^{3k}) \leqq 2^{-m-1}\left(\frac{r^k}{k!} + \frac{r^{3k}}{(3k)!}\right) < 2^{-m-1}e^r.$$

Put

$$\phi(r) = \mathcal{M}\left(\left(\sum_{j=1}^{m+1} h_j\right)^4, r\right), \quad \chi(r) = \mathcal{A}\left(\left(\sum_{j=1}^{m+1} h_j\right)^4, r\right), \quad \psi(r) = \mathcal{Q}\left(\left(\sum_{j=1}^{m+1} h_j\right)^4, r\right).$$

Now clearly we may fix γ so small that, for any $\delta \in (0, 1)$, $\psi(\lambda_l') - \psi(\lambda_l) > \varkappa_l$ ($l = 1, \ldots, m$). A straightforward calculation making use of the facts that $k/3 > \deg \sum_{j=1}^{m} h_j$ and

$$\int_0^{2\pi} \cos k_1\theta \cos k_2\theta \, d\theta = \int_0^{2\pi} \cos k_1\theta \sin k_2\theta \, d\theta = 0 \quad (k_1 \neq k_2)$$

yields

$$\phi(r) = \mathcal{M}(h_{m+1}^4, r) + o(r^{8k/3}) + \delta^2 \, o(r^{20k/3}) =$$
$$= \tfrac{1}{8}\gamma^4(3r^{4k} - 4\delta r^{6k} + 12\delta^2 r^{8k} + 3\delta^4 r^{12k}) + o(r^{8k/3}) + \delta^2 \, o(r^{20k/3})$$

and using (1) we get

$$\chi(r) = \tfrac{1}{8}\gamma^4\left(\frac{3r^{4k}}{2k+1} - \frac{4\delta r^{6k}}{3k+1} + \frac{12\delta^2 r^{8k}}{4k+1} + \frac{3\delta^4 r^{12k}}{6k+1}\right) + o(r^{8k/3}) + \delta^2 \, o(r^{20k/3}).$$

The limiting processes implied by the o-notation are independent of δ. Now choose a number ε satisfying

$$(15) \qquad 0 < \frac{\varepsilon}{2k+1} + \frac{\varepsilon}{16\sqrt{2}} < \frac{3}{8^4(6k+1)}.$$

Then there exists a number R depending only on ε (not on δ) satisfying $R > 2\lambda_m'$ such that when $r \geqq R$

$$|\phi(r) - \tfrac{1}{8}\gamma^4(3r^{4k} - 4\delta r^{6k} + 12\delta^2 r^{8k} + 3\delta^4 r^{12k})| < \tfrac{1}{8}\gamma^4\varepsilon(r^{4k} + \delta^{3/2} r^{7k})$$

and

$$\left|\chi(r) - \tfrac{1}{8}\gamma^4\left(\frac{3r^{4k}}{2k+1} - \frac{4\delta r^{6k}}{3k+1} + \frac{12\delta^2 r^{8k}}{4k+1} + \frac{3\delta^4 r^{12k}}{6k+1}\right)\right| < \tfrac{1}{8}\gamma^4\varepsilon\left(\frac{r^{4k}}{2k+1} + \delta^{3/2} r^{7k}\right).$$

Hence, when $r \geqq R$

$$\frac{\dfrac{3-\varepsilon}{2k+1} - \dfrac{4\delta r^{2k}}{3k+1} - \varepsilon\delta^{3/2} r^{3k} + \dfrac{12\delta^2 r^{4k}}{4k+1} + \dfrac{3\delta^4 r^{8k}}{6k+1}}{3 + \varepsilon - 4\delta r^{2k} + \varepsilon\delta^{3/2} r^{3k} + 12\delta^2 r^{4k} + 3\delta^4 r^{8k}} < \psi(r) <$$

$$< \frac{\dfrac{3+\varepsilon}{2k+1} - \dfrac{4\delta r^{2k}}{3k+1} + \varepsilon\delta^{3/2} r^{3k} + \dfrac{12\delta^2 r^{4k}}{4k+1} + \dfrac{3\delta^4 r^{8k}}{6k+1}}{3 - \varepsilon - 4\delta r^{2k} - \varepsilon\delta^{3/2} r^{3k} + 12\delta^2 r^{4k} + 3\delta^4 r^{8k}} .$$

Hence, there exists a number ε' such that $0 < \varepsilon' < \frac{1}{8}$ with the property that

$$\psi(r) < \frac{1}{2k+1} \frac{3+\varepsilon}{3-\varepsilon} + \frac{\varepsilon}{16\sqrt{2}} \tag{16}$$

whenever $r \geqq R$ and $\delta r^{2k} < \varepsilon'$. Now fix δ so small that $\delta R^{2k} < \varepsilon'$. Then, by (16) and the choice (15) of ε,

$$\psi(R) < \frac{1}{2k+1} + \frac{\varepsilon}{2k+1} + \frac{\varepsilon}{16\sqrt{2}} < \frac{1}{2k+1} + \frac{3}{8^4(6k+1)} < \frac{353}{352} \frac{1}{2k+1} .$$

Let $R' = (8\delta)^{-1/2k}$. Then $\delta R'^{2k} = \frac{1}{8} > \varepsilon' > \delta R^{2k}$, so $R' > R$ and therefore

$$\psi(R') > \frac{\dfrac{3-\varepsilon}{2k+1} - \dfrac{1}{2(3k+1)} - \dfrac{\varepsilon}{16\sqrt{2}} + \dfrac{3}{16(4k+1)} + \dfrac{3}{8^4(6k+1)}}{3 + \varepsilon - \dfrac{1}{2} + \dfrac{\varepsilon}{16\sqrt{2}} + \dfrac{3}{16} + \dfrac{3}{8^4}} .$$

By (15) and the inequality

$$\varepsilon + \frac{\varepsilon}{16\sqrt{2}} + \frac{3}{8^4} < \frac{1}{16},$$

which follows from (15), we obtain

$$\psi(R') > \frac{4}{11}\left(\frac{3}{2k+1} - \frac{1}{2(3k+1)} + \frac{3}{16(4k+1)}\right) >$$

$$> \frac{4}{11}\left(\frac{3}{2k+1} - \frac{1}{2(3k+1)} + \frac{3}{32(2k+1)}\right) =$$

$$= \frac{1}{88}\left(\frac{99}{2k+1} - \frac{16}{3k+1}\right).$$

Since $k > 21$, $64(2k+1) < 43(3k+1)$, from which it follows that

$$\psi(R') > \frac{353}{352} \frac{1}{2k+1} > \psi(R) .$$

The induction is completed by taking $\lambda_{m+1} = R$, $\lambda'_{m+1} = R'$ and

$$\varkappa_{m+1} = \tfrac{1}{2}(\psi(R') - \psi(R)) .$$

By (α) the series $\sum_{m=1}^{\infty} h_m$ is locally uniformly convergent in R^2. Let its sum be h. Then h is harmonic in R^2 and for each positive integer l we have by (γ).

$$\mathcal{Q}(h^4, \lambda_l') = \lim_{m\to\infty} \mathcal{Q}((\sum_{j=1}^{m} h_j)^4, \lambda_l') > \lim_{m\to\infty} \mathcal{Q}((\sum_{j=1}^{m} h_j)^4, \lambda_l) = \mathcal{Q}(h^4, \lambda_l).$$

Since $\lambda_m \to \infty$, $\lambda_m' \to \infty$ and $\lambda_m < \lambda_m'$, it follows that $\mathcal{Q}(h^4, \cdot)$ is not decreasing on any interval $(\varrho, +\infty)$. On the other hand, by Theorem 6, $\mathcal{Q}(h^4, r) \to 0$ $(r \to \infty)$, so $\mathcal{Q}(h^4, \cdot)$ is not increasing on any interval $(\varrho, +\infty)$.

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