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ON MINIMAL LAGRANGIAN SURFACES IN THE PRODUCT OF RIEMANNIAN TWO MANIFOLDS

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Abstract. Let (Σ_1, g_1) and (Σ_2, g_2) be connected, complete and orientable 2-dimensional Riemannian manifolds. Consider the two canonical Kähler structures $(G^\varepsilon, J, \Omega^\varepsilon)$ on the product 4-manifold $\Sigma_1 \times \Sigma_2$ given by $G^\varepsilon = g_1 \oplus \varepsilon g_2$, $\varepsilon = \pm 1$ and J is the canonical product complex structure. Thus for $\varepsilon = 1$ the Kähler metric G^+ is Riemannian while for $\varepsilon = -1$, G^- is of neutral signature. We show that the metric G^ε is locally conformally flat if and only if the Gauss curvatures $\kappa(g_1)$ and $\kappa(g_2)$ are both constants satisfying $\kappa(g_1) = -\varepsilon\kappa(g_2)$. We also give conditions on the Gauss curvatures for which every G^ε-minimal Lagrangian surface is the product $\gamma_1 \times \gamma_2 \subset \Sigma_1 \times \Sigma_2$, where γ_1 and γ_2 are geodesics of (Σ_1, g_1) and (Σ_2, g_2), respectively. Finally, we explore the Hamiltonian stability of projected rank one Hamiltonian G^ε-minimal surfaces.

1. Introduction. A submanifold of a symplectic manifold is said to be *Lagrangian* if it is half the ambient dimension and the symplectic form vanishes on it. A Lagrangian submanifold of a pseudo-Riemannian manifold is said to be *minimal* if it is a critical point of the volume functional associated with pseudo-Riemannian metric. A minimal submanifold is characterized by the vanishing of the trace of its second fundamental form, the *mean curvature*. Recently, an interest in minimal Lagrangian submanifolds in pseudo-Riemannian Kähler structures has grown amongst geometers [2], [20], while minimal Lagrangian submanifolds in Calabi-Yau manifolds are of great interest in theoretical physics because of their close relationship to the mirror symmety [19]. In addition, the space $\mathbb{L}(\mathbb{M}^3)$ of oriented geodesics in a 3-dimensional space form $(\mathbb{M}^3, g)$ admits a natural Kähler structure where the metric G is of neutral signature, scalar flat and locally conformally flat [1], [3], [11], [12].

The significance of these structures is that the identity component of the isometry group of G is isomorphic with the identity component of the isometry group of g. Moreover, Salvai has proved that the neutral Kähler metrics on $\mathbb{L}(\mathbb{E}^3)$ and $\mathbb{L}(\mathbb{H}^3)$ are the unique metrics with this property [16], [17]. The neutral Kähler structure on $\mathbb{L}(\mathbb{M}^3)$ plays an important role in the surface theory in $(\mathbb{M}^3, g)$. In particular, if S is a smoothly immersed surface in M, the set of oriented geodesics normal to S forms a Lagrangian surface in $\mathbb{L}(\mathbb{M}^3)$. A Lagrangian surface Σ in $\mathbb{L}(\mathbb{M}^3)$ is G-minimal if and only if Σ is locally the set of normal oriented geodesics of an equidistant tube along a geodesic in M [3], [6], [10].

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Oh in [14] has introduced a natural variational problem, apart from the classical variational problem of minimizing the volume functional in a homology class, consisting of minimizing the volume with respect to Hamiltonian compactly supported variations. An important property of these variations is that they preserve the Lagrangian constraint. A Lagrangian submanifold in a Kähler or a pseudo-Kähler manifold is said to be *a Hamiltonian minimal submanifold* if it is a critical point of the volume functional with respect to Hamiltonian compactly supported variations. A Hamiltonian minimal submanifold can be characterized by its mean curvature vector being the divergence-free.

For example, in the space $\mathbb{L}(\mathbb{E}^3)$ of oriented lines in the Euclidean 3-space, a Hamiltonian minimal surface is the set of oriented lines normal to a surface $S \subset \mathbb{E}^3$ that is a critical point of the functional

$$\mathcal{F}(S) = \int_S \sqrt{H^2 - K} dA ,$$

where H, K denote the mean and the Gauss curvatures of S, respectively [6]. The neutral Kähler structures on the space of oriented great circles in the three sphere $\mathbb{S}^3$ and the space of oriented space-like geodesics in the anti De Sitter 3-space Ad$\mathbb{S}^3$ can both be identified with the product structures, $\mathbb{L}(\mathbb{S}^3) = \mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{L}^+(\mathrm{Ad}\mathbb{S}^3) = \mathbb{H}^2 \times \mathbb{H}^2$.

More generally, one is led to consider the Kähler structures derived by the product structure of $\Sigma_1 \times \Sigma_2$, where (Σ_1, g_1) and (Σ_2, g_2) are complete, connected, orientable Riemannian 2-manifolds.

Let ω_1 and ω_2 be the symplectic two forms of (Σ_1, g_1) and (Σ_2, g_2), respectively, and j_1 and j_2 their complex structures as Riemann surfaces. For $\varepsilon = 1$ or -1, consider the product structures of the four-dimensional manifold $\Sigma_1 \times \Sigma_2$ endowed with the product metrics $G^\varepsilon = \pi_1^* g_1 + \varepsilon \pi_1^* g_2$, the almost complex structure $J = j_1 \oplus j_2$ and the symplectic two forms $\Omega^\varepsilon = \pi_1^* \omega_1 + \varepsilon \pi_2^* \omega_2$, where π_i are the projections of $\Sigma_1 \times \Sigma_2$ onto Σ_i, $i = 1, 2$. The quadruples $(\Sigma_1 \times \Sigma_2, G^\varepsilon, J, \Omega^\varepsilon)$ are easily seen to be 4-dimensional Kähler structures.

In this paper we study G^ε-minimal Lagrangian surfaces in the Kähler 4-manifold $(\Sigma_1 \times \Sigma_2, G^\varepsilon, J, \Omega^\varepsilon)$. In Section 2 we prove:

THEOREM 1. *The Kähler metric G^+ is Riemannian while the Kähler metric G^- is neutral. Moreover, the Kähler metric G^ε is conformally flat if and only if the Gauss curvatures $\kappa(g_1)$ and $\kappa(g_2)$ are both constants with $\kappa(g_1) = -\varepsilon\kappa(g_2)$.*

In Section 3, we first define the *projected rank* (see Definition 3.1) of a surface in $\Sigma_1 \times \Sigma_2$ and we prove that every Lagrangian surface is either of projected rank one or of projected rank two.

For the projected rank one case, we classify all Hamiltonian G^ε-minimal surfaces:

THEOREM 2. *Every projected rank one Lagrangian surface can be locally parametrised by $\Phi : S \to \Sigma_1 \times \Sigma_2 : (s, t) \mapsto (\phi(s), \psi(t))$, where ϕ and ψ are regular curves on Σ and the induced metric $\Phi^* G^\varepsilon$ is flat. Φ is Hamiltonian G^ε-minimal if and only if ϕ and ψ are*

Cornu spirals of parameters λ_ϕ *and* λ_ψ, *respectively, such that*

$$\lambda_\phi = -\varepsilon \lambda_\psi \,.$$

Φ *is a* G^ε*-minimal Lagrangian if and only if both* ϕ *and* ψ *are geodesics. Furthermore, every projected rank one* G^ε*-minimal Lagrangian surface in* $\Sigma_1 \times \Sigma_2$ *is totally geodesic.*

In the same section, the following theorem gives the conditions for the non-existence of projected rank two G^ε-minimal Lagrangian surfaces:

THEOREM 3. *Let* (Σ_1, g_1) *and* (Σ_2, g_2) *be Riemannian two manifolds and let* $(G^\varepsilon, J, \Omega^\varepsilon)$ *be the canonical Kähler product structures on* $\Sigma_1 \times \Sigma_2$. *Let* $\kappa(g_1), \kappa(g_2)$ *be the Gauss curvatures of* g_1 *and* g_2, *respectively. Assume that either of the following hold:*

(i) *The metrics* g_1 *and* g_2 *are both generically non-flat and* $\varepsilon\kappa(g_1)\kappa(g_2) < 0$ *away from flat points.*

(ii) *Only one of the metrics* g_1 *and* g_2 *is flat while the other is non-flat generically. Then every* G^ε*-minimal Lagrangian surface is of projected rank one.*

Here a generic property is one that holds almost everywhere. Note that Theorem 3.5 is no longer true when (Σ_1, g_1) and (Σ_2, g_2) are both flat, since there exist projected rank two minimal Lagrangian immersions in the complex Euclidean space $\mathbb{C}^2$ endowed with the pseudo-Hermitian product structure [6].

Minimality is the first order condition for a submanifold to be volume-extremizing in its homology class. Harvey and Lawson [13] have proven that minimal Lagrangian submanifolds of a Calabi-Yau manifold is calibrated, which implies by Stokes theorem, that are volume-extremizing. The second order condition for a minimal submanifold to be volume-extremizing was first derived by Simons [18].

Minimal submanifolds that are local extremizers of the volume are called *stable minimal submanifolds*. The stability of a minimal submanifold is determined by the monotonicity of the second variation of the volume functional. If the second variation of the volume functional of a Hamiltonian minimal submanifold is monotone for any Hamiltonian compactly supported variation, it is said to be *Hamiltonian stable*. In [14] and [15], the second variation formula of a Hamiltonian minimal submanifold has been derived in the case of a Kähler manifold, while for the pseudo-Kähler case it has been derived in [5].

The following theorem in Section 4 investigates the Hamiltonian stability of projected rank one Hamiltonian G^ε-minimal surfaces in $\Sigma_1 \times \Sigma_2$:

THEOREM 4. *Let* $\Phi = (\phi, \psi)$ *be of projected rank one Hamiltonian* G^ε*-minimal immersion in* $(\Sigma_1 \times \Sigma_2, G^\varepsilon)$ *such that* $\kappa(g_1) \leq -2k_\phi^2$ *and* $\kappa(g_2) \leq -2k_\psi^2$ *along the curves* ϕ *and* ψ *respectively. Then* Φ *is a local minimizer of the volume in its Hamiltonian isotopy class.*

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2. The product Kähler structure. Consider the Riemannian 2-manifolds (Σ_k, g_k) for $k = 1, 2$ and denote by j_k the rotation by an angle $+\pi/2$ in $T\Sigma_k$. Set $\omega_k(\cdot, \cdot) = g_k(j_k\cdot, \cdot)$ so that the quadruples $(\Sigma_k, g_k, j_k, \omega_k)$ are 2-dimensional Kähler manifolds.

Using the following identification,

$$X \in T(\Sigma_1 \times \Sigma_2) \simeq (X_1, X_2) \in T\Sigma_1 \oplus T\Sigma_2\,, \quad \text{where} \quad X_k \in T\Sigma_k\,,$$

we obtain the natural splitting $T(\Sigma_1 \times \Sigma_2) = T\Sigma_1 \oplus T\Sigma_2$. Let $(x, y) \in \Sigma_1 \times \Sigma_2$ and $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ be two tangent vectors in $T_{(x,y)}(\Sigma_1 \times \Sigma_2)$. Define the metric $G^{\varepsilon}_{(x,y)}$ by:

$$G^{\varepsilon}_{(x,y)}(X, Y) = g_1(X_1, Y_1)(x) + \varepsilon g_2(X_2, Y_2)(y)\,,$$

where $\varepsilon = 1$ or -1. The Levi-Civita connection ∇ with respect to the metric G^{ε} is

$$\nabla_X Y = (D^1_{X_1} Y_1, D^2_{X_2} Y_2)\,,$$

where $X = (X_1, X_2), Y = (Y_1, Y_2)$ are vector fields in $\Sigma_1 \times \Sigma_2$ and D^1, D^2 denote the Levi-Civita connections with respect to g_1 and g_2, respectively.

Consider the endomorphism $J \in \mathrm{End}(T\Sigma_1 \oplus T\Sigma_2)$ defined by $J = j_1 \oplus j_2$, i.e., $J(X) = (j_1 X_1, j_2 X_2)$. Clearly, J is an almost complex structure on $\Sigma_1 \times \Sigma_2$.

PROPOSITION 2.1. *The almost complex structure J is integrable.*

PROOF. The Nijenhuis tensor N_J is

$$N_J(X, Y) = [JX, JY]^{\nabla} - J[JX, Y]^{\nabla} - J[JX, Y]^{\nabla} - [X, Y]^{\nabla}\,,$$

where $X = (X_1, X_2), Y = (Y_1, Y_2)$ are vector fields in $\Sigma_1 \times \Sigma_2$ and $[\cdot, \cdot]^{\nabla}$ denotes the Lie bracket with respect to the Levi-Civita connection ∇. Then

$$[X, Y]^{\nabla} = ([X_1, Y_1]^{D^1}, [X_2, Y_2]^{D^2})\,,$$

where $[\cdot, \cdot]^{D^i}$ are the Lie brackets with respect to the Levi-Civita connections D^i. Thus,

$$\begin{aligned} N_J(X, Y) &= [JX, JY]^{\nabla} - J[JX, Y]^{\nabla} - J[JX, Y]^{\nabla} - [X, Y]^{\nabla} \\ &= (N_{j_1}(X_1, Y_1), N_{j_2}(X_2, Y_2))\,, \end{aligned}$$

and the proposition follows. □

Let $\pi_i : \Sigma_1 \times \Sigma_2 \to \Sigma_i$ be the i-th projection, and define the following two-forms

$$\Omega^{\varepsilon} = \pi_1^* \omega_1 + \varepsilon \pi_2^* \omega_2\,.$$

THEOREM 2.2. *The quadruples $(\Sigma_1 \times \Sigma_2, G^{\varepsilon}, J, \Omega^{\varepsilon})$ are 4-dimensional Kähler structures. The Kähler metric G^{ε} is conformally flat if and only if the Gauss curvatures $\kappa(g_1)$ and $\kappa(g_2)$ are both constants with $\kappa(g_1) = -\varepsilon\kappa(g_2)$.*

PROOF. We have already seen that the almost complex structure J is integrable. It is obvious that Ω^{ε} is closed, i.e., $d\Omega^{\varepsilon} = 0$ and hence a symplectic form on $\Sigma_1 \times \Sigma_2$.

Moreover, J is compatible with Ω^{ε} since for $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$, we have

$$\Omega^{\varepsilon}_{(x,y)}(JX, JY) = \Omega^{\varepsilon}_{(x,y)}((j_1 X_1, j_1 X_2), (j_2 Y_1, j_2 Y_2))$$

$$
\begin{aligned}
&= \omega_1(j_1X_1, j_1Y_1)(x) + \varepsilon\omega_2(j_2X_2, j_2Y_2)(y) \\
&= \omega_1(X_1, Y_1)(x) + \varepsilon\omega_2(X_2, Y_2)(y) \\
&= \Omega^{\varepsilon}_{(x,y)}(X, Y) .
\end{aligned}
$$

We proceed with the proof by considering the cases of G^+ and G^-.

THE CASE OF G^+: Assume that (e_1, e_2) and (v_1, v_2) are orthonormal frames on Σ_1 and Σ_2 respectively, both oriented in such a way $j_1e_1 = e_2$ and $j_2v_1 = v_2$. Consider the orthonormal frame (E_1, E_2, E_3, E_4) of $(\Sigma_1 \times \Sigma_2, G^+)$ defined by

$$
E_1 = \frac{1}{\sqrt{3}}(e_1, v_1 + v_2) , \quad E_2 = JE_1 = \frac{1}{\sqrt{3}}(e_2, v_2 - v_1)
$$

$$
E_3 = \frac{1}{\sqrt{3}}(e_1 - e_2, -v_1) , \quad E_4 = JE_3 = \frac{1}{\sqrt{3}}(e_1 + e_2, -v_2) .
$$

If Ric^+ denotes the Ricci curvature tensor with respect to the metric G^+, we have

$$
\mathrm{Ric}^+(E_1, E_1)_{(x,y)} = \mathrm{Ric}^+(E_2, E_2)_{(x,y)} = \frac{\kappa(g_1)(x) + 2\kappa(g_2)(y)}{3} ,
$$

$$
\mathrm{Ric}^+(E_3, E_3)_{(x,y)} = \mathrm{Ric}^+(E_4, E_4)_{(x,y)} = \frac{2\kappa(g_1)(x) + \kappa(g_2)(y)}{3} ,
$$

and the scalar curvatute R^+ is:

$$
R^+ = \sum_{i=1}^{4} Ric^+(E_i, E_i) = 2(\kappa(g_1)(x) + \kappa(g_2)(y)) . \tag{1}
$$

If G^ε is conformally flat, it is scalar flat [9] and thus, from (1), the Gauss curvatures $\kappa(g_1)$, $\kappa(g_2)$ are constants with $\kappa(g_1) = -\kappa(g_2)$.

Conversely, suppose that

$$
\kappa(g_1) = -\kappa(g_2) = c , \tag{2}
$$

where c is a real constant. Consider the corresponding coframe $\mathcal{B}_+ = (e_1, e_2, e_3, e_4)$ of the orthonormal frame (E_1, E_2, E_3, E_4). The Hodge star operator $* : \Lambda^2(\Sigma_1 \times \Sigma_2) \to \Lambda^2(\Sigma_1 \times \Sigma_2)$ defined by

$$
a \wedge *b = G^+(a, b)\mathrm{Vol} ,
$$

splits the bundle of 2-forms $\Lambda^2(\Sigma_1 \times \Sigma_2)$ into:

$$
\Lambda^2(\Sigma_1 \times \Sigma_2) = \Lambda^2_+(\Sigma_1 \times \Sigma_2) \oplus \Lambda^2_-(\Sigma_1 \times \Sigma_2) ,
$$

where $\Lambda^2_+(\Sigma_1 \times \Sigma_2)$, $\Lambda^2_-(\Sigma_1 \times \Sigma_2)$ are the self-dual and the anti-self-dual 2-form bundles, respectively, and $\mathrm{Vol} = e_1 \wedge e_2 \wedge e_3 \wedge e_4$ is the volume element.

With respect to this splitting the Riemann curvature operator $\mathcal{R} : \Lambda^2(\Sigma_1 \times \Sigma_2) \to \Lambda^2(\Sigma_1 \times \Sigma_2)$ defined by

$$
\mathcal{R}(e_i \wedge e_j)e_k \wedge e_l = G(R(E_i, E_j)E_k, E_l) ,
$$

is decomposed by:

$$\mathcal{R} = \begin{pmatrix} W^+ + \frac{R^+}{12} I & Z \\ Z^* & W^- + \frac{R^+}{12} I \end{pmatrix},$$

where $W^\pm : \Lambda^2_\pm(\Sigma_1 \times \Sigma_2) \to \Lambda^2_\pm(\Sigma_1 \times \Sigma_2)$ are the self-dual and the anti-self-dual part of the Weyl tensor W and Z is the traceless Ricci tensor. Note that $W = W^+ \oplus W^-$. An orthonormal basis for $\Lambda^2_\pm(\Sigma_1 \times \Sigma_2)$ is

$$e_1^\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \pm e_3 \wedge e_4)\,,$$
$$e_2^\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 \mp e_2 \wedge e_4)\,,$$
$$e_3^\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_4 \pm e_2 \wedge e_3)\,.$$

The metric G^+ is scalar flat and the self-dual part W^+ vanishes, since

$$W^+ = R^+ \begin{pmatrix} 1/3 & & \\ & -1/6 & \\ & & -1/6 \end{pmatrix}.$$

Substituting (2) into (1), the scalar curvature R^+ vanishes and thus $W^-(e_i^-, e_j^-) = \mathcal{R}(e_i^-)e_j^-$. A brief computation shows that $\mathcal{R}(e_i^-)e_j^- = 0$ for all i, j. For example,

$$\begin{aligned} \mathcal{R}(e_1^-)e_2^- &= \frac{1}{2}\mathcal{R}(e_1 \wedge e_2)e_1 \wedge e_2 + \frac{1}{2}\mathcal{R}(e_3 \wedge e_4)e_3 \wedge e_4 \\ &= \frac{1}{2}\big(G^+(R(E_1, E_2)E_1, E_2) + G^+(R(E_3, E_4)E_3, E_4)\big) \\ &= 0\,. \end{aligned}$$

Thus, the anti-self-dual part W^- also vanishes. Therefore, the Weyl tensor $W = 0$, or G^+ is locally conformally flat.

THE CASE OF G^-: We now prove that the neutral Kähler metric G^- is conformally flat if and only if the Gauss curvatures $\kappa(g_1), \kappa(g_2)$ are both constants with $\kappa(g_1) = \kappa(g_2)$. For this metric, consider the orthonormal frame (E_1, E_2, E_3, E_4) defined by:

$$E_1 = (e_1, v_1 + v_2)\,, \quad E_2 = JE_1 = (e_2, v_2 - v_1)\,,$$
$$E_3 = (e_1 - e_2, v_1)\,, \quad E_4 = JE_3 = (e_1 + e_2, v_2)\,.$$

In particular,

$$-|E_1|^2 = -|E_2|^2 = |E_3|^2 = |E_4|^2 = 1\,, \quad G(E_i, E_j) = 0\,, \ \ \forall i \neq j\,.$$

A brief computation gives

$$\mathrm{Ric}^-(E_1, E_1) = \mathrm{Ric}^-(E_2, E_2) = \kappa(g_1)(x) + 2\kappa(g_2)(y)\,,$$
$$\mathrm{Ric}^-(E_3, E_3) = \mathrm{Ric}^-(E_4, E_4) = 2\kappa(g_1)(x) + \kappa(g_2)(y)\,,$$

where Ric^- is the Ricci tensor of the metric G^-. Then, if R^- denotes the scalar curvature of G^-, we have

$$\begin{aligned} R^- &= \sum_{k=1}^{2}\big(-\mathrm{Ric}^-(E_k, E_k) + \mathrm{Ric}^-(E_{2+k}, E_{2+k})\big) \\ &= 2(\kappa(g_1)(x) - \kappa(g_2)(y))\,. \end{aligned} \tag{3}$$

If the neutral Kähler metric G^- is conformally flat, it is also scalar flat [7] and hence, from (3), the Gauss curvatures $\kappa(g_1)$ and $\kappa(g_2)$ are constants with $\kappa(g_1) = \kappa(g_2)$. Following the same argument as before, assume the converse, that is, $\kappa(g_1) = \kappa(g_2) = c$, where c is a real constant. Consider the corresponding coframe $\mathcal{B}_2 = (e_1, e_2, e_3, e_4)$ and the Hodge star operator $* : \Lambda^2(\Sigma_1 \times \Sigma_2) \to \Lambda^2(\Sigma_1 \times \Sigma_2)$. The Hodge star operator splits the Riemann curvature operator $\mathcal{R} : \Lambda^2(\Sigma_1 \times \Sigma_2) \to \Lambda^2(\Sigma_1 \times \Sigma_2)$ in the same way as in the Riemannian case. The Weyl $(0, 4)$-tensor W is given by:

$$W_{ijkl} = R^G_{ijkl} - \frac{1}{2}(-G_{jk}\,\mathrm{Ric}^{\mathrm{G}}_{\mathrm{il}} + G_{jl}\,\mathrm{Ric}^{\mathrm{G}}_{\mathrm{ik}} - G_{il}\,\mathrm{Ric}^{\mathrm{G}}_{\mathrm{jk}} + G_{ik}\,\mathrm{Ric}^{\mathrm{G}}_{\mathrm{jl}})\,,$$

where $R^G_{ijkl} = G(R^G(E_i, E_j)E_k, E_l)$. An orthonormal basis for $\Lambda^2_\pm(\Sigma_1 \times \Sigma_2)$, in the neutral case, is

$$\begin{aligned} e_1^\pm &= \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \pm e_3 \wedge e_4)\,, \\ e_2^\pm &= \frac{1}{\sqrt{2}}(e_1 \wedge e_3 \pm e_2 \wedge e_4)\,, \\ e_3^\pm &= \frac{1}{\sqrt{2}}(e_1 \wedge e_4 \mp e_2 \wedge e_3)\,. \end{aligned}$$

The metric G^- is scalar flat, following [7], and the anti-self-dual part W^- vanishes, since

$$W^- = R^- \begin{pmatrix} 1/3 & & \\ & 1/6 & \\ & & 1/6 \end{pmatrix}.$$

The self-dual part is

$$W^+ = \begin{pmatrix} W_{1212} + W_{3434} + 2W_{1234} & 2(W_{1213} + W_{1334}) & 2(W_{1214} + W_{1434}) \\ & 2(W_{1313} + W_{1324}) & 2(W_{1314} - W_{1323}) \\ & & 2(W_{1414} - W_{1423}) \end{pmatrix},$$

and a brief computation shows that W^+ vanishes. Therefore, the Weyl tensor W vanishes, or G is locally conformally flat. □

COROLLARY 2.3. *Let (Σ, g) be a Riemannian two manifold. The neutral Kähler metric G^- of the four dimensional Kähler manifold $\Sigma \times \Sigma$ is conformally flat if and only if the metric g is of constant Gaussian curvature.*

3. Surface theory in the 4-manifold $\Sigma_1 \times \Sigma_2$. Let $\Phi : S \to \Sigma_1 \times \Sigma_2$ be a smooth immersion of a surface S in $\Sigma_1 \times \Sigma_2$, where (Σ_1, g_1) and (Σ_2, g_2) are both Riemannian two manifolds and let π_i be the projections of $\Sigma_1 \times \Sigma_2$ onto Σ_i, $i = 1, 2$. We denote by ϕ and ψ the mappings $\pi_1 \circ \Phi$ and $\pi_2 \circ \Phi$, respectively, and we write $\Phi = (\phi, \psi)$. The rank of a mapping at a point is the rank of its derivative at that point.

DEFINITION 3.1. The immersion $\Phi = (\phi, \psi) : S \to \Sigma_1 \times \Sigma_2$ is said to be of *projected rank zero* at a point $p \in S$ if either $\mathrm{rank}(\phi(p)) = 0$ or $\mathrm{rank}(\psi(p)) = 0$. Φ is of *projected rank one* at p if either $\mathrm{rank}(\phi(p)) = 1$ or $\mathrm{rank}(\psi(p)) = 1$. Finally, Φ is of *projected rank two* at p if $\mathrm{rank}(\phi(p)) = \mathrm{rank}(\psi(p)) = 2$.

Note that, since it is an immersion, Φ must be of projected rank zero, one or two.

3.1. Projected rank zero case. Let $\Phi = (\phi, \psi)$ be of projected rank zero immersion in $\Sigma_1 \times \Sigma_2$. Assuming, without loss of generality, that $\mathrm{rank}(\phi) = 0$, the map ϕ is locally a constant function and the map ψ is a local diffeomorphism. We now give the following proposition:

PROPOSITION 3.2. *There are no Lagrangian immersions in $\Sigma_1 \times \Sigma_2$ of projected rank zero.*

PROOF. If $\Phi = (\phi, \psi) : S \to \Sigma_1 \times \Sigma_2$ were an immersed surface with $\mathrm{rank}(\phi) = 0$, then $\psi : S \to \Sigma_2$ is a local diffeomorphism and thus for any vector fields X, Y on S

$$\begin{aligned} \Phi^*\Omega^\varepsilon(X, Y) &= \Omega^\varepsilon(d\Phi(X), d\Phi(Y)) \\ &= \Omega^\varepsilon((0, d\psi(X)), (0, d\psi(Y))) \\ &= \varepsilon\, \omega(d\psi(X), d\psi(Y)) \\ &\neq 0, \end{aligned}$$

where the last line follows from the non-degeneracy of ω and the fact that $d\psi$ is a bundle isomorphism. □

3.2. Projected rank one Lagrangian surfaces. We begin by giving the definition of Cornu spirals in a Riemannian two manifold.

DEFINITION 3.3. Let (Σ, g) be a Riemannian two manifold. A regular curve γ of Σ is called a *Cornu spiral of parameter* λ if its curvature κ_γ is a linear function of its arclength parameter such that $\kappa_\gamma(s) = \lambda s + \mu$, where s is the arclength and λ, μ are real constants.

A Cornu spiral γ in $\mathbb{R}^2$ of parameter λ can be parametrised, up to congruences, by

$$\gamma(s) = \left(\int_0^s \cos(\lambda t^2/2) dt, \int_0^s \sin(\lambda t^2/2) dt \right),$$

and they are bounded but have infinite length [4].

Let $\Phi = (\phi, \psi) : S \to \Sigma_1 \times \Sigma_2$ be of projected rank one immersion in $\Sigma_1 \times \Sigma_2$. Then either ϕ or ψ is of rank one. The following theorem gives all rank one Hamiltonian G^ε-minimal surfaces:

THEOREM 3.4. *Every projected rank one Lagrangian surface can be locally parametrised by $\Phi : S \to \Sigma_1 \times \Sigma_2 : (s,t) \mapsto (\phi(s), \psi(t))$, where ϕ and ψ are regular curves on Σ and the induced metric $\Phi^* G^\varepsilon$ is flat. In addition, Φ is Hamiltonian G^ε-minimal if and only if ϕ and ψ are Cornu spirals of parameters λ_ϕ and λ_ψ, respectively, such that*

$$\lambda_\phi = -\varepsilon \lambda_\psi \, .$$

Moreover, Φ is a G^ε-minimal Lagrangian if and only if both ϕ and ψ are geodesics, and every projected rank one G^ε-minimal Lagrangian surface in $\Sigma_1 \times \Sigma_2$ is totally geodesic.

PROOF. Let $\Phi = (\phi, \psi) : S \to \Sigma_1 \times \Sigma_2$ be of projected rank one Lagrangian immersion. Assume, without loss of generality, that ϕ is of rank one. We now prove that ψ is of rank one.

Since Φ is an immersion of a surface, the map ψ cannot be of rank zero. Suppose that ψ is of rank two, i.e., a local diffeomorphism. Thus, Φ is locally parametrised by $\Phi : U \subset S \to \Sigma_1 \times \Sigma_2 : (s,t) \mapsto (\phi(s), \psi(s,t))$. Hence,

$$\Phi_s = (\phi'(s), \psi_s) \quad \Phi_t = (0, \psi_t) \, .$$

Since Φ is a Lagrangian immersion, we have that $\omega_2(\psi_s, \psi_t) = 0$. The fact that ψ is a local diffeomorphism implies that for any non-zero vector field X in Σ_2 can be written as $X = a\psi_s + b\psi_t$. Hence, we have that $\omega_2(\psi_s, X) = 0$. The nondegeneracy of ω_2 implies that ψ is cannot be a local diffeomorphism, since $\psi_s = 0$. Thus ψ is also a rank one immersion.

We now have that S is locally parametrised by $\Phi : U \subset S \to \Sigma_1 \times \Sigma_2 : (s,t) \mapsto (\phi(s), \psi(t))$, where ϕ and ψ are regular curves in Σ_1 and Σ_2, respectively. If s, t are the corresponding arc-length parameters of ϕ and ψ, the Frénet equtions give

$$D^1_{\phi'}\phi' = k_\phi j \phi' \quad D^2_{\psi'}\psi' = k_\psi j \psi' \, ,$$

where k_ϕ and k_ψ denote the curvatures of ϕ and ψ, respectively. Moreover, $\Phi_s = (\phi', 0)$ and $\Phi_t = (0, \psi')$ and thus

$$\nabla_{\Phi_s}\Phi_s = (D^1_{\phi'}\phi', 0) = (k_\phi j\phi', 0) \, , \quad \nabla_{\Phi_t}\Phi_t = (0, D^2_{\psi'}\psi') = (0, k_\psi j \psi') \, , \quad \nabla_{\Phi_t}\Phi_s = (0,0) \, .$$

The first fundamental form $G^\varepsilon_{ij} = G^\varepsilon(\partial_i \Phi, \partial_j \Phi)$ is given by

$$G_{ss} = \varepsilon G_{tt} = 1 \, , \quad G_{st} = 0 \, ,$$

which proves that the immersion Φ is flat.

The second fundamental form h^ε of Φ is completely determined by the following tri-symmetric tensor

$$h^\varepsilon(X, Y, Z) := G^\varepsilon(h^\varepsilon(X, Y), JZ) = \Omega^\varepsilon(X, \nabla_Y Z) \, .$$

We then have

$$h^\varepsilon_{sst} = \Omega^\varepsilon(\Phi_s, \nabla_{\Phi_s}\Phi_t) = 0 \, , \quad h^\varepsilon_{stt} = \Omega^\varepsilon(\Phi_s, \nabla_{\Phi_t}\Phi_t) = 0 \, .$$

Moreover,

$$h^\varepsilon_{sss} = \Omega^\varepsilon(\Phi_s, \nabla_{\Phi_s}\Phi_s) = \Omega^\varepsilon((\phi', 0) \, , (k_\phi j \phi', 0)) = G^\varepsilon((j\phi', 0), (k_\phi j\phi', 0)) = k_\phi \, ,$$

and similarly, $h^\varepsilon_{ttt} = \varepsilon k_\psi$. Denote the mean curvature of Φ with respect to the metric G^ε by $\vec{H}^\varepsilon$. Then

$$G^\varepsilon(2\vec{H}^\varepsilon, J\Phi_s) = \frac{h^\varepsilon_{sss}G^\varepsilon_{tt} + h^\varepsilon_{stt}G^\varepsilon_{ss} - 2h^\varepsilon_{sst}G^\varepsilon_{st}}{G^\varepsilon_{ss}G^\varepsilon_{tt} - (G^\varepsilon_{st})^2} = k_\phi\,,$$

and

$$G^\varepsilon(2\vec{H}^\varepsilon, J\Phi_t) = \frac{h^\varepsilon_{sst}G^\varepsilon_{tt} + h^\varepsilon_{ttt}G^\varepsilon_{ss} - 2h^\varepsilon_{stt}G^\varepsilon_{st}}{G^\varepsilon_{ss}G^\varepsilon_{tt} - (G^\varepsilon_{st})^2} = k_\psi\,.$$

Hence

$$2\vec{H}^\varepsilon = k_\phi J\Phi_s + \varepsilon k_\psi J\Phi_t\,.$$

It is not hard to see that the Lagrangian immersion Φ is G^ε-minimal if and only if the curves ϕ and ψ are geodesics. Moreover, if Φ is a G^ε-minimal Lagrangian it is totally geodesic, since the second fundamental form vanishes identically.

Note also that

$$\mathrm{div}^\varepsilon(\Phi_s) = -G^\varepsilon(\nabla_{\Phi_s}\Phi_s, \Phi_s) = -G^\varepsilon((k_\phi j\phi', 0)\,, (\phi', 0)) = -g(k_\phi j\phi', \phi') = 0\,.$$

In a similar way, we derive that $\mathrm{div}^\varepsilon(\Phi_t) = 0$.

Thus,

$$\begin{aligned}-\mathrm{div}^\varepsilon(2J\vec{H}^\varepsilon) &= G^\varepsilon(\nabla k_\phi, \Phi_s) + k_\phi \mathrm{div}^\varepsilon(\Phi_s) + \varepsilon G^\varepsilon(\nabla k_\psi, \Phi_t) + \varepsilon k_\psi \mathrm{div}^\varepsilon(\Phi_t)\\ &= \frac{D}{ds}k_\phi(s) + \varepsilon\frac{D}{dt}k_\psi(t)\,,\end{aligned}$$

and the theorem follows. □

3.3. Projected rank two Lagrangian surfaces. For the projected rank two case, we have the following theorem:

THEOREM 3.5. *Let (Σ_1, g_1) and (Σ_2, g_2) be Riemannian two manifolds and let $(G^\varepsilon, J, \Omega^\varepsilon)$ be the canonical Kähler product structures on $\Sigma_1 \times \Sigma_2$ constructed in Section 2. Let $\kappa(g_1), \kappa(g_2)$ be the Gauss curvatures of g_1 and g_2, respectively. Assume that one of the following holds:*

(i) The metrics g_1 and g_2 are both generically non-flat and $\varepsilon\kappa(g_1)\kappa(g_2) < 0$ away from flat points.

(ii) Only one of the metrics g_1 and g_2 is flat while the other is non-flat generically. Then every G^ε-minimal Lagrangian surface is of projected rank one.

PROOF. Assume that the G^ε-minimal Lagrangian immersion $\Phi = (\phi, \psi) : S \to \Sigma_1 \times \Sigma_2$ is of projected rank two. Then by definition the mappings $\phi : S \to \Sigma_1$ and $\psi : S \to \Sigma_2$ are both local diffeomorphisms. The Lagrangian assumption $\Phi^*\Omega^\varepsilon = 0$ yields

$$\phi^*\omega_1 = -\varepsilon\psi^*\omega_2\,. \tag{4}$$

Take an orthonormal frame (e_1, e_2) of Φ^*G^ε such that,

$$G^\varepsilon(d\Phi(e_1), d\Phi(e_1)) = \varepsilon G^\varepsilon(d\Phi(e_2), d\Phi(e_2)) = 1\,, \quad G^\varepsilon(d\Phi(e_1), d\Phi(e_2)) = 0\,.$$

The Lagrangian condition implies that the frame $(d\Phi(e_1), d\Phi(e_2), Jd\Phi(e_1), Jd\Phi(e_2))$ is orthonormal. Let (s_1, s_2) and (v_1, v_2) be oriented orthonormal frames of (Σ_1, g_1) and (Σ_2, g_2), respectively, such that $j_1 s_1 = s_2$ and $j_2 v_1 = v_2$. Then there exist smooth functions $\lambda_1, \lambda_2, \mu_1, \mu_2$ on Σ_1 and $\bar\lambda_1, \bar\lambda_2, \bar\mu_1, \bar\mu_2$ on Σ_2 such that

$$d\phi(e_1) = \lambda_1 s_1 + \lambda_2 s_2 \quad d\phi(e_2) = \mu_1 s_1 + \mu_2 s_2\,,$$
$$d\psi(e_1) = \bar\lambda_1 v_1 + \bar\lambda_2 v_2 \quad d\psi(e_2) = \bar\mu_1 v_1 + \bar\mu_2 v_2\,.$$

Hence

$$\phi^*\omega_1(e_1, e_2) = \lambda_1\mu_2 - \lambda_2\mu_1\,, \qquad \psi^*\omega_2(e_1, e_2) = \bar\lambda_1\bar\mu_2 - \bar\lambda_2\bar\mu_1\,.$$

Using the Lagrangian condition (4), we have

$$(\lambda_1\mu_2 - \lambda_2\mu_1)(\phi(p)) = -\varepsilon(\bar\lambda_1\bar\mu_2 - \bar\lambda_2\bar\mu_1)(\psi(p))\,, \quad \forall p \in S\,.$$

Moreover, the assumption that Φ is of projected rank two, implies that $\lambda_1\mu_2 - \lambda_2\mu_1 \neq 0$ for every $p \in S$.

For the mean curvature vector H^ε of the immersion Φ, consider the one form a_{H^ε} defined by $a_{H^\varepsilon} = G^\varepsilon(JH^\varepsilon, \cdot)$. It is known from [8] that since Φ is Lagrangian

$$da_{H^\varepsilon} = \Phi^*\rho^\varepsilon\,, \tag{5}$$

where ρ^ε is the Ricci form of G^ε. Since Φ is a G^ε-minimal Lagrangian immersion $\Phi^*\rho^\varepsilon$ vanishes and thus

$$\begin{aligned}
0 &= \rho^\varepsilon(d\Phi(e_1), d\Phi(e_2))\\
&= \mathrm{Ric}^\varepsilon(d\Phi(e_1), Jd\Phi(e_2))\\
&= \varepsilon G^\varepsilon(R(d\Phi e_1, d\Phi e_2)Jd\Phi e_2, d\Phi e_2) + G^\varepsilon(R(d\Phi e_1, d\Phi e_2)Jd\Phi e_1, d\Phi e_1)\\
&= \varepsilon g_1(R_1(d\phi e_1, d\phi e_2)j_1 d\phi e_2, d\phi e_2) + g_2(R_2(d\psi e_1, d\psi e_2)j_2 d\psi e_2, d\psi e_2)\\
&\qquad + g_1(R_1(d\phi e_1, d\phi e_2)j_1 d\phi e_1, d\phi e_1) + \varepsilon g_2(R_2(d\psi e_1, d\psi e_2)Jd\psi e_1, d\psi e_1)\\
&= \varepsilon\big((\lambda_1^2 + \lambda_2^2 + \varepsilon(\mu_1^2 + \mu_2^2)\big)(\mu_1\lambda_2 - \mu_2\lambda_1)\kappa(g_1)\\
&\qquad + \big(\bar\lambda_1^2 + \bar\lambda_2^2 + \varepsilon(\bar\mu_1^2 + \bar\mu_2^2)\big)(\bar\mu_1\bar\lambda_2 - \bar\mu_2\bar\lambda_1)\kappa(g_2)\\
&= \varepsilon(\mu_1\lambda_2 - \mu_2\lambda_1)\big[\big(\lambda_1^2 + \lambda_2^2 + \varepsilon(\mu_1^2 + \mu_2^2)\big)\kappa(g_1) - \big(\bar\lambda_1^2 + \bar\lambda_2^2 + \varepsilon(\bar\mu_1^2 + \bar\mu_2^2)\big)\kappa(g_2)\big]\,,
\end{aligned}$$

which finally gives,

$$\big(\lambda_1^2 + \lambda_2^2 + \varepsilon(\mu_1^2 + \mu_2^2)\big)\kappa(g_1) = \big(\bar\lambda_1^2 + \bar\lambda_2^2 + \varepsilon(\bar\mu_1^2 + \bar\mu_2^2)\big)\kappa(g_2)\,. \tag{6}$$

The condition $G^\varepsilon(d\Phi(e_1), d\Phi(e_2)) = 0$ yields

$$\lambda_1\mu_1 + \lambda_2\mu_2 = -\varepsilon(\bar\lambda_1\bar\mu_1 + \bar\lambda_2\bar\mu_2)\,. \tag{7}$$

Now, using (4) and (7), we have

$$(\lambda_1^2 + \lambda_2^2)(\mu_1^2 + \mu_2^2) = (\bar\lambda_1^2 + \bar\lambda_2^2)(\bar\mu_1^2 + \bar\mu_2^2)\,. \tag{8}$$

From $G^\varepsilon(d\Phi(e_1), d\Phi(e_1)) = \varepsilon G^\varepsilon(d\Phi(e_2), d\Phi(e_2)) = 1$ we obtain

$$\lambda_1^2 + \lambda_2^2 + \varepsilon(\bar\lambda_1^2 + \bar\lambda_2^2) = \varepsilon(\mu_1^2 + \mu_2^2) + \bar\mu_1^2 + \bar\mu_2^2 = 1\,. \tag{9}$$

Set $a := \lambda_1^2 + \lambda_2^2$, $b := \mu_1^2 + \mu_2^2$, $\bar a := \bar\lambda_1^2 + \bar\lambda_2^2$, $\bar b := \bar\mu_1^2 + \bar\mu_2^2$. The relations (7), (8) and (9) give

$$ab = \bar a\bar b\,, \quad a + \varepsilon\bar a = \varepsilon b + \bar b = 1\,.$$

Thus $a = -\varepsilon\bar{a} + 1$ and $b = \varepsilon - \varepsilon\bar{b}$, and from $ab = \bar{a}\bar{b}$ we have that $\bar{a} + \varepsilon\bar{b} = \varepsilon$. Moreover, $\bar{a} = \varepsilon - \varepsilon a$ and $\bar{b} = 1 - \varepsilon b$, and again from $ab = \bar{a}\bar{b}$ we have $a + \varepsilon b = 1$. Hence, relation (6) becomes

$$\kappa(g_1)(\phi(p)) = \varepsilon\kappa(g_2)(\psi(p)), \quad \text{for every} \ \ p \in S,$$

which implies that the metrics g_1 and g_2 can satisfy neither condition (i) nor condition (ii) of the statement. □

The following corollaries follow:

COROLLARY 3.6. *Every G^+-minimal Lagrangian surface immersed in $\mathbb{S}^2 \times \mathbb{H}^2$ is, up to isometry, the cylinder $\mathbb{S}^1 \times \mathbb{R}$. Moreover, every G^ε-minimal Lagrangian surface immersed in $\mathbb{R}^2 \times \mathbb{H}^2$ ($\mathbb{R}^2 \times \mathbb{S}^2$) is of projected rank one and thus it is $\gamma_1 \times \gamma_2$, where γ_1 is a straight line in $\mathbb{R}^2$ and γ_2 is a geodesic in $\mathbb{H}^2$* (γ_2 is a geodesic in $\mathbb{S}^2$), *respectively.*

COROLLARY 3.7. *Let (Σ, g) be a Riemannian two manifold such that the metric g is non-flat. Then every G^--minimal Lagrangian surface immersed in $\Sigma \times \Sigma$ is of projected rank one and consequently the product of two geodesics of (Σ, g).*

4. The Hamiltonian stability of minimal Lagrangian surfaces. The Hamiltonian stability of a Hamiltonian minimal surface S in a pseudo-Riemannian manifold $(\mathcal{M}, G)$ is given by the monotonicity of the second variation formula of the volume $V(S)$ under Hamiltonian deformations (see [14] and [5]). For a smooth compactly supported function $u \in C_c^\infty(S)$ the second variation $\delta^2 V(S)(X)$ formula in the direction of the Hamiltonian vector field $X = J\nabla u$ is:

$$\delta^2 V(S)(X) = \int_S \big((\Delta u)^2 - \mathrm{Ric}^{\mathrm{G}}(\nabla u, \nabla u) - 2G(h(\nabla u, \nabla u), nH) + G^2(nH, J\nabla u)\big) dV,$$

where h is the second fundamental form of S, $\mathrm{Ric}^{\mathrm{G}}$ is the Ricci curvature tensor of the metric G, and Δ with ∇ denote the Laplacian and gradient, respectively, with respect to the metric G induced on S. For the Hamiltonian stability of projected rank one Hamiltonian G^ε-minimal surfaces we give the following theorem:

THEOREM 4.1. *Let $\Phi = (\phi, \psi)$ be of projected rank one Hamiltonian G^ε-minimal immersion in $(\Sigma_1 \times \Sigma_2, G^\varepsilon)$ such that $\kappa(g_1) \leq -2k_\phi^2$ and $\kappa(g_2) \leq -2k_\psi^2$ along the curves ϕ and ψ respectively. Then Φ is a local minimizer of the volume in its Hamiltonian isotopy class.*

PROOF. Let $\Phi = (\phi, \psi) : S \to \Sigma_1 \times \Sigma_2$ be of projected rank one Hamiltonian G^ε-minimal immersion and let (s, t) be the corresponded arclengths of ϕ and ψ, respectively. Then $(\phi_s, j_1\phi_s)$ is an oriented orthonormal frame of (Σ_1, g_1) and $(\psi_t, j_2\psi_t)$ is an oriented orthonormal frame of (Σ_2, g_2). Therefore,

$$\begin{aligned}\mathrm{Ric}^\varepsilon(\Phi_s, \Phi_s) &= \varepsilon G^\varepsilon(R(\Phi_t, \Phi_s)\Phi_s, \Phi_t) + G^\varepsilon(R(J\Phi_s, \Phi_s)\Phi_s, J\Phi_s) \\ &\qquad + \varepsilon G^\varepsilon(R(J\Phi_t, \Phi_s)\Phi_s, J\Phi_t) \\ &= G^\varepsilon(R(J\Phi_s, \Phi_s)\Phi_s, J\Phi_s)\end{aligned}$$

$$
\begin{aligned}
&= G^{\varepsilon}((R_1(j_1\phi_s, \phi_s)\phi_s, R_2(j_2\psi_s, \psi_s)\psi_s), (j_1\phi_s, j_2\psi_s)) \\
&= G^{\varepsilon}((R_1(j_1\phi_s, \phi_s)\phi_s, 0), (j_1\phi_s, 0)) \\
&= g_1(R_1(j_1\phi_s, \phi_s)\phi_s, j_1\phi_s) \\
&= \kappa(g_1) .
\end{aligned}
$$

Moreover, a similar computation gives

$$\mathrm{Ric}^{\varepsilon}(\Phi_t, \Phi_t) = \kappa(g_2) \quad \text{and} \quad \mathrm{Ric}^{\varepsilon}(\Phi_s, \Phi_t) = 0 .$$

Then, for every $u(s,t) \in C_c^{\infty}(S)$, we have

$$\mathrm{Ric}^{\varepsilon}(\nabla u, \nabla u) = \kappa(g_1)u_s^2 + \kappa(g_2)u_t^2 .$$

Furthermore,

$$G^{\varepsilon}(h^{\varepsilon}(\nabla u, \nabla u), 2\vec{H}^{\varepsilon}) = u_s^2 k_{\phi}^2 + u_t^2 k_{\psi}^2$$

and

$$G^{\varepsilon}(2\vec{H}^{\varepsilon}, J\nabla u) = u_s k_{\phi} + \varepsilon u_t k_{\psi} .$$

The second variation formula for the volume functional with respect to the Hamiltonian vector field $X = J\nabla u$ becomes

$$
\begin{aligned}
\delta^2 V(S)(X) &= \int_S (\Delta^{\varepsilon} u)^2 - \mathrm{Ric}^{\varepsilon}(\nabla u, \nabla u) - 2G^{\varepsilon}(h^{\varepsilon}(\nabla u, \nabla u), 2\vec{H}^{\varepsilon}) + G^{\varepsilon}(2\vec{H}^{\varepsilon}, J\nabla u)^2 \\
&= \int_S (u_{ss} + \varepsilon u_{tt})^2 - u_s^2\kappa(g_1) - u_t^2\kappa(g_2) - (u_s k_{\phi} - \varepsilon u_t k_{\psi})^2 \\
&= \int_S (u_{ss} + \varepsilon u_{tt})^2 + u_s^2(-\kappa(g_1) - k_{\phi}^2) + u_t^2(-\kappa(g_2) - k_{\psi}^2) + 2\varepsilon u_s u_t k_{\phi} k_{\psi} .
\end{aligned}
$$

Assuming that $\kappa(g_1) \leq -2k_{\phi}^2$ and $\kappa(g_2) \leq -2k_{\psi}^2$ along the curves ϕ and ψ, respectively, we conclude that the second variation formula is nonnegative. □

Every minimal Lagrangian surface in a pseudo-Kähler 4-manifold is unstable [2]. The following corollary explores the Hamiltonian stability of G^{-}-minimal Lagrangian surfaces in $\Sigma_1 \times \Sigma_2$:

COROLLARY 4.2. *Let (Σ_1, g_1) and (Σ_2, g_2) be Riemannian two manifolds such that their Gauss curvatures $\kappa(g_1)$ and $\kappa(g_2)$ are both negative. Then every G^{-}-minimal Lagrangian surface is a local minimizer of the volume in its Hamiltonian isotopy class.*

PROOF. From Theorem 3.5 every G^{-}-minimal Lagrangian immersion must be of projected rank one and thus it is parametrised by $\Phi = (\phi, \psi) : S \to \Sigma_1 \times \Sigma_2$, where $\phi = \phi(s)$ and $\psi = \psi(t)$, where s, t are arclengths. Assuming that $\kappa(g_1), \kappa(g_2)$ are both negative, we have that:

$$\kappa(g_1)(s) \leq -2k_{\phi}^2(s) = 0 , \quad \kappa(g_2)(t) \leq -2k_{\psi}^2(t) = 0 ,$$

and from Theorem 4.1 the G^{-}-minimal Lagrangian immersion Φ is stable under Hamiltonian deformations. □

We also have the next corollary:

COROLLARY 4.3. *Let (Σ, g) be a Riemannian two manifold of negative Gaussian curvature. Then every G^--minimal Lagrangian surface immersed in $\Sigma \times \Sigma$ is a local minimizer of the volume in its Hamiltonian isotopy class.*

EXAMPLE 1. It is easy to see that if (Σ, g) is a Riemannian two manifold of constant Gauss curvature $c \neq 0$, then every G^--minimal Lagrangian surface immersed in $\Sigma \times \Sigma$ is a local minimizer of the volume in its Hamiltonian isotopy class if and only if $c < 0$.

EXAMPLE 2. Let $L(\mathbb{S}^3)$ and $L^+(\mathrm{Ad}\,\mathbb{S}^3)$ be the spaces of oriented closed geodesics in the three sphere and anti-De Sitter 3-space, respectively. Then $L(\mathbb{S}^3) = \mathbb{S}^2 \times \mathbb{S}^2$ and $L^+(\mathrm{Ad}\,\mathbb{S}^3) = \mathbb{H}^2 \times \mathbb{H}^2$ (see [1] and [3]). The previous example generalises a result obtained in [5] which states that every minimal Lagrangian surface in the space of closed oriented geodesics $L(\mathbb{S}^3)$ is Hamiltonian unstable and every Lagrangian minimal surface in $L^+(\mathrm{Ad}\,\mathbb{S}^3)$ is Hamiltonian stable.

The following proposition investigates the Hamiltonian stability of G^+-minimal Lagrangian surfaces:

PROPOSITION 4.4. *Let (Σ_1, g_1) and (Σ_2, g_2) be Riemannian two manifolds with Gaussian curvatures satisfying*

$$c_1 \leq |\kappa(g_1)(x)| \leq C_1\,, \quad c_2 \leq |\kappa(g_2)(y)| \leq C_2\,, \quad \textit{and} \quad \kappa(g_1)(x)\kappa(g_2)(y) < 0\,,$$

for every pair $(x, y) \in \Sigma_1 \times \Sigma_2$ and for some positive constants c_1, c_2, C_1, C_2. Then, every G^+-minimal Lagrangian surface is Hamiltonian unstable and hence G^+-unstable.

PROOF. Consider again a Lagrangian minimal immersion $\Phi = (\phi, \psi) : S \to \Sigma_1 \times \Sigma_2$. From Theorem 3.5, we have that $\phi = \phi(s)$ and $\psi = \psi(t)$ are geodesics of Σ_1 and Σ_2, respectively, with (s, t) chosen to be the corresponding arc-lengths. Then $(\phi_s, j_1\phi_s)$ is an oriented orthonormal frame of (Σ_1, g_1) and $(\psi_t, j_2\psi_t)$ is an oriented orthonormal frame of (Σ_2, g_2). A computation similar to that in Theorem 4.1 gives

$$\mathrm{Ric}^+(\Phi_s, \Phi_s) = \kappa(g_1)\,, \quad \mathrm{Ric}^+(\Phi_t, \Phi_t) = \kappa(g_2)\,, \quad \mathrm{Ric}^+(\Phi_s, \Phi_t) = 0\,,$$

and the second variation formula for the volume of S in the direction of the Hamiltonian vector field $X = J\nabla u$ is

$$\delta^2 V(S)(X) = \int_S \big((u_{ss} - u_{tt})^2 - \kappa(g_1)u_s^2 - \kappa(g_2)u_t^2\big)dV\,.$$

Assume that $\kappa(g_1) < 0$. Then, $\kappa(g_2) > 0$ and

$$\delta^2 V(S)(X) \geq \int_S \big((u_{ss} - u_{tt})^2 - C_1u_s^2 + c_2u_t^2\big)dV\,.$$

Thus, for the quadratic functional

$$Q_1(u) := \int_S -C_1u_s^2 + c_2u_t^2\,,$$

there exists $u^1 \in C_c^\infty(S)$ such that $Q_1(u^1) \geq 0$. Therefore, $\delta^2 V(S)(J\nabla u^1) \geq 0$.

On the other hand, for every $u \in C_c^\infty(S)$

$$\delta^2 V(S)(J\nabla u) \le \int_S \big((u_{ss} + u_{tt})^2 - c_1 u_s^2 + C_2 u_t^2\big) dV\,.$$

Then, for the quadratic functional

$$Q_2(u) := \int_S -c_1 u_s^2 + C_2 u_t^2\,,$$

there exists $u^2 \in C_c^\infty(S)$ such that $Q_2(u^2) \le 0$. An argument similar to that in the proof of Theorem 3 of [5] establishes the existence of $u^3 \in C_c^\infty(S)$ such that

$$\int_S \big((u^3_{ss} + u^3_{tt})^2 - c_1(u^3_s)^2 + C_2(u^3_t)^2\big) dV \le 0\,,$$

which implies that $\delta^2 V(S)(J\nabla u^3) \le 0$. Therefore the second variation formula for the volume of S under Hamiltonian deformations is indefinite. □

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