W. BERGWEILER
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ON MEROMORPHIC FUNCTIONS THAT SHARE THREE VALUES AND ON THE EXCEPTIONAL SET IN WIMAN-VALIRON THEORY

BY WALTER BERGWEILER[1)]

1. Introduction and Results

Two meromorphic functions are said to share a value a if they have the same a-points. We distinguish the cases that we count multiplicities (CM) and that we ignore multiplicities (IM). One of the main tools that has been used in the study of functions that share values is Nevanlinna's theory on the distribution of values (cf. [3, 6, 7]). Here it is important to have relations between the Nevanlinna characteristics $T(r, f)$ and $T(r, g)$ if f and g share values.

It is well-known [2, Theorem 2; 5, Satz 3] that

$$T(r, f)\sim T(r, g) \qquad (r\notin E) \tag{1.1}$$

if f and g share four values IM. Here and in the following E denotes an exceptional set of finite measure. If f and g share three values IM, then

$$\frac{1}{3}-o(1)\leqq\frac{T(r, f)}{T(r, g)}\leqq 3+o(1) \qquad (r\notin E). \tag{1.2}$$

This was proved by Gundersen [2, Theorem 3] who also gave an example which shows that the bounds 1/3 and 3 are sharp.

This paper is concerned with the question what can be said about the relation between $T(r, f)$ and $T(r, g)$ if f and g share three values CM. A recent result of Brosch [1, Satz 5.7] says that (1.2) can be improved in this case. He proved that

$$\frac{3}{8}-o(1)\leqq\frac{T(r, f)}{T(r, g)}\leqq\frac{8}{3}+o(1) \qquad (r\notin E). \tag{1.3}$$

It is not known whether these bounds are sharp. Osgood and Yang [8, Theorem 3] proved that $T(r, f)\sim T(r, g)$ if f and g are entire functions of finite order and they conjectured that this remains true for arbitrary entire functions. The

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question whether (1.1) holds for meromorphic f and g has been raised by Gundersen [2]. Brosch [1, Satz 2.1] proved that $T(r, f) \sim T(r, g)$ if f and g are meromorphic functions of finite lower order. We shall prove that (1.1) does not hold in general. More specifically, we shall prove that the constant 8/3 in (1.3) cannot be replaced by any constant less than 2 (and 3/8 cannot be replaced by any constant greater than 1/2).

THEOREM 1. *Let* $\phi(t)$ *be defined for* $t \geqq t_0 > 0$ *and assume that* $\phi(t) \to \infty$ *as* $t \to \infty$. *Then there exist a set* F *satisfying*

$$\int_F \frac{\phi(t)}{t} dt = \infty \tag{1.4}$$

and meromorphic functions f *and* g *that share* 0, 1 *and* ∞ *CM such that*

$$\liminf_{\substack{r \to \infty \\ r \in F}} \frac{T(r, f)}{T(r, g)} \geqq 2 . \tag{1.5}$$

The lemma used in the proof of Theorem 1 applies to another problem. Let f be an entire function, $M(r, f)$ its maximum modulus and $v = v(r, f)$ its central index. Classical theorems due to Wiman [10] and Valiron [9] say that

$$f(z_1) \sim \left(\frac{z_1}{z}\right)^v f(z) \tag{1.6}$$

if z_1 is in a certain neighborhood of z, if $|f(z)| = M(|z|, f)$ and if $|z| = r \notin F$, where F is an exceptional set of finite logarithmic measure, i. e.,

$$\int_F \frac{dt}{t} < \infty . \tag{1.7}$$

The sharpest result of this type seems to be due to Hayman [4]. A simple consequence (cf. [4, 9, 10]) is

$$\lim_{\substack{r \to \infty \\ r \notin F}} \frac{A(r, f)}{M(r, f)} = 1 , \tag{1.8}$$

where $A(r, f) = \max\{\mathrm{Re}\, f(z);\ |z| = r\}$.

Considering a gap series one easily sees that an exceptional set of some form is necessary in (1.6). We show that the condition (1.7) for the exceptional set in (1.8) (and hence in (1.6)) cannot be weakened in a certain sense.

THEOREM 2. *Let* ϕ *be as in Theorem* 1. *There exist a set* F *satisfying* (1.4) *and an entire function* f *such that*

$$\lim_{\substack{r \to \infty \\ r \in F}} \frac{A(r, f)}{M(r, f)} = 0 . \tag{1.9}$$

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2. A fundamental lemma

The key step in the proof of Theorems 1 and 2 is

LEMMA 1. *Let ϕ be as in Theorem 1 and let (A_k) be a sequence of real, continuous, periodic functions with period 2π such that*

$$\int_{-\pi}^{\pi} A_k(\theta)d\theta=0 \qquad (k\in\boldsymbol{N}). \tag{2.1}$$

Let (ε_k) be a sequence of positive real numbers satisfying $\varepsilon_k\to 0$ as $k\to\infty$.

Then there exist a sequence (m_k) of positive integers, sequences (s_k), (s_k') and (M_k) of real numbers which tend to infinity and an entire function h such that

$$\int_{s_k}^{s_k'} \frac{\phi(t)}{t}dt\geqq 1 \tag{2.2}$$

for $k\geqq 2$ and that

$$|\operatorname{Re} h(se^{i\theta})-M_k A_k(m_k\theta)|\leqq M_k\varepsilon_k \tag{2.3}$$

for $s_k\leqq s\leqq s_k'$, $\theta\in\boldsymbol{R}$ and $k\geqq 2$.

Proof. We define

$$a_k(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi} A_k(\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta .$$

Then a_k is holomorphic in $|z|<1$ and $a_k(0)=0$ by (2.1). It follows from Poisson's integral formula that there exists R_k satisfying $R_k<1$ such that

$$|\operatorname{Re} a_k(re^{i\theta})- A_k(\theta)|\leqq\frac{\varepsilon_k}{2} \tag{2.4}$$

for $R_k\leqq r<1$ and $\theta\in\boldsymbol{R}$. We choose R_k' such that $R_k<R_k'<1$ and denote the n-th partial sum of the Maclaurin series of a_k by $p_{k,n}$.

Now we define the sequences (m_k) and (M_k), a sequence (h_k) of polynomials and a real sequence (r_k) by recursion.

We choose $r_1=m_1=M_1=1$, $h_1\equiv 0$ and assume that $k\geqq 2$ and that m_j, M_j, h_j and r_j have been specified for $1\leqq j\leqq k-1$. We denote the degree of h_j by l_j and define $N_k=\max\{l_j;\, 1\leqq j\leqq k-1\}$ and $m_k=8N_k$. Then we choose r_k such that $r_k>r_{k-1}+1$.

$$\left(\frac{r_{k-1}}{r_k}\right)^{m_k}\leqq R_k' , \tag{2.5}$$

$$\frac{\phi(R_k r)}{m_k}\log\frac{R'_k}{R_k}\geqq 1 \qquad (r\geqq r_k) \tag{2.6}$$

and

$$\sum_{j=1}^{k-1} M(r, h_j)\leqq r^{2N_k} \qquad (r\geqq r_k). \tag{2.7}$$

Moreover, we define $M_k=(r_k)^{4N_k}$ and we can achieve by a suitable choice of r_k that

$$1\leqq (M_k)^{1/2}\leqq \frac{\varepsilon_k}{4}M_k \tag{2.8}$$

and

$$M_k M\left(\left(\frac{r_{k-1}}{r_k}\right)^{m_k}, a_k\right)\leqq \frac{1}{2^{k+1}}. \tag{2.9}$$

To see that (2.9) is satisfied if r_k is large enough we note that $M(r, a_k)\leqq C_k r$ for some constant C_k and sufficiently small r since $a_k(0)=0$. This implies together with the definition of m_k and M_k that (2.9) is satisfied if r_k is large enough.

Now we choose n_k such that

$$M_k|p_{k,n_k}(re^{i\theta})-a_k(re^{i\theta})|\leqq \frac{1}{2^{k+1}} \tag{2.10}$$

for $r\leqq R'_k$ and $\theta\in\boldsymbol{R}$. It follows from (2.5), (2.9) and (2.10) that

$$M_k M\left(\left(\frac{r_{k-1}}{r_k}\right)^{m_k}, p_k, n_k\right)\leqq \frac{1}{2^{k+1}}+\frac{1}{2^{k+1}}=\frac{1}{2^k}. \tag{2.11}$$

Finally we define

$$h_k(z)=M_k p_{k,n_k}\left(\left(\frac{z}{r_k}\right)^{m_k}\right). \tag{2.12}$$

Now let (m_k), (M_k), (h_k) and (r_k) be the sequences defined above and let (n_k) and (N_k) be the corresponding auxiliary sequences. We define h by

$$h(z)=\sum_{k=1}^{\infty} h_k(z).$$

We note that $|h_k(z)|\leqq 2^{-k}$ for $|z|\leqq r_{k-1}$ by (2.11) and (2.12) and that $r_k\to\infty$ as $k\to\infty$ since $r_k>r_{k-1}+1$. Hence it follows that the series for h converges uniformly on compact sets so that h is an entire function.

Now we define s_k and s'_k by

$$\left(\frac{s_k}{r_k}\right)^{m_k}=R_k \quad\text{and}\quad \left(\frac{s'_k}{r_k}\right)^{m_k}=R'_k.$$

Then $r_k R_k<s_k<s'_k<r_k$ and

$$\frac{R'_k}{R_k}=\left(\frac{s'_k}{s_k}\right)^{m_k}.$$

It follows that

$$\log\frac{s'_k}{s_k}=\frac{1}{m_k}\log\frac{R'_k}{R_k}$$

and hence that

$$\int_{s_k}^{s'_k}\frac{\phi(t)}{t}dt\geqq\inf_{t\in[s_k,s'_k]}\phi(t)\log\frac{s'_k}{s_k}$$

$$\geqq\inf_{t\geqq r_kR_k}\phi(t)\frac{1}{m_k}\log\frac{R'_k}{R_k}\geqq 1$$

by (2.6).

It remains to prove (2.3). Choose s such that $s_k\leqq s\leqq s'_k$ and define

$$r=\left(\frac{s}{r_k}\right)^{m_k}.$$

Then $R_k\leqq r\leqq R'_k$ and it follows from (2.12), (2.4) and (2.10) that

$$\begin{aligned}(2.13)\qquad |Re\, h_k(se^{i\theta})-M_kA_k(m_k\theta)|&=M_k|Re\, p_{k,n_k}(re^{im_k\theta})-A_k(m_k\theta)|\\ &\leqq M_k|p_{k,n_k}(re^{im_k\theta})-a_k(re^{im_k\theta})|+M_k|Re\, a_k(re^{im_k\theta})-A_k(m_k\theta)|\\ &\leqq\frac{1}{2^{k+1}}+M_k\frac{\varepsilon_k}{2}.\end{aligned}$$

Furthermore we have

$$(2.14)\qquad \sum_{j=1}^{k-1}M(s,h_j)\leqq\sum_{j=1}^{k-1}M(r_k,h_j)\leqq r_k^{2N_k}=(M_k)^{1/2}$$

by (2.7) and from (2.11) and (2.12) we can deduce that

$$(2.15)\qquad M(s,h_j)\leqq M(r_k,h_j)\leqq M(r_{j-1},h_j)\leqq\frac{1}{2^j}$$

for $j\geqq k+1$. Combining (2.13), (2.14), (2.15) and (2.8) we find that

$$\begin{aligned}&|\mathrm{Re}\, h(se^{i\theta})-M_kA_k(m_k\theta)|\\ &\leqq\Big|\sum_{j=1}^{k-1}h_j(se^{i\theta})\Big|+|Re\, h_k(se^{i\theta})-M_kA_k(m_k\theta)|+\Big|\sum_{j=k+1}^{\infty}h_j(se^{i\theta})\Big|\\ &\leqq(M_k)^{1/2}+\frac{1}{2^{k+1}}+M_k\frac{\varepsilon_k}{2}+\sum_{j=k+1}^{\infty}\frac{1}{2^j}\\ &\leqq 2(M_k)^{1/2}+M_k\frac{\varepsilon_k}{2}\leqq M_k\varepsilon_k.\end{aligned}$$

This completes the proof of Lemma 1.

3. Proof of Theorems 1 and 2

The proof of Theorem 1 requires the following lemma.

LEMMA 2. *Let α and β be entire functions and define f by*

$$f(z)=\frac{e^{\alpha(z)}-1}{e^{\beta(z)}-1}.$$

Denote the counting function of the common zeros of $e^{\alpha}-1$ and $e^{\beta}-1$ by $N_0(r)$ and define

$$H(\theta)=\max\{Re\,\alpha(re^{i\theta}),\ Re\,\beta(re^{i\theta}),\ 0\}.$$

Then

$$T(r,\ f)=(1+o(1))\frac{1}{2\pi}\int_0^{2\pi}H(\theta)d\theta-N_0(r) \qquad (r\notin E)$$

where E has finite measure.

Proof. We have

$$N(r,\ f)=N\left(r,\ \frac{1}{e^{\beta}-1}\right)-N_0(r)$$

$$=(1+o(1))T(r,\ e^{\beta})-N_0(r) \qquad (r\notin E).$$

To compute $m(r,\ f)$ we define $S_1=\{\theta;\ |\theta|\leqq\pi,\ Re\,\alpha(re^{i\theta})\leqq 1\}$, $S_2=\{\theta;\ |\theta|\leqq\pi,\ Re\,\alpha(re^{i\theta})>1,\ Re\,\beta(re^{i\theta})\leqq 1\}$ and $S_3=\{\theta;\ |\theta|\leqq\pi,\ Re\,\alpha(re^{i\theta})>1,\ Re\,\beta(re^{i\theta})>1\}$. Then we have

$$\int_{S_1}\log^+|f(re^{i\theta})|d\theta\leqq\int_{S_1}\log^+\frac{1}{|e^{\beta(re^{i\theta})}-1|}d\theta+O(1)$$

$$\leqq m\left(r,\ \frac{1}{e^{\beta}-1}\right)+O(1)=o(T(r,\ e^{\beta})) \qquad (r\notin E).$$

We also have

$$\int_{S_2}\log^+|f(re^{i\theta})|d\theta$$

$$=\int_{S_2}\log^+|e^{\alpha(re^{i\theta})}-1|d\theta+\int_{S_2}\log^+\frac{1}{|e^{\beta(re^{i\theta})}-1|}d\theta+O(1)$$

$$=\int_{S_2}Re^+\alpha(re^{i\theta})d\theta+o(T(r,\ e^{\beta})) \qquad (r\notin E)$$

and

$$\int_{S_3} \log^+|f(re^{i\theta})|\,d\theta=\int_{S_3} Re^+(\alpha(re^{i\theta})-\beta(re^{i\theta}))d\theta+O(1).$$

It follows that

$$\begin{aligned}T(r, f)&=N(r, f)+m(r, f)\\&=N(r, f)+\sum_{j=1}^{3}\frac{1}{2\pi}\int_{S_j}\log^+|f(re^{i\theta})|\,d\theta\\&=(1+o(1))T(r, e^{\beta})+\frac{1}{2\pi}\int_{S_2} Re^+\alpha(re^{i\theta})d\theta\\&\quad+\frac{1}{2\pi}\int_{S_3} Re^+(\alpha(re^{i\theta})-\beta(re^{i\theta}))d\theta-N_0(r) \qquad (r\notin E).\end{aligned}$$

The conclusion follows since

$$\begin{aligned}T(r, e^{\beta})&=\frac{1}{2\pi}\int_{-\pi}^{\pi} Re^+\beta(re^{i\theta})d\theta\\&=\frac{1}{2\pi}\int_{S_1} Re^+\beta(re^{i\theta})d\theta+\frac{1}{2\pi}\int_{S_3} Re^+\beta(re^{i\theta})d\theta+O(1)\end{aligned}$$

and

$$Re^+(\alpha(re^{i\theta})-\beta(re^{i\theta}))+Re^+\beta(re^{i\theta})=\max\{Re\,\alpha(re^{i\theta}),\ Re\,\beta(re^{i\theta})\}$$

for $\theta\in S_3$.

Proof of Theorem 1. We can choose a sequence (A_k) which satisfies the hypothesis of Lemma 1 such that

$$A_k(\theta)=\begin{cases}0 & \text{if } -\pi+2^{-k}\leqq\theta\leqq-\frac{\pi}{2}-2^{-k}\\ -1 & \text{if } -\frac{\pi}{2}+2^{-k}\leqq\theta\leqq\frac{\pi}{2}-2^{-k}\\ 2 & \text{if } \frac{\pi}{2}+2^{-k}\leqq\theta\leqq\pi-2^{-k}\end{cases}$$

and $-1\leqq A_k(\theta)\leqq 2$ for all $\theta\in\boldsymbol{R}$. Let h be the function of Lemma 1 and define $\alpha(z)=h(z)$, $\beta(z)=\overline{h(\bar{z})}$.

$$f(z)=\frac{e^{\alpha(z)}-1}{e^{\beta(z)}-1} \quad\text{and}\quad g(z)=\frac{e^{-\alpha(z)}-1}{e^{-\beta(z)}-1}.$$

Then f and g share 0, 1 and ∞ *CM*. Define

(3.1) $$G=\bigcup_{k=1}^{\infty}[s_k, s_k'].$$

Lemma 2 and a simple computation yield that

$$T(s,\ f)=(1+o(1))M_k-N_0(s)$$

and

$$T(s,\ g)=(1+o(1))\frac{M_k}{2}-N_0(s)$$

if $s_k\leqq s\leqq s_k'$ and if $s\notin E$.

We conclude that

$$T(s,\ f)\geqq(2-o(1))T(s,\ g) \qquad (s\in G\setminus E).$$

Hence (1.5) is satisfied if we define $F=G\setminus E$. Since we may assume without loss of generality that $\phi(t)\leqq t$, we have

$$\int_E \frac{\phi(t)}{t}dt<\infty .$$

On the other hand, we have

$$\int_G \frac{\phi(t)}{t}dt=\infty .$$

Hence (1.4) follows and this completes the proof of Theorem 1.

Proof of Theorem 2. We apply Lemma 1 for a sequence (A_k) which satisfies

$$\lim_{k\to\infty}\frac{\max\{A_k(\theta);\ \theta\in\boldsymbol{R}\}}{-\min\{A_k(\theta);\ \theta\in\boldsymbol{R}\}}=0.$$

If we define G by (3.1), then we find

$$\lim_{\substack{r\to\infty\\ r\in G}}\frac{A(r,\ h)}{B(r,\ h)}=0$$

where $B(r,\ h)=-\min\{Re\,h(z);\ |z|=r\}$. The conclusion follows since $B(r,\ h)\leqq M(r,\ h)$.

Concluding Remarks. Our method does not yield examples of functions f and g that share three values CM such that (1.5) holds for a set F of infinite logarithmic measure. The question how large the set F can be remains open. It does not seem unlikely that this set is always small in some sense and that there exists an unbounded set G such that $T(r,\ f)\sim T(r,\ g)$ for $r\in G$.

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CORNELL UNIVERSITY
DEPARTMENT OF MATHEMATICS
WHITE HALL
ITHACA, NY 14853
U. S. A.

present address:
Lehrstuhl II für Mathematik
RWTH Aachen
Templergraben 55
D-5100 Aachen
Federal Republic of Germany