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ON MINIMAL SEQUENCES OF TYPE l_+
AND BOUNDED BIORTHOGONAL SYSTEMS IN BANACH SPACES

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A sequence $\{x_n\}$ in a Banach space E is said to be *minimal* if there exists a (not necessarily unique) sequence of functionals $\{f_n\} \subset E^*$ such that $f_i(x_j) = \delta_{ij}$ ($i, j = 1, 2, \dots$); any such pair of sequences (x_n, f_n) is called a *biorthogonal system*. A minimal sequence is called [5] (E, E^*) -*bounded* if there exists a sequence $\{f_n\} \subset E^*$ such that (x_n, f_n) is a biorthogonal system which is *bounded* in the sense of V. PTÁK [3], i.e., $\sup_n \|x_n\| < \infty$ and $\sup_n \|f_n\| < \infty$. A minimal sequence $\{x_n\}$ is said to be of *type l_+* , if it is (E, E^*) -bounded and if there exists a constant $\eta > 0$ such that we have, for every finite sequence $\alpha_1, \dots, \alpha_n \geq 0$,

$$(1) \quad \left\| \sum_{i=1}^n \alpha_i x_i \right\| \geq \eta \sum_{i=1}^n \alpha_i.$$

This notion was introduced by V. Pták [3] (for the term “of type l_+ ” see [4], [5]), who has shown in [3] that minimal sequences of type l_+ exist both in every non-reflexive Banach space and in the Hilbert space $L^2([0, 1])$. In [5] the question was raised ([5], page 166, problem 3.9) whether a minimal sequence $\{x_n\}$ of type l_+ exists in *every* Banach space. In the present Note we shall prove that the answer to this problem is affirmative and that for a wide class of *separable* Banach spaces (including all spaces having a finite dimensional decomposition) $\{x_n\}$ can be chosen to be also *complete* in E , i.e. such that the closed linear span $[x_n]$ of $\{x_n\}$ coincides with E . The problem whether every separable Banach space E has a complete minimal sequence $\{x_n\}$ of type l_+ remains still open and it turns out to be equivalent to the problem whether every separable Banach space E has an (E, E^*) -bounded complete minimal sequence ([5], p. 169, problem 3.10).

Theorem 1. *For a Banach space E the following two statements are equivalent:*

- 1°. E has a complete minimal sequence $\{x_n\}$ of type l_+ .
- 2°. E has an (E, E^*) -bounded complete minimal sequence $\{y_n\}$.

Proof. The implication $1^\circ \Rightarrow 2^\circ$ is obvious since by definition every minimal sequence of type l_+ is (E, E^*) -bounded.

Conversely, assume that $(y_n, g_n) (\{y_n\} \subset E, \{g_n\} \subset E^*)$ is a bounded biorthogonal system such that $[y_n] = E$. Put

$$(2) \quad x_n = y_1 + y_{n+1} \quad (n = 1, 2, \dots),$$

$$(3) \quad f_n = g_{n+1} \quad (n = 1, 2, \dots).$$

Then (x_n, f_n) is a bounded biorthogonal system and for every finite sequence $\alpha_1, \dots, \alpha_n \geq 0$ we have

$$(4) \quad \left\| \sum_{i=1}^n \alpha_i x_i \right\| \geq \frac{\|g_1\|}{\|g_1\|} \left(\sum_{i=1}^n \alpha_i x_i \right) \Big| = \frac{1}{\|g_1\|} \left| \sum_{i=1}^n \alpha_i g_1(x_i) \right| = \frac{1}{\|g_1\|} \sum_{i=1}^n \alpha_i,$$

i.e. (1) with $\eta = 1/\|g_1\|$. Therefore, if $[x_n] = E$, then $\{x_n\}$ is a complete minimal sequence of type l_+ . If $[x_n] \neq E$, then there exists a $g \in E^*$ such that $g \neq 0, g(x_n) = 0$ ($n = 1, 2, \dots$), whence $g(y_1) = -g(y_2) = -g(y_3) = \dots$ and thus, since $[y_n] = E$ and $g \neq 0$, it follows that $g(y_1) \neq 0$. Put

$$(5) \quad x_0 = y_1, \quad f_0 = \frac{1}{g(y_1)} g.$$

Then $[x_n]_0^\infty = E$ and $(x_n, f_n)_0^\infty$ is a bounded biorthogonal system such that for every finite sequence $\alpha_0, \alpha_1, \dots, \alpha_n \geq 0$ we have (4) with $\sum_{i=1}^n$ replaced by $\sum_{i=0}^n$. Therefore $\{x_n\}_0^\infty$ is a complete minimal sequence of type l_+ , which completes the proof of theorem 1.

Remark 1. The problem, whether every separable Banach space has property 2° ([5], p. 169, problem 3.10), is apparently slightly "easier" than the unsolved problem of S. BANACH [1], whether in every separable Banach space E there exists an (E, E^*) -bounded M -basis $\{y_n\}$, that is, a bounded biorthogonal system (y_n, g_n) such that $[y_n] = E$ and that $\{g_n\}$ is total on E (i.e., $\{x \in E \mid g_n(x) = 0 \ (n = 1, 2, \dots)\} = \{0\}$).

We shall give now a class of separable Banach spaces having property 2° (and hence 1°) of theorem 1. We recall that a sequence of finite-dimensional subspaces $\{E_n\}$ of a Banach space E is called a *finite dimensional decomposition* (f.d.d.) of E if for every $x \in E$ there exists a unique sequence $\{z_n\} \subset E$ with $z_n \in E_n$ ($n = 1, 2, \dots$) such that $x = \sum_{i=1}^{\infty} z_i$. It is well known that in this case for each n the operator $P_n(x) = \sum_{i=1}^n z_i$ ($x = \sum_{i=1}^{\infty} z_i \in E$) is a bounded linear projection, called "the natural projection" of E onto E_n , and that $\sup_n \|P_n\| < \infty$. Let us also recall that a sequence $\{z_n\} \subset E$

is called a *basis* of E if the one-dimensional subspaces $E = \{\alpha z_n \mid \alpha \text{ scalar}\}$ constitute an f.d.d. of E , i.e., if for every $x \in E$ there exists a unique sequence of scalars $\{\alpha_n\}$ such that $x = \sum_{i=1}^{\infty} \alpha_i z_i$. In this case $\{z_n\}$ is a complete minimal sequence in E , namely, for $h_n(x) = \alpha_n$ ($x = \sum_{i=1}^{\infty} \alpha_i z_i \in E$) we have $h_n \in E^*$ ($n = 1, 2, \dots$) and $h_i(x_j) = \delta_{ij}$ ($i, j = 1, 2, \dots$). Moreover, it is also known that if $\inf_n \|z_n\| > 0$ then $\sup_n \|h_n\| < \infty$.

Lemma 1. *Every Banach space E with an f.d.d. $\{E_n\}$ has an (E, E^*) -bounded M -basis $\{y_n\}$.*

Proof. Since $\dim E_n < \infty$ ($n = 1, 2, \dots$), for each n there exists (see e.g. [1]) a biorthogonal system $(y_i, \varphi_i)_{i=m_{n-1}+1}^{m_n}$ ($\{y_i\}_{i=m_{n-1}+1}^{m_n} \subset E_n$, $\{\varphi_i\}_{i=m_{n-1}+1}^{m_n} \subset E_n^*$) such that

$$(6) \quad \|y_i\| = \|\varphi_i\| = 1 \quad (i = m_{n-1} + 1, \dots, m_n; n = 1, 2, \dots; m_0 = 0).$$

For each $i = m_{n-1} + 1, \dots, m_n$ ($n = 1, 2, \dots$) put

$$(7) \quad g_i(x) = \begin{cases} \varphi_i(x) & \text{for } x \in E_n \\ 0 & \text{for } x \in \bigcup_{j \neq n} E_j \end{cases}$$

and extend g_i by linearity to the (dense) linear subspace of E spanned by $\bigcup_{j=1}^{\infty} E_j$; this is possible, since $E_n \cap \bigcup_{j \neq n} E_j = \{0\}$. Then for every finite sum $x = \sum_{k=1}^p z_k \in E$ with $z_k \in E_k$ ($k = 1, \dots, p$) we have

$$|g_i(x)| = |g_i(\sum_{k=1}^p z_k)| = \begin{cases} |\varphi_i(z_n)| & \text{for } i = m_{n-1} + 1, \dots, m_n; n = 1, \dots, p \\ 0 & \text{for } i = m_{n-1} + 1, \dots, m_n; n = p + 1, p + 2, \dots \end{cases}$$

whence, by (6), we obtain for $i = m_{n-1} + 1, \dots, m_n$ and $n = 1, \dots, p$

$$|g_i(x)| \leq \|\varphi_i\| \|z_n\| = \|z_n\| = \|P_n(x)\| \leq \sup_j \|P_j\| \|x\|$$

where P_n is the natural projection of E onto E_n ($n = 1, \dots, p$). Since the set of all finite sums $\sum_{k=1}^p z_k$ with $z_k \in E_k$ ($k = 1, \dots, p$) is dense in E , it follows that $\{g_n\} \subset E^*$ and that $\sup_n \|g_n\| < \infty$. Furthermore, obviously $[y_n] = E$ and $g_i(y_j) = \delta_{ij}$ ($i, j = 1, 2, \dots$). Finally, $\{g_n\}$ is total on E , because $x = \sum_{k=1}^{\infty} z_k \in E$ and $g_n(x) = 0$ ($n = 1, 2, \dots$) imply, by (7), $\varphi_i(z_n) = 0$ ($i = m_{n-1} + 1, \dots, m_n; n = 1, 2, \dots$), whence $z_n = 0$ ($n = 1, 2, \dots$) and $x = 0$.

From theorem 1 and lemma 1 it follows

Corollary 1. *Every Banach space E with an f.d.d. (in particular, every Banach space E with a basis) has a complete minimal sequence $\{x_n\}$ of type l_+ .*

Dropping the assumption that E is separable (and hence the requirement that $\{x_n\}$ be complete in E), we have

Theorem 2. *Every Banach space E contains a minimal sequence $\{x_n\}$ of type l_+ .*

Proof. It is well known (see e.g. [1]) that every Banach space has a “basic sequence” $\{z_n\}$ (i.e., a sequence $\{z_n\}$ which is a basis of $[z_n]$). Then $\{y_n\} = \{z_n/\|z_n\|\}$ is a basis of $[y_n] = [z_n]$ with $\|y_n\| = 1$ ($n = 1, 2, \dots$) and hence for $\{g_n\} \subset [y_n]^*$ with $g_i(y_j) = \delta_{ij}$ ($i, j = 1, 2, \dots$) we have $\sup_n \|g_n\| < \infty$. Hence, by theorem 1, the subspace $[y_n]$ of E has a complete minimal sequence $\{x_n\}$ of type l_+ . Since $\{x_n\}$ is obviously a minimal sequence of type l_+ in E , the proof of theorem 2 is complete.

Remark 2. Note that in a direct proof of theorem 2 the case $[x_n] \neq E$ of the proof of theorem 1 can be omitted.

Remark 3. Such a result is no longer true if we require, in addition, that $\{f_n|_{[x_j]}\}$ be total on $[x_j]$. Indeed, V. Pták has observed [3] that if a Banach space E has a minimal sequence $\{x_n\}$ of type l_+ with this additional property, then E is non-reflexive. The converse of this latter statement is also true, since every non-reflexive Banach space E has ([4], [2]) even a basic sequence $\{x_n\}$ of type l_+ .

Added in proof. Our problem mentioned before theorem 1 (and in remark 1) has been solved in the affirmative by W. J. Davis and W. B. Johnson (to appear in *Studia Math.*).

References

- [1] S. Banach: Théorie des opérations linéaires. Monografie Matematyczne, Warszawa, 1932.
- [2] A. Pelczyński: A note on the paper of I. Singer “Basic sequences and reflexivity of Banach spaces”, *Studia Math.*, 21, 1962, 371—374.
- [3] V. Pták: Biorthogonal systems and reflexivity of Banach spaces. *Czechoslovak Math. J.* 9, 1959, 319—326.
- [4] I. Singer: Basic sequences and reflexivity of Banach spaces. *Studia Math.* 21, 1962, 351—369.
- [5] I. Singer: Bases in Banach spaces. II (in Romanian) *Studii și cercet. mat.* 15, 1964, 157—208.

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