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On Minimal Doubly Resolving Sets of Circulant Graphs

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Abstract: Consider a simple connected undirected graph $G = (V_G,E_G)$, where V_G represents the vertex set and E_G represents the edge set respectively. A subset B of V_G is called a resolving set if for every two distinct vertices x, y of G there is a vertex v in set B such that $d(x,v) \neq d(y,v)$. The resolving set of minimum cardinality is called metric basis of graph G. This minimal cardinality of metric basis is denoted by $\beta(G)$, and is called metric dimension of G. A subset D of V is called doubly resolving set if for every two vertices x, y of G there are two vertices $u, v \in D$ such that $d(u,x) - d(u,y) \neq d(v,x) - d(v,y)$. A doubly resolving set with minimum cardinality is called minimal doubly resolving set. This minimum cardinality is denoted by $\psi(G)$.

Some partial cases for metric dimension of circulant graph $C_n(1,2,3)$ for $n \geq 12$ has been discussed in [21]. Afterwards, problem of finding metric dimension for circulant graph $C_n(1,2,3)$, $n \geq 12$ has been completely solved by Borchert et al., in [7].

In this paper, we prove that $\psi(C_n(1,2,3)) = \beta(C_n(1,2,3)) = \begin{cases} 4 \text{ if } n \neq 1 (\text{mod } 6), \\ 5 \text{ otherwise.} \end{cases}$

Keywords: resolving set, metric dimension, minimal doubly resolving set, circulant graph.

1. Introduction and preliminary results

Slater and Harary introduced the metric dimension problem independently in [1] and in [2] respectively. This problem has been considered and solved completely/partially for many families of graphs. As an example, one can consult from [10-28]. The applications of metric dimension includes network discovery and verification [3], geographic routing protocols and robot navigation [4], connected joints in graphs, and chemistry.

Consider a simple connected undirected graph $G = (V_G,E_G)$, where V_G and E_G denote the set of vertices and set of edges of G, respectively. The distance between vertices x and y is denoted by $d(x,y)$ which is the length of the shortest path between x and y. We say that a vertex v resolve two vertices x and y of G if $d(x,v) \neq d(y,v)$. A subset B of V_G is called a resolving set if every two distinct vertices of graph G are resolved by some vertex in set B. This concept can be explained in another terminology also, which is as follows:

Consider an ordered subset $B=\{x_1, x_2, ..., x_p\}$ of vertices of G. For an arbitrary vertex y of G, we have the following p-tuple

$$r(y|B)=(d(y,x_1), d(y,x_2), ..., d(y,x_p))$$

which is called the representation of vertex y or vector of metric coordinates of y with respect to B. The set B is called a resolving set if vector of metric coordinates of each vertex with respect to set B is unique. The resolving set of minimum cardinality is called metric basis of graph G. This minimal cardinality of metric basis is denoted by $\beta(G)$, and is called metric dimension of G.

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The notion of doubly resolving set of graph G was introduced by Caceres et al. [10] in the following way. Consider a graph G of order at least 2. Two vertices x, y are said to doubly resolved by vertices x', y' of G if

$$d(x,x') - d(x,y') \neq d(y,x') - d(y,y').$$

An ordered subset $D=\{x_1, x_2, ..., x_q\}$ of V_G is called a doubly resolving set if every two distinct vertices of G are doubly resolved by some two vertices in set D, i.e., for each pair of vertices $x, y \in V_G$ we have

$$r(x|D) - r(y|D) \neq \lambda I,$$

where λ is an integer and I denotes the unit vector (1,1,...,1). Minimal doubly resolving set of graph G is a doubly resolving set with minimal cardinality. This minimum cardinality is denoted by $\psi(G)$. Observe that, if vertices x', y' doubly resolve the vertices x, y, then $d(x,x')-d(x,y') \neq 0$ or $d(y,x')-d(y,y') \neq 0$. This shows that x' or y' resolve x, y, which follows that a doubly resolving set is also a resolving set, hence $\beta(G) \leq \psi(G)$. In this way, these sets constitute a useful tool for obtaining upper bounds on the metric dimension of graphs. The metric dimension problem and minimal doubly resolving set problem are NP-hard. The proofs can be found in [6] and [8], respectively. The problem of finding minimal doubly resolving set for different families of graphs has been studied in [5] and [9].

The circulant graphs constitute an important class of graphs which can be used in the design of local area networks [29]. Let n, m and $t_1, t_2, ..., t_m$ be positive integers, $1 \leq t_i \leq \lfloor n/2 \rfloor$ and $t_i \neq t_j$ for all $1 \leq i \leq j \leq m$. The circulant graph $C_n(t_1,...,t_m)$ can be form by taking set of vertices $V=\{v_1,...,v_n\}$ and set of edges $E=\{v_i v_{i+tj}, 1 \leq i \leq n, 1 \leq j \leq m\}$ with indices taken modulo n. The numbers $t_1,...,t_m$ are called the generators and we say that the edge $v_i v_{i+tj}$ is of type t_j. The graph $C_{12}(1,2,3)$ is shown in Fig. 1 below.

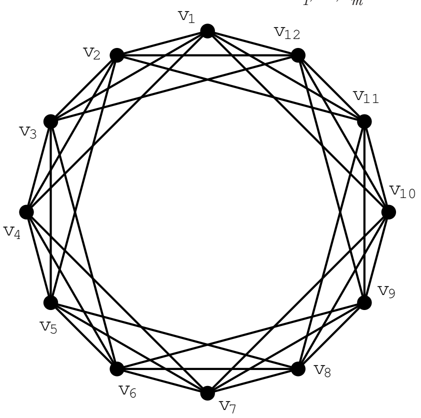

Fig. 1: *The circulant graph* $C_{12}(1,2,3)$.

The circulant graph $C_n(t_1,...,t_m)$ is a regular graph of degree r, where

$$r = \begin{cases} 2m-1 \text{ if } \frac{n}{2} \in \{t_1, \dots, t_m\}, \\ 2m \text{ otherwise.} \end{cases}$$

In this paper, we explicitly determine the minimal doubly resolving sets for circulant graph $C_n(1,2,3)$ for $n \geq 12$. Also we prove that $\beta(C_n(1,2,3)) = \psi(C_n(1,2,3))$ for $n \geq 12$.

2. The minimal doubly resolving sets for circulant graph $C_n(1,2,3)$ for $n \geq 12$

We have

$$\psi(C_n(1,2,3)) \geq \beta(C_n(1,2,3)) = \begin{cases} 4 \text{ if } n \neq 1 (\bmod 6), \\ 5 \text{ otherwise.} \end{cases}$$

for $n \geq 12$, see [7].

Let us denote the set $S_i(v_1) = \{w \in V(C_n(1,2,3)): d(v_1,w)=i\}$, which is the set of vertices in $V(C_n(1,2,3))$ at distance i from v_1. The Table 1 displays the sets $S_i(v_1)$ for circulant graph $C_n(1,2,3)$, where $n \geq 12$.

Table 1: $S_i(v_1)$ *for* $C_n(1,2,3)$ *for* $n \geq 12$.

n	i	$S_i(v_1)$
$6k$ $(k \geq 2)$	0 $1 \leq i \leq k-1$ k $k+1 \leq i$	$\{v_1\}$ $\{v_{3i-1}, v_{3i}, v_{3i+1}, v_{6k+3-3i}, v_{6k+2-3i}, v_{6k+1-3i}\}$ $\{v_{3k-1}, v_{3k}, v_{3k+1}, v_{3k+2}, v_{3k+3}\}$ $\emptyset$
$6k+1$ $(k \geq 2)$	0 $1 \leq i \leq k$ $k+1 \leq i$	$\{v_1\}$ $\{v_{3i-1}, v_{3i}, v_{3i+1}, v_{6k+4-3i}, v_{6k+3-3i}, v_{6k+2-3i}\}$ $\emptyset$
$6k+2$ $(k \geq 2)$	0 $1 \leq i \leq k$ $k+1$ $k+2 \leq i$	$\{v_1\}$ $\{v_{3i-1}, v_{3i}, v_{3i+1}, v_{6k+5-3i}, v_{6k+4-3i}, v_{6k+3-3i}\}$ $\{v_{3k+2}\}$ $\emptyset$
$6k+3$ $(k \geq 2)$	0 $1 \leq i \leq k$ $k+1$ $k+2 \leq i$	$\{v_1\}$ $\{v_{3i-1}, v_{3i}, v_{3i+1}, v_{6k+6-3i}, v_{6k+5-3i}, v_{6k+4-3i}\}$ $\{v_{3k+2}, v_{3k+3}\}$ $\emptyset$
$6k+4$ $(k \geq 2)$	0 $1 \leq i \leq k$ $k+1$ $k+2 \leq i$	$\{v_1\}$ $\{v_{3i-1}, v_{3i}, v_{3i+1}, v_{6k+7-3i}, v_{6k+6-3i}, v_{6k+5-3i}\}$ $\{v_{3k+2}, v_{3k+3}, v_{3k+4}\}$ $\emptyset$
$6k+5$ $(k \geq 2)$	0 $1 \leq i \leq k$ $k+1$ $k+2 \leq i$	$\{v_1\}$ $\{v_{3i-1}, v_{3i}, v_{3i+1}, v_{6k+8-3i}, v_{6k+7-3i}, v_{6k+6-3i}\}$ $\{v_{3k+2}, v_{3k+3}, v_{3k+4}, v_{3k+5}\}$ $\emptyset$

Theorem 2.1. $\psi(C_n(1,2,3)) = 4$ for $n \equiv 0, 2, 3, 4, 5 \pmod 6$, $n \geq 12$.

Proof. We need to show that $\psi(C_n(1,2,3)) \leq 4$ for $n \equiv 0, 2, 3, 4, 5 \pmod 6$, $n \geq 12$. So it suffices to find a doubly resolving set of cardinality 4 in each case. Let us first consider the case when $n \equiv 0 \pmod 6$, i.e., $n = 6k$ for $k \geq 2$. Using the sets $S_i(v_1)$ from Table 1, the Table 2 displays the vectors of metric coordinates of every vertex of $C_n(1,2,3)$ with respect to the set $D^* = \{v_1, v_3, v_5, v_{3k+1}\}$.

Table 2: *Vectors of metric coordinates for $C_n(1,2,3)$ for $n=6k$, $k \geq 3$.*

i	$S_i(v_1)$	metric coordinates w.r.t. $D^* = \{v_1, v_3, v_5, v_{3k+1}\}$
0	v_1	$(0, 1, 2, k)$
1	v_2	$(1, 1, 1, k)$
	v_3	$(1, 0, 1, k)$
	v_4	$(1, 1, 1, k\text{-}1)$
	v_{6k}	$(1, 1, 2, k)$
	v_{6k-1}	$(1, 2, 2, k)$
	v_{6k-2}	$(1, 2, 3, k\text{-}1)$
$2 \leq i \leq k\text{-}2$ ($k \geq 4$)	v_{3i-1}	$(i, i\text{-}1, i\text{-}2, k+1\text{-}i)$
	v_{3i}	$(i, i\text{-}1, i\text{-}1, k+1\text{-}i)$
	v_{3i+1}	$(i, i, i\text{-}1, k\text{-}i)$
	$v_{6k+3-3i}$	$(i, i, i+1, k+1\text{-}i)$
	$v_{6k+2-3i}$	$(i, i+1, i+1, k+1\text{-}i)$
	$v_{6k+1-3i}$	$(i, i+1, i+2, k\text{-}i)$
$k\text{-}1$	v_{3k-4}	$(k\text{-}1, k\text{-}2, k\text{-}3, 2)$
	v_{3k-3}	$(k\text{-}1, k\text{-}2, k\text{-}2, 2)$
	v_{3k-2}	$(k\text{-}1, k\text{-}1, k\text{-}2, 1)$
	v_{3k+6}	$(k\text{-}1, k\text{-}1, k, 2)$
	v_{3k+5}	$(k\text{-}1, k, k, 2)$
	v_{3k+4}	$(k\text{-}1, k, k, 1)$
k	v_{3k-1}	$(k, k\text{-}1, k\text{-}2, 1)$
	v_{3k}	$(k, k\text{-}1, k\text{-}1, 1)$
	v_{3k+1}	$(k, k, k\text{-}1, 0)$
	v_{3k+2}	$(k, k, k\text{-}1, 1)$
	v_{3k+3}	$(k, k, k, 1)$

Note that for $k = 2$, it can be checked directly that $\{v_1, v_3, v_5, v_7\}$ is a doubly resolving set.

In the same way, using Table 1, the Tables 3 - 6 display the vectors of metric coordinates of vertices of $C_n(1,2,3)$ for $n \equiv 2, 3, 4, 5 \pmod 6$ with respect to the set $D^*=\{v_1, v_2, v_{3k+1}, v_{3k+4}\}$, $D^*=\{v_1, v_2, v_{3k+1}, v_{3k+5}\}$, $D^*=\{v_1, v_2, v_{3k+1}, v_{6k+4}\}$ and $D^*=\{v_1, v_2, v_{3k+1}, v_{3k+7}\}$, respectively.

From Tables 2 to 6, it can be verified directly that if two vertices x, y belongs to $S_i(v_1)$ for some i, then

$$r(x|D^*) - r(y|D^*) \neq 0I,$$

where I denotes the unit vector. Also if $x \in S_i(v_1)$ and $x \in S_j(v_1)$ for $i \neq j$, then

$$r(x|D^*) - r(y|D^*) \neq (i-j)I.$$

Table 3: *Vectors of metric coordinates for $C_n(1,2,3)$ for $n=6k+2$, $k \geq 2$.*

i	$S_i(v_1)$	metric coordinates w.r.t. $D^* = \{v_1, v_2, v_{3k+1}, v_{3k+4}\}$
0	v_1	$(0, 1, k, k)$
1	v_2	$(1, 0, k, k)$
	v_3	$(1, 1, k, k+1)$
	v_4	$(1, 1, k\text{-}1, k)$
	v_{6k+2}	$(1, 1, k+1, k)$
	v_{6k+1}	$(1, 1, k, k\text{-}1)$
	v_{6k}	$(1, 2, k, k\text{-}1)$
$2 \leq i \leq k\text{-}1$ ($k \geq 3$)	v_{3i-1}	$(i, i\text{-}1, k+1\text{-}i, k+2\text{-}i)$
	v_{3i}	$(i, i, k+1\text{-}i, k+2\text{-}i)$
	v_{3i+1}	$(i, i, k\text{-}i, k+1\text{-}i)$
	$v_{6k+5-3i}$	$(i, i, k+2\text{-}i, k+1\text{-}i)$
	$v_{6k+4-3i}$	$(i, i, k+1\text{-}i, k\text{-}i)$
	$v_{6k+3-3i}$	$(i, i+1, k+1\text{-}i, k\text{-}i)$
k	v_{3k-1}	$(k, k\text{-}1, 1, 2)$
	v_{3k}	$(k, k, 1, 2)$
	v_{3k+1}	$(k, k, 0, 1)$
	v_{3k+5}	$(k, k, 2, 1)$
	v_{3k+4}	$(k, k, 1, 0)$
	v_{3k+3}	$(k, k+1, 1, 1)$
$k+1$	v_{3k+2}	$(k+1, k, 1, 1)$

Table 4: *Vectors of metric coordinates for $C_n(1,2,3)$ for $n=6k+3$, $k \geq 2$.*

i	$S_i(v_1)$	metric coordinates w.r.t. $D^* = \{v_1, v_2, v_{3k+1}, v_{3k+5}\}$
0	v_1	$(0, 1, k, k)$
1	v_2	$(1, 0, k, k)$
	v_3	$(1, 1, k, k+1)$
	v_4	$(1, 1, k\text{-}1, k+1)$
	v_{6k+3}	$(1, 1, k+1, k)$
	v_{6k+2}	$(1, 1, k+1, k\text{-}1)$
	v_{6k+1}	$(1, 2, k, k\text{-}1)$
$2 \leq i \leq k\text{-}1$ ($k \geq 3$)	v_{3i-1}	$(i, i\text{-}1, k+1\text{-}i, k+2\text{-}i)$
	v_{3i}	$(i, i, k+1\text{-}i, k+2\text{-}i)$
	v_{3i+1}	$(i, i, k\text{-}i, k+2\text{-}i)$
	$v_{6k+6-3i}$	$(i, i, k+2\text{-}i, k+1\text{-}i)$
	$v_{6k+5-3i}$	$(i, i, k+2\text{-}i, k\text{-}i)$
	$v_{6k+4-3i}$	$(i, i+1, k+1\text{-}i, k\text{-}i)$
k	v_{3k-1}	$(k, k\text{-}1, 1, 2)$
	v_{3k}	$(k, k, 1, 2)$
	v_{3k+1}	$(k, k, 0, 2)$
	v_{3k+6}	$(k, k, 2, 1)$
	v_{3k+5}	$(k, k, 2, 0)$
	v_{3k+4}	$(k, k+1, 1, 1)$
$k+1$	v_{3k+2}	$(k+1, k, 1, 1)$
	v_{3k+3}	$(k+1, k+1, 1, 1)$

Table 5: *Vectors of metric coordinates for $C_n(1,2,3)$ for $n=6k+4$, $k\geq 2$.*

i	$S_i(v_1)$	metric coordinates w.r.t. $D^*=\{v_1, v_2, v_{3k+1}, v_{6k+4}\}$
0	v_1	$(0, 1, k, 1)$
$1\leq i\leq k$	v_{3i-1} v_{3i} v_{3i+1} $v_{6k+7-3i}$ $v_{6k+6-3i}$ $v_{6k+5-3i}$	$(i, i\text{-}1, k+1\text{-}i, i)$ $(i, i, k+1\text{-}i, i)$ $(i, i, k\text{-}i, i+1)$ $(i, i, k+2\text{-}i, i\text{-}1)$ $(i, i, k+2\text{-}i, i)$ $(i, i+1, k+2\text{-}i, i)$
$k+1$	v_{3k+2} v_{3k+3} v_{3k+4}	$(k+1, k, 1, k+1)$ $(k+1, k+1, 1, k+1)$ $(k+1, k+1, 1, k)$

Table 6: *Vectors of metric coordinates for $C_n(1,2,3)$ for $n=6k+5$, $k\geq 2$.*

i	$S_i(v_1)$	metric coordinates w.r.t. $D^*=\{v_1, v_2, v_{3k+1}, v_{3k+7}\}$
0	v_1	$(0, 1, k, k)$
1	v_2 v_3 v_4 v_{6k+5} v_{6k+4} v_{6k+3}	$(1, 0, k, k)$ $(1, 1, k, k+1)$ $(1, 1, k\text{-}1, k+1)$ $(1, 1, k+1, k)$ $(1, 1, k+1, k\text{-}1)$ $(1, 2, k+1, k\text{-}1)$
$2\leq i\leq k\text{-}1$ $(k\geq 3)$	v_{3i-1} v_{3i} v_{3i+1} $v_{6k+8-3i}$ $v_{6k+7-3i}$ $v_{6k+6-3i}$	$(i, i\text{-}1, k+1\text{-}i, k+3\text{-}i)$ $(i, i, k+1\text{-}i, k+3\text{-}i)$ $(i, i, k\text{-}i, k+2\text{-}i)$ $(i, i, k+3\text{-}i, k+1\text{-}i)$ $(i, i, k+2\text{-}i, k\text{-}i)$ $(i, i+1, k+2\text{-}i, k\text{-}i)$
k	v_{3k-1} v_{3k} v_{3k+1} v_{3k+8} v_{3k+7} v_{3k+6}	$(k, k\text{-}1, 1, 3)$ $(k, k, 1, 3)$ $(k, k, 0, 2)$ $(k, k, 3, 1)$ $(k, k, 2, 0)$ $(k, k+1, 2, 1)$
$k+1$	v_{3k+2} v_{3k+3} v_{3k+4} v_{3k+5}	$(k+1, k, 1, 2)$ $(k+1, k+1, 1, 2)$ $(k+1, k+1, 1, 1)$ $(k+1, k+1, 2, 1)$

Thus $D^*=\{v_1, v_3, v_5, v_{3k+1}\}$, $D^*=\{v_1, v_2, v_{3k+1}, v_{3k+4}\}$, $D^*=\{v_1, v_2, v_{3k+1}, v_{3k+5}\}$, $D^*=\{v_1, v_2, v_{3k+1}, v_{6k+4}\}$ and $D^*=\{v_1, v_2, v_{3k+1}, v_{3k+7}\}$ are doubly resolving sets (indeed minimal doubly resolving sets) of $C_n(1,2,3)$ for $n \equiv 0, 2, 3, 4, 5 \pmod 6$ respectively. Hence Theorem 2.1 holds.

Theorem 2.2. $\psi(C_n(1,2,3)) = 5$ for $n \equiv 1 \pmod 6$, $n\geq 12$.

Proof. The proof is same as the proof of Theorem 2.1. We need to show that $\psi(C_n(1,2,3)) \leq 5$ for $n \equiv 1 \pmod 6$, $n \geq 12$. For this, consider $n=6k+1$, $k \geq 2$, so the target is to find a doubly resolving set of cardinality 5 for each k. It can be proved by a direct check that $D^*=\{v_1, v_3, v_5, v_6, v_8\}$ and $D^*=\{v_1, v_3, v_5, v_6, v_{11}\}$ are doubly resolving sets corresponding to $k = 2$ and $k = 3$, respectively. Using the sets $S_i(v_1)$ from Table 1, the Table 7 displays the vectors of metric coordinates of every vertex of $C_n(1,2,3)$, with respect to the set $D^*=\{v_1, v_3, v_5, v_6, v_{3k+2}\}$.

Table 7: *Vectors of metric coordinates for $C_n(1,2,3)$ for $n=6k+1$, $k\geq 4$.*

i	$S_i(v_1)$	metric coordinates w.r.t. $D^*=\{v_1, v_3, v_5, v_6, v_{3k+2}\}$
0	v_1	$(0, 1, 2, 2, k)$
1	v_2 v_3 v_4 v_{6k+1} v_{6k} v_{6k-1}	$(1, 1, 1, 2, k)$ $(1, 0, 1, 1, k)$ $(1, 1, 1, 1, k)$ $(1, 1, 2, 2, k)$ $(1, 2, 2, 3, k)$ $(1, 2, 3, 3, k\text{-}1)$
2	v_5 v_6 v_7 v_{6k-2} v_{6k-3} v_{6k-4}	$(2, 1, 0, 1, k\text{-}1)$ $(2, 1, 1, 0, k\text{-}1)$ $(2, 2, 1, 1, k\text{-}1)$ $(2, 2, 3, 3, k\text{-}1)$ $(2, 3, 3, 4, k\text{-}1)$ $(2, 3, 4, 4, k\text{-}2)$
$3\leq i\leq k\text{-}2$ $(k\geq 5)$	v_{3i-1} v_{3i} v_{3i+1} $v_{6k+4-3i}$ $v_{6k+3-3i}$ $v_{6k+2-3i}$	$(i, i\text{-}1, i\text{-}2, i\text{-}2, k+1\text{-}i)$ $(i, i\text{-}1, i\text{-}1, i\text{-}2, k+1\text{-}i)$ $(i, i, i\text{-}1, i\text{-}1, k+1\text{-}i)$ $(i, i, i+1, i+1, k+1\text{-}i)$ $(i, i+1, i+1, i+2, k+1\text{-}i)$ $(i, i+1, i+2, i+2, k\text{-}i)$
$k\text{-}1$	v_{3k-4} v_{3k-3} v_{3k-2} v_{3k+7} v_{3k+6} v_{3k+5}	$(k\text{-}1, k\text{-}2, k\text{-}3, k\text{-}3, 2)$ $(k\text{-}1, k\text{-}2, k\text{-}2, k\text{-}3, 2)$ $(k\text{-}1, k\text{-}1, k\text{-}2, k\text{-}2, 2)$ $(k\text{-}1, k\text{-}1, k, k, 2)$ $(k\text{-}1, k, k, k, 2)$ $(k\text{-}1, k, k, k, 1)$
k	v_{3k-1} v_{3k} v_{3k+1} v_{3k+4} v_{3k+3} v_{3k+2}	$(k, k\text{-}1, k\text{-}2, k\text{-}2, 1)$ $(k, k\text{-}1, k\text{-}1, k\text{-}2, 1)$ $(k, k, k\text{-}1, k\text{-}1, 1)$ $(k, k, k, k, 1)$ $(k, k, k, k\text{-}1, 1)$ $(k, k, k\text{-}1, k\text{-}1, 0)$

From Table 7, it can be seen that the difference between vectors of metric coordinates, of any two chosen vertices, is not an integer multiple of unit vector I. Therefore, the set $D^*=\{v_1, v_3, v_5, v_6, v_{3k+2}\}$ is doubly resolving set (indeed minimal doubly resolving set) of the circulant graph $C_n(1,2,3)$ and hence Theorem 2.2 holds.

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