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ON MINIMAL SURFACES IN A RIEMANNIAN MANIFOLD OF CONSTANT CURVATURE

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In the present paper, we shall study minimal surfaces in a Riemannian manifold \widehat{M} of (non-zero) constant curvature c . When \widehat{M} is a sphere $S(1)$ of constant curvature 1, we find the conditions such that the immersion is uniquely determined up to a rigid motion of $S(1)$. In §2, we define the $(n+1)$ -th shape operator (n -th torsion operator) and the n -th torsion index (T_n -index) for minimal surfaces in \widehat{M} . In §3, we study complete flat minimal surfaces in \widehat{M} . When \widehat{M} is $S(1)$, we find the conditions such that the immersion is uniquely determined up to a rigid motion of $S(1)$. In §4, we give examples of minimal immersions of the Euclidean plane into a sphere. In the last section, we study compact minimal surfaces of non-negative curvature ($\neq 0$) in \widehat{M} and prove that they are generalized Veronese surfaces if they are immersed in $\widehat{M} = S(1)$ by the immersions with full torsion.

§ 1. **Preliminaries.** Let \widehat{M} be a $(2+\nu)$ -dimensional Riemannian manifold of constant curvature c , and M a 2-dimensional Riemannian manifold which is isometrically immersed in \widehat{M} by an immersion $x: M \rightarrow \widehat{M}$. The geodesic codimension of M in \widehat{M} is defined to be the minimum of codimension of M in totally geodesic submanifolds of \widehat{M} , see [7]. We denote by ∇ (resp. $\widehat{\nabla}$) the covariant differentiation on M (resp. \widehat{M}). Then the (first) shape operator (second fundamental form) φ_1 of the immersion is given by $\varphi_1(X, Y) = \widehat{\nabla}_X Y - \nabla_X Y$ for any tangent vector fields X and Y of M . There holds then $\varphi_1(X, Y) = \varphi_1(Y, X)$.

$F(\widehat{M})$ and $F(M)$ denote the orthonormal frame bundles over \widehat{M} and M , respectively. Let B be the set of all elements $b = (p, e_1, e_2, \dots, e_{2+\nu}) \in F(\widehat{M})$ such that $(p, e_1, e_2) \in F(M)$, identifying $p \in M$ with $x(p)$ and e_i with $dx(e_i)$, $i=1, 2$. Then B is a smooth submanifold of $F(\widehat{M})$. Let $\widehat{\omega}_A, \widehat{\omega}_{AB} = -\widehat{\omega}_{BA}$, $A, B = 1, 2, \dots, 2+\nu$, be the basic and connection forms of \widehat{M} on $F(\widehat{M})$ which satisfy the following structure equations

$$(1.1) \quad \begin{aligned} d\widehat{\omega}_A &= \sum_B \widehat{\omega}_{AB} \wedge \widehat{\omega}_B, \\ d\widehat{\omega}_{AB} &= \sum_C \widehat{\omega}_{AC} \wedge \widehat{\omega}_{CB} - c\widehat{\omega}_A \wedge \omega_B. \end{aligned}$$

In this paper, we use the following convention on the range of indices

$$i, j, \dots = 1, 2, \quad \alpha, \beta, \dots = 3, 4, \dots, 2 + \nu.$$

Deleting the hats of $\hat{\omega}_A, \hat{\omega}_{AB}$ on B , as is well known, we have

$$\begin{aligned} \omega_\alpha &= 0, \\ \omega_{i\alpha} &= \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha, \\ d\omega_i &= \omega_{ij} \wedge \omega_j, \quad i \neq j, \\ (1.2) \quad d\omega_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj} - \Omega_{ij}, \quad \Omega_{ij} = \frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l, \\ R_{ijkl} &= c(\partial_{ik} \partial_{jl} - \partial_{il} \partial_{jk}) + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \\ d\omega_{\alpha\beta} &= \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \Omega_{\alpha\beta} \quad \Omega_{\alpha\beta} = \frac{1}{2} \sum R_{\alpha\beta\gamma\delta} \omega_\gamma \wedge \omega_\delta, \\ R_{\alpha\beta\gamma\delta} &= \sum_k (h_{ik}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{ik}^\beta). \end{aligned}$$

M is said to be *minimal* if its mean curvature vector $\frac{1}{2} \sum_{j,\alpha} h_{jj}^\alpha e_\alpha$ vanishes identically, i. e., if trace $H_\alpha = 0$ for all α , $H_\alpha = (h_{ij}^\alpha)$. The *minimal index* at $p \in M$ ($m\text{-index}_p M$) is defined to be the dimension of the linear space of all second fundamental forms corresponding to normal vectors at p with vanishing trace. We easily have $m\text{-index}_p M \leq 2$ at every point $p \in M$. We denote the *square of the norm of the second fundamental form* by

$$S = \frac{1}{2} \sum h_{ij}^\alpha h_{ij}^\alpha$$

The *normal scalar curvature* K_N of M in \hat{M} is defined by

$$K_N = \sum_{\substack{\alpha < \beta \\ \alpha < \beta}} (R_{\alpha\beta\gamma\delta})^2 = \sum_{\substack{\alpha < \beta \\ \alpha < \beta}} \left\{ \sum_k (h_{ik}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{ik}^\beta) \right\}^2.$$

§ 2. The $(n + 1)$ -th shape operator and the n -th torsion index (T_n -index). In this section, we assume that M is *minimal* in \hat{M} . We define the $(n + 1)$ -th shape operator (the n -th torsion operator) and the n -th torsion index for minimal surfaces in \hat{M} by induction on n .

The (first) shape operator φ_1 can be written as

$$\varphi_1(X, Y) = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i(X) \omega_j(Y) e_\alpha.$$

Since $\sum_{j=1}^2 h_{jj}^a = 0$, for $X = e_1 \cos \theta + e_2 \sin \theta$, we have

$$(2.1) \quad \widetilde{\varphi}_1(X) := \varphi_1(X, X) = \cos 2\theta \cdot F_1 + \sin 2\theta \cdot G_1,$$

where $F_1 = \sum_{\alpha} h_{11}^{\alpha} e_{\alpha}$ and $G_1 = \sum_{\alpha} h_{12}^{\alpha} e_{\alpha}$. Let S_p^1 be the unit circle in a tangent space M_p to M at $p \in M$. It is clear from (2.1) that the image of S_p^1 under $\widetilde{\varphi}_1$ is a point, a segment, or an ellipse (a circle) according as $m\text{-index}_p M = 0, 1$, or 2 . We easily see that

$$(2.2) \quad S^2 - K_N = (\|G_1\|^2 - \|F_1\|^2)^2 + 4 \langle G_1, F_1 \rangle^2,$$

which is geometrically stated as follows :

$$\begin{aligned} S^2 - K_N &= \{(\text{length of major axis})^2 - (\text{length of minor axis})^2\}^2 && \text{if } m\text{-index}_p M = 2, \\ S^2 - K_N &= (\text{length of the segment})^4 && \text{if } m\text{-index}_p M = 1, \\ S^2 &= K_N = 0 && \text{if } m\text{-index}_p M = 0. \end{aligned}$$

If $m\text{-index}_p M = 0$ at every point $p \in M$, then M is totally geodesic in \widehat{M} . If $m\text{-index}_p M = 1$ at every point $p \in M$, then the geodesic codimension of M is 1.

Henceforth, we consider the case that $m\text{-index}_p M = 2$ at every point $p \in M$. Then, we can choose a local frame field $b \in B$ such that

$$(2.3) \quad \omega_{j\alpha_1} \neq 0, \quad \omega_{j\beta} = 0, \quad \alpha_1 \in \{3, 4\}, \quad \text{for } \beta > 4.$$

From (1.1), (1.2) and (2.3) we may write

$$\sum_{\alpha_1} h_{ij}^{\alpha_1} \omega_{\alpha_1 \beta} = \sum_k h_{ijk}^{\beta} \omega_k, \quad \beta > 4,$$

where h_{ijk}^{β} are symmetric in the indices i, j, k and $\sum_j h_{jjk}^{\beta} = 0$. Then, we can consider the 3-linear mapping from $M_p \times M_p \times M_p$ into the normal space N_p at p as follows

$$\varphi_2(X_1, X_2, X_3) = \sum_{\substack{\beta > 4 \\ i, j, k}} h_{ijk}^{\beta} \omega_i(X_1) \omega_j(X_2) \omega_k(X_3) e_{\beta}, \quad X_j \in M_p.$$

We call this mapping φ_2 the *second shape operator (first torsion operator)* of M in \widehat{M} . Putting $\widetilde{\varphi}_2(X) = \varphi_2(X, X, X)$ for $X \in M_p$, we get the mapping $\widetilde{\varphi}_2$ from M_p into N_p . For $X = e_1 \cos \theta + e_2 \sin \theta$, we have

$$(2.4) \quad \widetilde{\varphi}_2(X) = \cos 3\theta \cdot F_2 + \sin 3\theta \cdot G_2,$$

where $F_2 = \sum_{\beta > 4} h_{111}^{\beta} e_{\beta}$ and $G_2 = \sum_{\beta > 4} h_{112}^{\beta} e_{\beta}$. We call the dimension of the

image of M_p under $\tilde{\varphi}_2$ the *first torsion index* of M in \hat{M} at $p \in M$ and denote it by $T_1\text{-index}_p M$. It is clear from (2.4) that the image of S_p^1 under $\tilde{\varphi}_2$ is a point, a segment, or an ellipse (a circle) according as $T_1\text{-index}_p M = 0, 1, \text{ or } 2$. We easily see that

$$\left(\|G_2\|^2 - \|F_2\|^2 \right)^2 + 4 \langle G_2, F_2 \rangle^2 = \begin{cases} \left\{ (\text{length of major axis})^2 - (\text{length of minor axis})^2 \right\}^2 & \text{if } T_1\text{-index}_p M = 2, \\ (\text{length of the segment})^4 & \text{if } T_1\text{-index}_p M = 1, \\ 0 & \text{if } T_1\text{-index}_p M = 0. \end{cases}$$

Thus, $(\|G_2\|^2 - \|F_2\|^2)^2 + 4 \langle G_2, F_2 \rangle^2$ is invariant under the rotation of frames $\{e_1, e_2\}$ and $\{e_3, e_4\}$, so it is a differentiable function on M . If $T_1\text{-index}_p M = 0$ at every point $p \in M$, then we see that the geodesic codimension of M is 2. If $T_1\text{-index}_p M = 1$ at every point $p \in M$, then we see that the geodesic codimension of M is 3, this fact was proved in Theorem 4 in [7].

We consider the case that $T_1\text{-index}_p M = 2$ at every point $p \in M$. Then, we can choose a local frame field satisfying (2.3) and

$$(2.5) \quad \omega_{\alpha_1 \alpha_2} \neq 0, \quad \omega_{\alpha_1 \gamma} = 0, \quad \alpha_1 \in \{3, 4\}, \quad \alpha_2 \in \{5, 6\}, \quad 6 < \gamma.$$

It follows from (1.1), (1.2) and (2.5) that we may write

$$\sum_{\alpha_2} h_{i j k}^{\alpha_2} \omega_{\alpha_2 \gamma} = \sum_l h_{i j k l} \omega_l, \quad 6 < \gamma,$$

where $h_{i j k l}^{\alpha_2}$ are symmetric in the indices i, j, k, l and $\sum_i h_{i i j k}^{\alpha_2} = 0$. Then, we can consider the 4-linear mapping from $M_p \times \cdots \times M_p$ into N_p as follows: for $X_j \in M_p, j = 1, 2, 3, 4$,

$$\varphi_3(X_1, X_2, X_3, X_4) = \sum_{\substack{i, j, k, l \\ \tau > 6}} h_{i j k l} \omega_i(X_1) \omega_j(X_2) \omega_k(X_3) \omega_l(X_4) e_\tau.$$

We call this mapping φ_3 the *third shape operator (second torsion operator)* of M in \hat{M} . Putting $\tilde{\varphi}_3(X) = \varphi_3(X, X, X, X)$ for $X \in M_p$, we get the mapping $\tilde{\varphi}_3$ from M_p into N_p . Particularly, for a unit tangent vector $X = e_1 \cos \theta + e_2 \sin \theta$, we have

$$(2.6) \quad \tilde{\varphi}_3(X) = \cos 4\theta \cdot F_3 + \sin 4\theta \cdot G_3,$$

where $F_3 = \sum_{\tau > 6} h_{1111}^\tau e_\tau$ and $G_3 = \sum_{\tau > 6} h_{1112}^\tau e_\tau$. We call the dimension of the image of M_p under $\tilde{\varphi}_3$ the *second torsion index* of M in \hat{M} at p and denote it by $T_2\text{-index}_p M$. From (2.6), we see that the image of S_p^1 under

$\tilde{\varphi}_3$ is a point, a segment, or an ellipse (a circle) according as $T_2\text{-index}_p M = 0, 1,$ or 2 . By the same reason as the case of $\tilde{\varphi}_2$, we easily see that $(\|G_3\|^2 - \|F_3\|^2)^2 + 4\langle F_3, G_3 \rangle^2$ is a differentiable function on M .

Now, we assume that $T_{n-1}\text{-index}_p M = 2$ at every point $p \in M, n \geq 2$. Then, we can choose a local frame field such that

$$(2.7) \quad \begin{cases} \omega_{\alpha_t \alpha_{t+1}} \neq 0, & \omega_{\alpha_t \gamma} = 0, & \alpha_t \in I_t = \{2t+1, 2t+2\}, \\ \alpha_{t+1} \in I_{t+1}, & 2t+4 < \gamma, & t = 0, 1, 2, \dots, n-1. \end{cases}$$

From (1.1), (1.2) and (2.7), we may write

$$\sum_{\alpha_{t+1}} h_{j_1 j_2 \dots j_{t+2}}^{\alpha_{t+1}} \omega_{\alpha_{t+1} \gamma} = \sum_{j_{t+3}=1}^2 h_{j_1 j_2 \dots j_{t+3}}^{\gamma} \omega_{j_{t+3}}, \quad 2t+4 < \gamma,$$

where $h_{j_1 j_2 \dots j_{t+3}}^{\gamma}$ are symmetric in the indices j_1, j_2, \dots, j_{t+3} and $\sum_j h_{j_1 j_2 \dots j_{t+1}}^{\alpha_{t+1}} = 0$ for $t = 0, 1, 2, \dots, n-1$. Hence, for $t = n-1$, we can consider the $(n+2)$ -linear mapping φ_{n+1} from $M_p \times \dots \times M_p$ into N_p as follows: for $X_j \in M_p, j = 1, 2, \dots, n+2$,

$$\varphi_{n+1}(X_1, \dots, X_{n+2}) = \sum_{\gamma > 2n+2} h_{j_1 j_2 \dots j_{n+2}}^{\gamma} \omega_{j_1}(X_1) \dots \omega_{j_{n+2}}(X_{n+2}) e_{\gamma}.$$

Putting $\tilde{\varphi}_{n+1}(X) = \varphi_{n+1}(X, \dots, X)$ for $X \in M_p$, we get the mapping $\tilde{\varphi}_{n+1}$ from M_p into N_p . We call this mapping φ_{n+1} (or $\tilde{\varphi}_{n+1}$) the $(n+1)$ -th shape operator (n -th torsion operator) of M in \hat{M} . Particularly, for a unit tangent vector $X = e_1 \cos \theta + e_2 \sin \theta$, we have

$$(2.8) \quad \tilde{\varphi}_{n+1}(X) = \cos(n+2)\theta \cdot F_{n+1} + \sin(n+2)\theta \cdot G_{n+1},$$

where $F_{n+1} = \sum_{\gamma > 2n+2} h_{1 \dots 1}^{\gamma} e_{\gamma}$ and $G_{n+1} = \sum_{\gamma > 2n+2} h_{1 \dots 1}^{\gamma} e_{\gamma}$. We call the dimension of the image of M_p under $\tilde{\varphi}_{n+1}$ the n -th torsion index of M in \hat{M} at $p \in M$ and denote it by $T_n\text{-index}_p M$. It is clear from (2.8) that the image of S_p^1 under $\tilde{\varphi}_{n+1}$ is a point, a segment, or an ellipse (a circle) according as $T_n\text{-index}_p M = 0, 1,$ or 2 . By the same reason as the case of $\tilde{\varphi}_2$, we see easily that $(\|G_{n+1}\|^2 - \|F_{n+1}\|^2)^2 + 4\langle F_{n+1}, G_{n+1} \rangle^2$ is a function on M .

§ 3. Complete flat minimal surfaces in \hat{M} . In this section, we assume that M is a complete, connected and oriented 2-dimensional Riemannian manifold which is minimally immersed in a Riemannian manifold \hat{M} of constant curvature $c (\neq 0)$ and that the Gaussian curvature K of M is identically zero. Then, as is well known, M may be considered as a Riemann surface. Since $K \equiv 0$ and M is complete, as is stated in [1], M is parabolic, i. e., a negative subharmonic function

on M must be constant. We first have

Lemma 1. *On M we have only one of the following cases :*

S_0 -case $S = c$ and $K_N = 0$,

C_0 -case $S = c$ and $K_N = S^2 = c^2$,

E_0 -case $S = c$, $K_N = \text{constant} > 0$ and $S^2 - K_N > 0$.

Proof. Since $K = 0$ and $K = c - S$, we have

$$S = c = \text{constant} > 0 \text{ on } M.$$

We shall prove that K_N is constant on M . Since $K = 0$ on M , we can choose a neighborhood U of a point $p \in M$ in which there exist isothermal coordinates (u, v) and a frame field $b \in B$ such that

$$ds^2 = du^2 + dv^2, \quad \omega_1 = du, \quad \omega_2 = dv,$$

where ds is the line element of M . We may write $\omega_{i\alpha}$ ($i = 1, 2$ and $3 \leq \alpha$) as follows

$$\omega_{1\alpha} = f_\alpha \omega_1 + g_\alpha \omega_2, \quad \omega_{2\alpha} = g_\alpha \omega_1 - f_\alpha \omega_2,$$

where f_α and g_α are differentiable functions on U . Using the structure equations, we can see that the complex-valued function

$$w(z, \bar{z}) = \|G_1\|^2 - \|F_1\|^2 + 2i \langle F_1, G_1 \rangle$$

is holomorphic in $z = u + iv$, where $F_1 = \sum_{\alpha=3}^n f_\alpha e_\alpha$ and $G_1 = \sum_{\alpha=3}^n g_\alpha e_\alpha$. Hence, $|w(z, \bar{z})|^2$ is a subharmonic function on M . Since $S^2 - K_N = |w(z, z)|^2$ and $S = c$, we see that K_N is a non-negative superharmonic function on M , so it must be constant on M , because M is parabolic. Thus, we have only one of three cases in Lemma.

By Lemma 1, at every point $p \in M$, the image of S_p^1 under $\tilde{\varphi}_1$ is a segment of the constant length or an ellipse with axes (a circle with radius) of the constant lengths according as S_0 -case or E_0 -case (C_0 -case).

From now on, by S -case, C -case, or E -case we mean the case where the image of S_p^1 under the t -th shape operator (the $(t-1)$ -th torsion operator) $\tilde{\varphi}_t$ is a segment, a circle, or an ellipse respectively.

In the S_0 -case, $m\text{-index}_p M = 1$ at every point $p \in M$, so the geodesic codimension of M is 1 by Theorem 1 in [6].

In the C_0 -case and the E_0 -case, $m\text{-index}_p M = 2$ at every point $p \in M$. In the E_0 -case, we can choose a local frame field $b \in B$ such that

$$(3.1) \quad H_3 = \begin{pmatrix} k_1 & 0 \\ 0 & -k_1 \end{pmatrix}, \quad H_4 = \begin{pmatrix} 0 & \sigma_1 k_1 \\ \sigma_1 k_1 & 0 \end{pmatrix}, \quad H_\beta = 0, \quad 4 < \beta,$$

where k_1 and σ_1 are real constants ($\neq 0$) on M , $\sigma_1^2 \neq 1$. From (3.1) we have

$$\omega_{12} = \omega_{34} = 0,$$

which implies that on a neighborhood U of a point $p \in M$ we can choose isothermal coordinates (u, v) such that

$$(3.2) \quad ds^2 = du^2 + dv^2, \quad \omega_1 = du, \quad \omega_2 = dv.$$

In the C_0 -case, since $K \equiv 0$, on a neighborhood of a point of M we can choose isothermal coordinates and a frame field satisfying (3.1) and (3.2), where $\sigma_1 = 1$. Thus, we may consider that the C_0 -case is a special case of the E_0 -case. It follows from (3.1) and (3.2) that we have

$$(3.3) \quad \begin{aligned} \omega_{13} + i\omega_{23} &= k_1 d\bar{z}, & z &= u + iv, \\ \omega_{14} + i\omega_{24} &= i\sigma_1 k_1 d\bar{z}, & \omega_1 + i\omega_2 &= dz. \end{aligned}$$

Since $\omega_{i\beta} = 0$ ($4 < \beta$), we may write

$$(3.4) \quad \omega_{3\beta} + i\sigma_1 \omega_{4\beta} = (f_\beta + ig_\beta) d\bar{z}, \quad \beta < 4,$$

where f_β and g_β are differentiable functions on U . Then, for a unit tangent vector $X = e_1 \cos \theta + e_2 \sin \theta$, the second shape operator $\tilde{\varphi}_2$ is written as

$$\tilde{\varphi}_2(X) = \cos 3\theta \cdot F_2 + \sin 3\theta \cdot G_2,$$

where $F_2 = k_1 \sum_{\beta > 4} f_\beta e_\beta$ and $G_2 = k_1 \sum_{\beta > 4} g_\beta e_\beta$ are normal vector fields on U . Using the structure equations, from (3.2) and (3.4) we see that the complex-valued function

$$w_1(z, \bar{z}) = -k_1^2 \sum_{\beta} (f_\beta - ig_\beta)^2 = \|G_2\|^2 - \|F_2\|^2 + 2i \langle G_2, F_2 \rangle$$

is holomorphic in z , because k_1 is constant on M . Since, as be stated in § 2, $|w_1(z, \bar{z})|^2$ is a differentiable function on M , $|w_1(z, \bar{z})|^2$ is a subharmonic function on M . Since $\omega_{12} = \omega_{34} = 0$, $|w_1(z, \bar{z})|^2 \leq k_1^4 \left\{ \sum_{\beta > 4} (f_\beta^2 + g_\beta^2) \right\}^2 = 4\sigma_1^4 k_1^8 = \text{constant} (> 0)$. Hence $|w_1(z, \bar{z})|^2$ must be constant on M , because M is parabolic. Then, for the second shape operator we have S -case, C -case or E -case on M .

In the S -case for $\tilde{\varphi}_2$ on M , $T_1\text{-index}_p M = 1$ at every point $p \in M$,

so the geodesic codimension of M is 3, see § 2.

We next consider the C -case and E -case for $\tilde{\varphi}_2$. Since the image of S_p^1 under $\tilde{\varphi}_2$ is an ellipse with axes (a circle with radius) of constant length at every point $p \in M$, we can choose a neighborhood U of a point $p \in M$ in which there exist isothermal coordinates (u, v) and a local frame field $b \in B$ satisfying (3. 3) and

$$(3. 5) \quad \begin{cases} \omega_{35} + i\sigma_1\omega_{45} = k_2 d\bar{z}, & \omega_{37} = 0, \\ \omega_{36} + i\sigma_1\omega_{46} = i\sigma_2 k_2 d\bar{z}, & \omega_{47} = 0, \quad 6 < \gamma, \end{cases}$$

where k_2 is a non-zero complex constant on M and σ_2 is a non-zero real constant on M . In the C -case for $\tilde{\varphi}_2$, we may assume that k_2 is a non-zero real constant on M and $\sigma_2 = 1$. From (3. 3) and (3. 5), we have $\omega_{56} = 0$ and we may write

$$\omega_{57} + i\sigma_2\omega_{67} = (f_7 + ig_7)d\bar{z}, \quad 6 < \gamma.$$

Hence, for $X = e_1 \cos \theta + e_2 \sin \theta \in M_p$, we write the third shape operator $\tilde{\varphi}_3$ as

$$\tilde{\varphi}_3(X) = \cos 4\theta \cdot F_3 + \sin 4\theta \cdot G_3,$$

where F_3 and G_3 are real normal vector fields such that $F_3 + iG_3 = k_1 k_2 \sum_{r>6} (f_r + ig_r) e_r$. Continuing this way, we have

Lemma 2. *If the image of S_p^1 under the t -th shape operator $\tilde{\varphi}_t$ is an ellipse with axes (a circle with radius) of constant length at every point $p \in M$, for $t = 1, 2, \dots, s, 2 \leq s$, then the image of S_p^1 under the $(s+1)$ -th shape operator $\tilde{\varphi}_{s+1}$ is a segment of constant length or an ellipse with axes (a circle with radius) of constant length at every point $p \in M$.*

Proof. By induction, from the assumption we can verify that on a neighborhood U of a point $p \in M$ there exist isothermal coordinates (u, v) and a frame field $b \in B$ such that

$$(3. 6)_s \quad \begin{cases} \omega_{\alpha_1\beta_1} + i\sigma_t\omega_{\alpha_2\beta_1} = k_{t+1} d\bar{z}, & dz = du + idv = \omega_1 + i\omega_2, \\ \omega_{\alpha_1\beta_2} + i\sigma_t\omega_{\alpha_2\beta_2} = i\sigma_{t+1} k_{t+1} d\bar{z}, & \omega_{\alpha_1\gamma} = \omega_{\alpha_2\gamma} = 0, \\ \alpha_1 = 2t + 1, & \alpha_2 = 2t + 2, & \beta_1 = 2t + 3, & \beta_2 = 2t + 4, \\ 2t + 4 < \gamma, & t = 0, 1, 2, \dots, s-1, \end{cases}$$

where k_t ($2 \leq t \leq s$) are non-zero complex constant on M and k_1 and σ_t ($1 \leq t \leq s, \sigma_0 = 1$) are non-zero real constant on M . Using the structure equations, from (3. 6), we obtain

$$(3.7)_s \quad \omega_{a_1 a_2} = 0, \quad \alpha_1 = 2t + 1, \quad \alpha_2 = 2t + 2 \quad \text{for } t = 0, 1, 2, \dots, s$$

and we may write

$$(3.8) \quad \omega_{a_1 a_2} + i \sigma_s \omega_{a_1 a_2} = (f_t + i g_t) d\bar{z}, \quad a_1 = 2s + 1, \quad a_2 = 2s + 2 < \gamma.$$

Let F_{s+1} and G_{s+1} be real normal vector fields on U such that $F_{s+1} + i G_{s+1} = k_1 k_2 \cdots k_s \sum_{\gamma > 2s+2} (f_\gamma + i g_\gamma) e_\gamma$, then for a unit tangent vector $X = e_1 \cos \theta + e_2 \sin \theta \in M_p$ the $(s+1)$ -th shape operator $\tilde{\varphi}_{s+1}$ is written as

$$\tilde{\varphi}_{s+1}(X) = \cos(s+2)\theta \cdot F_{s+1} + \sin(s+2)\theta \cdot G_{s+1}.$$

From (3.6)_s, (3.7)_s and (3.8), we see that the complex-valued function

$$w_s(z, \bar{z}) = -(k_1 \bar{k}_2 \cdots \bar{k}_s)^2 \sum_{\gamma > 2s+2} (f_\gamma - i g_\gamma)^2 = |G_{s+1}|^2 - |F_{s+1}|^2 + 2i \langle G_{s+1}, F_{s+1} \rangle$$

is holomorphic in z , because k_t ($t=1, 2, \dots, s$) are constant on M . Since, as be stated in § 2, $|w_s(z, \bar{z})|^2$ is a differentiable function on M , $|w_s(z, \bar{z})|^2$ is a subharmonic function on M . On the other hand, from (3.7)_s we have $|w_s(z, \bar{z})|^2 \leq |k_1 k_2 \cdots k_s|^4 \{ \sum (f_\gamma^2 + g_\gamma^2) \}^2 = |k_1 k_2 \cdots k_s|^4 \{ (1 + \sigma_{s-1}^2) \sigma_s^2 \cdot |k_s|^2 / \sigma_{s-1}^2 \}^2 = \text{constant} (> 0)$ on M . Hence the subharmonic function $|w_s(z, \bar{z})|^2$ must be constant on M , because M is parabolic. Thus, at every point $p \in M$ the image of a unit tangent circle S_p^1 to M under $\tilde{\varphi}_{s+1}$ is a segment of constant length, a circle with constant length or an ellipse with axes of constant length on M .

If the image of S_p^1 under $\tilde{\varphi}_{s+1}$ is a segment of constant length at every point $p \in M$, then $T_s\text{-index}_p M = 1$ at every point p so the geodesic codimension of M is $2s + 1$.

If the geodesic codimension of M is even $2s$, using the structure equations, from (3.6)_s and (3.8) we have contradiction. Hence, the geodesic codimension of M is odd $2m + 1$ and the images of unit tangent circles to M under $\tilde{\varphi}_{m+1}$ are segments of constant length on M . Thus, we have proved the following

Theorem 1. *Let M be a 2-dimensional, connected, oriented and complete Riemannian manifold which is minimally immersed in a $(2 + \nu)$ -dimensional Riemannian manifold \hat{M} of non-zero constant curvature c . If Gaussian curvature of M is identically zero and the image of M under the immersion is not contained in a totally geodesic submanifold of \hat{M} , i. e., ν is the geodesic codimension of M , then ν is odd $2m + 1$. Furthermore, the images of unit tangent circles to M*

under the t -th shape operators ($1 \leq t \leq m$) are ellipses with axes (or circles with radius) of constant length and the images of unit tangent circles to M under the $(m+1)$ -th shape operator are segments of constant length on M .

In view of Theorem 1, we see that on a neighborhood of a point of M there exist isothermal coordinates (u, v) and a local frame field satisfying (3.6) _{$m+1$} , where $\sigma_{m+1}=0$ and k_{m+1} is a positive constant on M . In general, however, the constants k_t ($1 \leq t \leq m+1$) and σ_t ($1 \leq t \leq m$) depend on the immersions. Using the structure equations, from (3.7) _{m} we get

$$(3.9)_m \quad (1 + \sigma_{t+1}^2) |k_{t+1}|^2 = 2c\sigma_t^2 / (1 + \sigma_t^2), \quad \sigma_0 = 1, \quad \sigma_{m+1} = 0, \quad 0 \leq t \leq m.$$

Since $c > 0$ from Lemma 1, we consider the case where \widehat{M} is a $(2m+3)$ -dimensional sphere $S^{2m+3}(c)$ of constant curvature c . We may consider $S^{2m+3}(c) \subset E^{2m+4}$ and set $\sqrt{c}x = e_{2m+4}$. Let $E_t = e_{\alpha_1} + i\sigma_t e_{\alpha_2}$ and $E_t^* = \sigma_t e_{\alpha_1} - i e_{\alpha_2}$, $\alpha_1 = 2t + 1$, $\alpha_2 = 2t + 2$, where $t = 0, 1, 2, \dots, m$. Since $E_t^* = ((1 + \sigma_t^2)/2\sigma_t)\bar{E}_t - ((1 - \sigma_t^2)/2\sigma_t)E_t$, we have the following Frenet formulas of M

$$\begin{aligned} dx &= \frac{1}{2} (\bar{E}_0 dz + E_0 d\bar{z}), & z &= u + iv, \\ dE_0 &= -cx dz + k_1 E_1 d\bar{z}, \\ dE_1 &= -((1 + \sigma_1^2)k_1/2)E_0 dz - \{((1 - \sigma_1^2)k_1/2)\bar{E}_0 - k_2 E_2\} d\bar{z}, \\ dE_2 &= -((1 + \sigma_2^2)k_2/2\sigma_1)\bar{E}_1^* dz - \{((1 - \sigma_2^2)k_2/2\sigma_1) E_1^* - k_3 E_3\} d\bar{z}, \\ (3.10)_m & \dots\dots\dots \\ dE_t &= -((1 + \sigma_t^2)k_t/2\sigma_{t-1})\bar{E}_{t-1}^* dz - \{((1 - \sigma_t^2)k_t/2\sigma_{t-1}) E_{t-1}^* - k_{t+1} E_{t+1}\} d\bar{z}, \\ & \dots\dots\dots \\ dE_{m-1} &= -((1 + \sigma_{m-1}^2)k_{m-1}/2\sigma_{m-2})\bar{E}_{m-2}^* dz \\ & \quad - \{((1 - \sigma_{m-1}^2)k_{m-1}/2\sigma_{m-2}) E_{m-2}^* - k_m E_m\} d\bar{z}, \\ dE_m &= -((1 + \sigma_m^2)k_m/2\sigma_{m-1})\bar{E}_{m-1}^* dz \\ & \quad - \{((1 - \sigma_m^2)k_m/2\sigma_{m-1}) E_{m-1}^* - k_{m+1} e_{2m+3}\} d\bar{z}, \\ de_{2m+3} &= -(k_{m+1}/2\sigma_m)\bar{E}_m^* dz - (k_{m+1}/2\sigma_m) E_m^* d\bar{z}. \end{aligned}$$

From (3.9) _{m} and (3.10) _{m} we easily see that the vector fields $E_0, E_1, E_2, \dots, E_m, e_{2m+3}$ and $x = e_{2m+4}$ satisfy the following equation

$$(3.11) \quad \partial^2 Y / (\partial z \cdot \partial \bar{z}) = -(c/2)Y.$$

Remark. In Theorem 1, it seems to be difficult to prove the rigidity for the minimal immersion of the Euclidean plane into a sphere without

any assumption.

Under the assumption of Theorem 1, if the images of unit tangent circles to M under the t -th shape operators ($1 \leq t \leq m$) are circles with radius of constant length on M , then we can choose a neighborhood of a point of M in which there exist isothermal coordinates and a local frame field $b \in B$ satisfying (3.6) $_{m+1}$ such that $\sigma_t = 1$ ($1 \leq t \leq m$), $\sigma_{m+1} = 0$ and k_t ($1 \leq t \leq m+1$) are non-zero real constants on M . These constants are independent of the immersion. When $c=1$, in the same way as in §8 of [7], from (3.10) $_m$ we can verify that M is the surface given by

$$(3.12) \quad x = \frac{1}{\sqrt{2(m+2)}} \sum_{j=1}^{m+2} \{A_j \exp i\sqrt{2}(u \cdot \sin \frac{2j-1+\varepsilon}{2(m+2)}\pi + v \cdot \cos \frac{2j-1+\varepsilon}{2(m+2)}\pi) \\ + \bar{A}_j \exp(-i\sqrt{2})(u \cdot \sin \frac{2j-1+\varepsilon}{2(m+2)}\pi + v \cdot \cos \frac{2j-1+\varepsilon}{2(m+2)}\pi)\},$$

where A_1, A_2, \dots, A_{m+2} are constant vectors in $C^{m+2} = E^{2m+4}$ such that

$$A_j \cdot A_j = A_j \cdot A_k = A_j \cdot \bar{A}_k = 0, \quad A_j \cdot \bar{A}_j = 1, \quad j \neq k,$$

and $\varepsilon = 0$ or 1 according as $m = \text{odd}$ or even . Thus, we have proved the following

Theorem 2. *Under the assumption of Theorem 1, if \hat{M} is a unit sphere and the images of unit tangent circles to M under the t -th shape operators ($1 \leq t \leq m$) are circles with radius of constant length on M , then the immersion is uniquely determined up to a rigid motion of a sphere.*

We are interested in examples of flat minimal surfaces other than (3.12). In the next section, we shall find examples other than (3.12).

§ 4. Examples of minimal immersions of the Euclidean plane into a sphere of constant curvature 1. In this section, we give examples of flat minimal surfaces in $S^3(1)$ and $S^7(1)$ other than (3.12).

We first give examples in $S^3(1)$, that is, find solutions of (3.10) $_1$ in the case $c = 1$. Noticing (3.11), we choose three fixed constant vectors A_1, A_2, A_3 in $C^3 = E^6$ such that

$$(4.1) \quad A_j \cdot A_j = A_j \cdot A_k = A_j \cdot \bar{A}_k = 0, \quad \sum_{j=1}^3 A_j \cdot \bar{A}_j = 1, \\ j, k = 1, 2, 3, \quad j \neq k,$$

and let

$$(4.2) \quad x = \frac{1}{\sqrt{2}} \sum_{j=1}^3 \{A_j \exp \frac{1}{\sqrt{2}}(ze^{i\alpha_j} - \bar{z}e^{-i\alpha_j}) + \bar{A}_j \exp \frac{1}{\sqrt{2}}(-ze^{i\alpha_j} + \bar{z}e^{-i\alpha_j})\},$$

where the bar denotes the conjugate and α_j ($j = 1, 2, 3$) are real constant numbers. It is clear that $x = \bar{x}$ and $x \cdot x = 1$ by (4.1). In this case, we set

$$E_0 = 2 \frac{\partial x}{\partial \bar{z}} = - \sum_{j=1}^3 e^{-i\alpha_j} \{ A_j \exp \frac{1}{\sqrt{2}} (ze^{i\alpha_j} - \bar{z}e^{-i\alpha_j}) - \bar{A}_j \exp \frac{1}{\sqrt{2}} (\bar{z}e^{-i\alpha_j} - ze^{i\alpha_j}) \},$$

$$E_1 = \frac{\partial E_0}{k_1 \partial \bar{z}} = \frac{1}{k_1 \sqrt{2}} \sum_{j=1}^3 e^{-2i\alpha_j} \{ A_j \exp \frac{(ze^{i\alpha_j} - \bar{z}e^{-i\alpha_j})}{\sqrt{2}} + \bar{A}_j \exp \frac{(-ze^{i\alpha_j} + \bar{z}e^{-i\alpha_j})}{\sqrt{2}} \},$$

$$e_s = \frac{\partial E_1}{k_2 \partial \bar{z}} + \frac{(1 - \sigma_1^2) k_1 \bar{E}_0}{2k_2} = \frac{1}{2k_1 k_2} \sum_{j=1}^3 \left[\frac{1 - \sigma_1^2}{1 + \sigma_1^2} e^{i\alpha_j} - e^{-3i\alpha_j} \right] \cdot \{ A_j \exp \frac{1}{\sqrt{2}} (ze^{i\alpha_j} - \bar{z}e^{-i\alpha_j}) - \bar{A}_j \exp \frac{1}{\sqrt{2}} (-ze^{i\alpha_j} + \bar{z}e^{-i\alpha_j}) \},$$

where k_1 , k_2 and σ_1 are non-zero real constant numbers satisfying (3.9)₁. Then, we easily see that these vectors satisfy the equation (3.10)₁ in the case $c = 1$. From the above equations we have

$$x \cdot E_0 = E_0 \cdot E_1 = E_0 \cdot \bar{E}_1 = E_1 \cdot e_s = e_s \cdot x = 0,$$

$$E_0 \cdot \bar{E}_0 = 2, \quad E_1 \cdot \bar{E}_1 = 1 + \sigma_1^2.$$

Hence, we see that (4.2) is a solution of (3.10)₁ if and only if there hold the following equalities

$$x \cdot E_1 = E_0 \cdot E_0 = E_0 \cdot e_s = 0, \quad E_1 \cdot E_1 = 1 - \sigma_1^2,$$

$$e_s = \bar{e}_s, \quad e_s \cdot e_s = 1.$$

By the above definitions of x , E_0 , E_1 and e_s , these equations are equivalent to the following

$$(4.3) \quad \sum_{j=1}^3 A_j \cdot \bar{A}_j e^{-2i\alpha_j} = 0,$$

$$(4.4) \quad \sum_{j=1}^3 A_j \cdot \bar{A}_j e^{-4i\alpha_j} = k_1^2 (1 - \sigma_1^2),$$

$$(4.5) \quad \sum_{j=1}^3 A_j \cdot \bar{A}_j e^{-6i\alpha_j} = -2k_1^2 k_2^2,$$

$$(4.6) \quad \cos 3\alpha_j - ((1 - \sigma_1^2)/(1 + \sigma_1^2)) \cos \alpha_j = 0 \quad \text{for } j = 1, 2, 3.$$

Now, we shall find constant vectors A_j ($j = 1, 2, 3$) and real constant numbers α_j ($j = 1, 2, 3$) satisfying (4.3) ~ (4.6) under the condition

(3.9)₁. We consider the following special case:

$$(4.7) \quad A_1 \cdot \bar{A}_1 = A_2 \cdot \bar{A}_2, \quad -\alpha_1 = \alpha = \alpha_2, \quad 2\alpha_3 = \pi.$$

Then, from (4. 1), (4. 3) and (4. 7) we have

$$(4. 8) \quad A_1 \cdot \bar{A}_1 = A_2 \cdot \bar{A}_2 = \frac{1}{2(1 + \cos 2\alpha)}, \quad A_3 \cdot \bar{A}_3 = \frac{\cos 2\alpha}{1 + \cos 2\alpha}.$$

Hence, in this special case, if there exist A_j and α_j ($j = 1, 2, 3$) satisfying (4. 3) ~ (4. 6), the solutions (4. 2) are determined only by one parameter α . Using $(1 + \sigma_i^2)k_i^2 = 1$, from (4. 4) and (4. 8) we have

$$(4. 9) \quad \cos 2\alpha = \{(1 - \sigma_i^2)/(1 + \sigma_i^2) + 1\}/2 = 1/(1 + \sigma_i^2) = k_i^2.$$

Since $k_1 \neq 0$, $\sigma_1 \neq 0$ and $(1 + \sigma_i^2)k_i^2 = 1$, we have $0 < k_i^2 < 1$, which implies $0 < \cos 2\alpha < 1$ by (4. 9), so that we may assume that $0 < \alpha < \pi/4$. Then, from (4. 9) we have

$$\cos \alpha = \sqrt{(k_i^2 + 1)/2}, \quad \sin \alpha = \sqrt{(1 - k_i^2)/2}.$$

and from (4. 8) we have

$$(4. 10) \quad A_1 \cdot \bar{A}_1 = A_2 \cdot \bar{A}_2 = \frac{1}{2(1 + k_i^2)}, \quad A_3 \cdot \bar{A}_3 = \frac{k_i^2}{1 + k_i^2}.$$

We can easily see that these constants satisfy (4. 5) and (4. 6). Thus, we obtain examples given by

$$(4. 11) \quad x = \frac{1}{\sqrt{2}} [A_1 \exp i(v\sqrt{k_1^2 + 1} - u\sqrt{1 - k_1^2}) + \bar{A}_1 \exp i(u\sqrt{1 - k_1^2} - v\sqrt{1 + k_1^2}) \\ + A_2 \exp i(v\sqrt{1 + k_1^2} + u\sqrt{1 - k_1^2}) + \bar{A}_2 \exp(-i(v\sqrt{1 + k_1^2} + u\sqrt{1 - k_1^2})) \\ + A_3 \exp(\sqrt{2}iu) + \bar{A}_3 \exp(-\sqrt{2}iu)],$$

where A_1, A_2, A_3 are fixed constant vectors in $C^3 = E^8$ satisfying (4. 1) and (4. 10) and k_1 is a positive constant smaller than 1.

Next, we shall give examples in $S^7(1)$, that is, find solutions of (3. 10)₂ in the case where $c = 1$ and $k_2 =$ real constant $\neq 0$. Noticing (3. 11), we choose four fixed vectors A_1, A_2, A_3, A_4 in $C^4 = E^8$ such that

$$(4. 12) \quad A_j \cdot A_j = A_j \cdot A_k = A_j \cdot \bar{A}_k = 0, \quad \sum_{j=1}^4 A_j \cdot \bar{A}_j = 1, \quad j, k = 1, 2, 3, 4, \quad j \neq k.$$

Let

$$(4. 13) \quad x = \frac{1}{\sqrt{2}} \sum_{j=1}^4 \{A_j \exp \frac{1}{\sqrt{2}}(ze^{i\alpha_j} - \bar{z}e^{-i\alpha_j}) + \bar{A}_j \exp \frac{1}{\sqrt{2}}(-ze^{i\alpha_j} + \bar{z}e^{-i\alpha_j})\},$$

where α_j ($j = 1, 2, 3, 4$) are real constant numbers. It is clear from (4. 12) and (4. 13) that $x = \bar{x}$ and $x \cdot x = 1$. We define vectors E_0, E_1, E_2

and e_7 as follows

$$(4.14) \quad \begin{cases} E_0 = 2 \frac{\partial x}{\partial \bar{z}} = - \sum_{j=1}^4 e^{-i\alpha_j} B_j^* , \\ E_1 = \frac{\partial E_0}{k_1 \partial \bar{z}} = \frac{1}{k_1 \sqrt{2}} \sum_{j=1}^4 e^{-2i\alpha_j} B_j , \\ E_2 = \frac{\partial E_1}{k_2 \partial \bar{z}} + \frac{(1-\sigma_1^2)k_1 \bar{E}_0}{2k_2} = \frac{1}{2k_1 k_2} \sum_{j=1}^4 \left\{ \frac{1-\sigma_1^2}{1+\sigma_1^2} e^{i\alpha_j} - e^{-3i\alpha_j} \right\} B_j^* , \\ e_7 = \frac{\partial E_2}{k_3 \partial \bar{z}} + \frac{(1-\sigma_2^2)k_2}{2k_3 \sigma_1} \left\{ \frac{1+\sigma_1^2}{2\sigma_1} \bar{E}_1 - \frac{1-\sigma_1^2}{2\sigma_1} E_1 \right\} \\ = \frac{1}{2\sqrt{2}} \frac{1}{k_1 k_2 k_3} \sum_{j=1}^4 \left\{ e^{-4i\alpha_j} - \frac{1-\sigma_1^2}{1+\sigma_1^2} + \frac{1-\sigma_2^2}{1+\sigma_2^2} e^{3i\alpha_j} - \frac{(1-\sigma_1^2)(1-\sigma_2^2)}{(1+\sigma_1^2)(1+\sigma_2^2)} e^{-2i\alpha_j} \right\} B_j , \end{cases}$$

where $B_j = A_j \exp((ze^{i\alpha_j} - \bar{z}e^{-i\alpha_j})/\sqrt{2}) + \bar{A}_j \exp((-ze^{i\alpha_j} + \bar{z}e^{-i\alpha_j})/\sqrt{2})$ and $B_j^* = A_j \exp((ze^{i\alpha_j} - \bar{z}e^{-i\alpha_j})/\sqrt{2}) - \bar{A}_j \exp((-ze^{i\alpha_j} + \bar{z}e^{-i\alpha_j})/\sqrt{2})$ and k_1, k_2, k_3, σ_1 and σ_2 are real constant numbers satisfying (3.9)₂. From (4.14) we easily see that

$$\begin{aligned} x \cdot E_0 = E_0 \cdot E_1 = E_0 \cdot \bar{E}_1 = E_0 \cdot e_7 = E_1 \cdot E_2 = E_1 \cdot \bar{E}_2 = E_2 \cdot x = E_2 \cdot e_7 = 0, \\ E_0 \cdot \bar{E}_0 = 2, \quad E_1 \cdot \bar{E}_1 = 1 + \sigma_1^2, \quad E_2 \cdot \bar{E}_2 = 1 + \sigma_2^2. \end{aligned}$$

Hence, (4.13) is a solution of (3.10)₂ if and only if the following conditions are satisfied:

$$(4.15) \quad \begin{aligned} x \cdot E_1 = x \cdot e_7 = E_0 \cdot E_0 = E_0 \cdot E_2 = E_0 \cdot \bar{E}_2 = E_1 \cdot e_7 = 0, \\ E_1 \cdot \bar{E}_1 = 1 - \sigma_1^2, \quad E_2 \cdot \bar{E}_2 = 1 - \sigma_2^2, \end{aligned}$$

$$(4.16) \quad e_7 \text{ is a unit real vector.}$$

Using (4.12), we see that the conditions (4.15) are equivalent to the following equalities:

$$(4.17) \quad \begin{cases} \sum_{j=1}^4 A_j \cdot \bar{A}_j e^{-2i\alpha_j} = 0, \\ \sum_{j=1}^4 A_j \cdot \bar{A}_j e^{-4i\alpha_j} = k_1^2 (1 - \sigma_1^2), \\ \sum_{j=1}^4 A_j \cdot \bar{A}_j e^{-6i\alpha_j} = -2k_1^2 k_2^2 (1 - \sigma_2^2). \end{cases}$$

In order to find constant vectors A_j in C^4 and constant numbers α_j ($j = 1, 2, 3, 4$) satisfying (4.16) and (4.17), we consider the following special case:

$$(4.18) \quad A_1 \cdot \bar{A}_1 = A_3 \cdot \bar{A}_3, \quad -\alpha_1 = \alpha_3 = \alpha, \quad \alpha_2 = 0, \quad 2\alpha_4 = \pi.$$

By means of (3. 9)₂ and (4. 12), from (4. 17) we have

$$(4. 19) \quad \cos 2\alpha = k_2^2(1 - \sigma_2^2)/(2\sigma_1^2) = k_1^2(1 - k_3^2) =: k,$$

$$(4. 20) \quad \begin{aligned} A_1 \cdot \bar{A}_1 &= A_3 \cdot \bar{A}_3 = (k_1^2 - 1)/(2k^2 - 2), \\ A_2 \cdot \bar{A}_2 &= (k - k_1^2)/(2k - 2), \quad A_4 \cdot \bar{A}_4 = (k + k_1^2)/(2k + 2). \end{aligned}$$

Therefore, we have seen that if there exist A_j and α_j ($j = 1, 2, 3, 4$) satisfying (4. 12), (4. 16) and (4. 17), then (4. 13) is determined only by two parameters k_1 and k_3 . In this case, we can show that e_7 is a unit real vector. Since $k_3^2 = 2\sigma_2^2/(1 + \sigma_2^2)$, $k_1^2(1 + \sigma_1^2) = 1$, $\sigma_1 \neq 0$ and $\sigma_2 \neq 0$, we have

$$0 < k_1^2 < 1 \quad \text{and} \quad 0 < k_3^2 < 2,$$

which implies $|\cos 2\alpha| < 1$ by (4. 19). Hence, we may assume that $0 < \alpha < \pi/2$. Thus, we obtain examples of minimal immersions of the Euclidean plane into a sphere $S^7(1)$ given by

$$(4. 21) \quad \begin{aligned} x &= \frac{1}{\sqrt{2}} [A_1 \exp i(v\sqrt{1+k} - u\sqrt{1-k}) + \bar{A}_1 \exp i(u\sqrt{1-k} - v\sqrt{1+k}) \\ &+ A_2 \exp(\sqrt{2} iv) + \bar{A}_2 \exp(-\sqrt{2} iv) \\ &+ A_3 \exp i(v\sqrt{1+k} - u\sqrt{1-k}) + \bar{A}_3 \exp(-i(v\sqrt{1+k} - u\sqrt{1-k})) \\ &+ A_4 \exp(\sqrt{2} iu) + \bar{A}_4 \exp(-\sqrt{2} iu)], \end{aligned}$$

where A_1, A_2, A_3, A_4 are fixed constant vectors in $C^4 = E^8$ satisfying (4. 12) and (4. 20) and k is a constant real number such that

$$k := k_1^2(1 - k_3^2), \quad 0 < k_1^2 = \text{const.} < 1, \quad 0 < k_3^2 = \text{const.} < 2.$$

Remark. We have obtained many examples of the Euclidean plane minimally immersed into the Euclidean unit spheres S^5 and S^7 other than Otsuki's surfaces. When $m = 1$ and $m = 2$, Otsuki's surfaces (3. 12) are included in these examples as the special case where $\sigma_1^2 = 1$ and $\sigma_2^2 = 1$.

§ 5. Compact minimal surfaces of non-negative ($\neq 0$) curvature in \hat{M} . In this section, we shall consider *connected compact minimal surfaces of non-negative curvature K ($\neq 0$)* in a $(2 + \nu)$ -dimensional Riemannian manifold \hat{M} of constant curvature c .

Let U be a neighborhood of a point $p \in M$ in which there exist isothermal coordinates (u, v) and a frame field $b \in B$ such that

$$(5. 1) \quad ds^2 = E\{du^2 + dv^2\}, \quad \omega_1 = \sqrt{E} du, \quad \omega_2 = \sqrt{E} dv,$$

where ds is the line element of M and $E = E(u, v)$ is a positive function on U . In this case, we may write

$$\omega_{1\alpha} = f_\alpha \omega_1 + g_\alpha \omega_2, \quad \omega_{2\alpha} = g_\alpha \omega_1 - f_\alpha \omega_2, \quad 3 < \alpha,$$

where f_α and g_α are functions on U . Then, using the structure equations, we can verify that the complex valued function

$$(5.2) \quad w(z, \bar{z}) = E^2 (\|G_1\|^2 - \|F_1\|^2) + 2iE^2 \langle F_1, G_1 \rangle, \quad F_1 = \sum f_\alpha e_\alpha, \quad G_1 = \sum g_\alpha e_\alpha,$$

is holomorphic in $z = u + iv$. Then, we have the following

Lemma 3. *We have $S^2 - K_N = 0$ on M .*

Proof. By an easy computation, we see that $S^2 - K_N = |w(z, \bar{z})|^2 / E^4$ on M . If $S^2 - K_N = 0$ does not hold identically on M , $S^2 - K_N$ takes its positive maximum A at some $p_0 \in M$. Let U be a neighborhood of p_0 in which $S^2 - K_N > 0$ and there exist isothermal coordinates (u, v) and a frame field $b \in B$ satisfying (5.1). Then, from (5.2) we have

$$(5.3) \quad \Delta \log (S^2 - K_N) = -4\Delta \log E = 8EK, \quad \Delta = \partial^2 / \partial u^2 + \partial^2 / \partial v^2,$$

because the Gaussian curvature K is given by $K = -(1/2E)\Delta \log E$. If $K \geq 0$, the function $\log(S^2 - K_N)$ is a subharmonic function on U , so it must be constant A on U . Therefore, the closed set $\{p \in M | S^2 - K_N = A \text{ at } p\}$ of M is open in M . Since M is connected, $S^2 - K_N$ is identically a positive constant A on M . It follows from this fact and (5.3) that K is identically zero, which contradicts $K \neq 0$ on M .

By Lemma 3, if $m\text{-index}_p M \neq 0$ at every point $p \in M$, then we can choose a neighborhood U of a point $p \in M$ in which there exist isothermal coordinates and frame fields satisfying (5.1) and

$$(5.4) \quad \begin{cases} \omega_{13} = k_1 \omega_1 = \omega_{24}, & \omega_{1\beta} = \omega_{2\beta} = 0, \\ \omega_{23} = -k_1 \omega_2 = -\omega_{14}, & 4 < \beta, \end{cases}$$

where k_1 is a positive differentiable function M . Using the structure equations, from (5.4) we have

$$\omega_{34} = 2\omega_{12} - (\log k_1)_2 \omega_1 + (\log k_1)_1 \omega_2,$$

where $d(\log k_1) = \sum_{j=1}^2 (\log k_1)_j \omega_j$. Furthermore, from (5.4) we may write

$$(5.5) \quad \omega_{3\beta} = f_\beta \omega_1 + g_\beta \omega_2, \quad \omega_{4\beta} = g_\beta \omega_1 - f_\beta \omega_2, \quad 4 < \beta,$$

and define two normal vector fields $F_2 = \sum_{\beta>4} f_\beta e_\beta$ and $G_2 = \sum_{\beta>4} g_\beta e_\beta$ on U .

Then, we can write the second shape operator $\tilde{\varphi}_2$ as

$$\tilde{\varphi}_2(X) = k_1\{\cos 3\theta \cdot F_2 + \sin 3\theta \cdot G_2\}, \quad X = e_1 \cos \theta + e_2 \sin \theta \in M_p.$$

Using the structure equations, from (5.5) and (5.6) we see that the complex-valued function

$$\omega_1(z, \bar{z}) = E^3 k_1^2 (\|G_2\|^2 - \|F_2\|^2) + 2iE^3 k_1^2 \langle F_2, G_2 \rangle$$

is holomorphic in z . As stated in § 2, we see that $|\omega_1(z, \bar{z})|^2/E^6 = k_1^4 \{(\|G_2\|^2 - \|F_2\|^2)^2 + 4 \langle G_2, F_2 \rangle^2\}$ is a differentiable function on M . Hence, by the same reason as the proof of Lemma 3, we can prove

Lemma 4. *If $m\text{-index}_p M \neq 0$ at every point $p \in M$, then, at each point $p \in M$, the image of S_p^1 under the second shape operator is a point p or a circle according as $T_1\text{-index}_p M = 0$ or $\neq 0$.*

By Lemma 3 and Lemma 4, if $m\text{-index}_p M \neq 0$ and $T_1\text{-index}_p M \neq 0$ at every point $p \in M$, then we can choose a neighborhood U of a point $p \in M$ in which there exist isothermal coordinates (u, v) and a frame field $b \in B$ satisfying (5.1), (5.4) and

$$(5.7) \quad \begin{aligned} \omega_{35} &= k_2 \omega_1 = \omega_{46}, & \omega_{37} &= \omega_{47} = 0, \\ \omega_{36} &= k_2 \omega_2 = -\omega_{45}, & 6 &< \gamma, \end{aligned}$$

where k_2 is a positive differentiable function on M . Let $\lambda_2 = k_1 k_2$ and $d(\log \lambda_2) = \sum_{j=1}^2 (\log \lambda_2)_j \omega_j$, then from (5.7) we have

$$(5.8) \quad \omega_{56} = 3\omega_{12} - (\log \lambda_2)_2 \omega_1 + (\log \lambda_2)_1 \omega_2$$

and we may write

$$\omega_{57} = f_7 \omega_1 + g_7 \omega_2, \quad \omega_{67} = g_7 \omega_1 - f_7 \omega_2, \quad 6 < \gamma.$$

Hence, for a unit tangent vector $X = e_1 \cos \theta + e_2 \sin \theta \in M_p$, the third shape operator $\tilde{\varphi}_3$ is written as

$$\tilde{\varphi}_3(X) = \lambda_2 \{\cos 4\theta \cdot F_3 + \sin 4\theta \cdot G_3\},$$

where $F_3 = \sum_{r>6} f_r e_r$ and $G_3 = \sum_{r>6} g_r e_r$ are normal vector fields on U .

Continuing this way, we have the following

Theorem 3. *Let M be a 2-dimensional, connected and compact Riemannian manifold of non-negative curvature ($\neq 0$) which is minimally immersed in a $(2 + \nu)$ -dimensional Riemannian manifold \hat{M} of constant*

curvature c . If we have

- (A) the image of M is not contained in a totally geodesic submanifold of \widehat{M} , i. e., ν is the geodesic codimension of M ,
 - (B) $m\text{-index}_p M \neq 0$ at every point $p \in M$,
 - (C) $T_n\text{-index}_p M$ ($n = 1, 2, \dots$) are defined at every point $p \in M$ and $T_n\text{-index}_p M \neq 0$ at every point $p \in M$ for $n = 1, 2, \dots, [\frac{\nu}{2}] - 1$,
- then ν must be even $2m$ and the image of unit tangent circles to M under the n -th shape operators $\tilde{\varphi}_n$ ($1 \leq n \leq m$) are circles.

Proof. For the (first) shape operator and the second shape operator, we have proved our latter assertion in Lemma 3 and Lemma 4. By the induction on n , we shall prove that the image of a unit tangent circle S_p^1 under the n -th shape operators ($1 \leq n \leq [\frac{\nu}{2}]$) are circles for every $p \in M$. Now, we assume that the above assertion holds for all $t \leq s - 1$. Then, we can choose a neighborhood U of a point $p \in M$ in which there exist isothermal coordinates (u, v) and a frame field $b \in B$ satisfying (5.1) and

$$(5.9) \quad \begin{cases} \omega_{\alpha_1\beta_1} = k_t\omega_1 = \omega_{\alpha_2\beta_2}, & \omega_{\alpha_1\gamma} = \omega_{\alpha_2\gamma} = 0, \\ \omega_{\alpha_1\beta_2} = k_t\omega_2 = -\omega_{\alpha_2\beta_1}, & 2t + 2 < \gamma, \\ \alpha_1 = 2t - 1, \quad \alpha_2 = 2t, \quad \beta_1 = 2t + 1, \quad \beta_2 = 2t + 2, \\ t = 1, 2, \dots, s - 1, \end{cases}$$

where k_t ($1 \leq t \leq s - 1$) are positive differentiable functions on M . Using the structure equations, from (5.9) we have

$$(5.10)_{s-1} \quad \omega_{\beta_1\beta_2} = (t + 1)\omega_{12} - (\log \lambda_t)_2\omega_1 + (\log \lambda_t)_1\omega_2,$$

where $\beta_1 = 2t + 1, \beta_2 = 2t + 2, \lambda_t := k_1 \cdot k_2 \cdots k_t$ and $d(\log \lambda_t) = \sum_{j=1}^2 (\log \lambda_t)_j \omega_j$ for $t = 1, 2, \dots, s - 1$. From (5.9), we may write

$$\begin{aligned} \omega_{\alpha_1\gamma} &= f_\gamma\omega_1 + g_\gamma\omega_2, & a_1 &= 2s - 1, \\ \omega_{\alpha_2\gamma} &= g_\gamma\omega_1 - f_\gamma\omega_2, & a_2 &= 2s, \quad 2s < \gamma. \end{aligned}$$

Hence, for a unit tangent vector $X = e_1 \cos \theta + e_2 \sin \theta \in M_p$, the s -th shape operator $\tilde{\varphi}_s$ is written as

$$\tilde{\varphi}_s(X) = \lambda_{s-1} \{ \cos(s + 1)\theta \cdot F_s + \sin(s + 1)\theta \cdot G_s \},$$

where $F_s = \sum_{r>2s} f_r e_r$ and $G_s = \sum_{r>2s} g_r e_r$ are normal vector fields on a neighborhood U of $p \in M$. Using the structure equations, from (5.9) and (5.10)_{s-1} we can verify the complex-valued function

$$(5.11) \quad w_{s-1}(z, \bar{z}) = E^{s+1} \lambda_{s-1}^2 (\|G_s\|^2 - \|F_s\|^2) + 2i E^{s+1} \lambda_{s-1}^2 \langle G_s, F_s \rangle$$

is holomorphic in $z = u + iv$. Since λ_{s-1} is a differentiable function on M , as be stated in §2, $|w_{s-1}(z, \bar{z})|^2 / E^{2s+2} = \lambda_{s-1}^4 \{(\|G_s\|^2 - \|F_s\|^2)^2 + 4 \langle G_s, F_s \rangle^2\}$ is a differentiable function on M . Therefore, by the same way as the proof of Lemma 3, we see that at each point $p \in M$ the image of a unit tangent circle S_p^1 to M under the s -th shape operator $\tilde{\varphi}_s$ is a point p or a circle according as $T_{s-1}\text{-index}_p M = 0$ or $\neq 0$. Since $T_{s-1}\text{-index}_p M \neq 0$ at every point $p \in M$ if $s \leq \left\lfloor \frac{\nu}{2} \right\rfloor$ from (C), at each point $p \in M$ we can choose a neighborhood U of a point p in which there exist isothermal coordinates (u, v) and a frame field $b \in B$ satisfying (5.1), (5.4), (5.9) for every $t \leq s$ and

$$\begin{aligned} \omega_{a_1 b_1} &= k_s \omega_1 = \omega_{a_2 b_2}, & \omega_{a_1 r} &= \omega_{a_2 r} = 0, \\ \omega_{a_1 b_2} &= k_s \omega_2 = -\omega_{a_2 b_1}, & b_1 &= 2s + 1, \quad b_2 = 2s + 2, \quad 2s + 2 < r, \end{aligned}$$

where k_s is a positive differentiable function on M .

Thus, it is clear that the geodesic codimension ν of M is even $2m$ (m a positive integer).

Using the structure equations, from (5.4), (5.9) and (5.10)_m we obtain

$$(5.12) \quad \begin{cases} \Delta(\log \lambda_t) = E \{(t+1)K - 2k_t^2 + 2k_{t+1}^2\}, & t = 1, 2, \dots, m-1, \\ \Delta(\log \lambda_m) = E \{(m+1)K - 2k_m^2\}, \end{cases}$$

where $\Delta = \partial^2 / \partial u^2 + \partial^2 / \partial v^2$. From (5.12) we have

$$(5.13) \quad \Delta \log (\lambda_1 \cdot \lambda_2 \cdots \lambda_m) = E \left\{ \frac{m(m+3)}{2} K - 2k_1^2 \right\} = E \left\{ \frac{(\nu+2)(\nu+4)}{8} K - c \right\},$$

because $K = c - 2k_1^2$ and $\nu = 2m$. Therefore, if $(\nu+2)(\nu+4)K - 8c$ does not change its sign, $\log(\lambda_1 \cdot \lambda_2 \cdots \lambda_m)$ is a subharmonic or superharmonic function on M , so it must be constant, because M is compact. Hence, $K = 8c / (\nu+2)(\nu+4) = \text{constant} > 0$ and so k_t ($1 \leq t \leq m$) are constant on M . Supposing $K = 1$, from (5.12) we get

$$\begin{aligned} k_t^2 &= (m-t+1)(m+t+2)/4 \quad \text{for } t = 1, 2, \dots, m, \\ c &= (m+1)(m+2)/2. \end{aligned}$$

Let $E_t = e_{2t+1} + i e_{2t+2}$, $t = 0, 1, 2, \dots, m$. Then, the Frenet formulas of M can be written as follows

$$dx = \frac{1}{h} (\bar{E}_0 dz + E_0 d\bar{z}), \quad z = u + iv,$$

$$\begin{aligned}
 DE_0 &= \frac{1}{h} E_0(\bar{z}dz - zd\bar{z}) + \frac{2k_1}{h} E_1 d \cdot, \\
 DE_1 &= -\frac{2k_1}{h} E_0 dz + \frac{2}{h} E_1(\bar{z}dz - zd\bar{z}) + \frac{2k_2}{h} E_2 d\bar{z}, \\
 &\dots\dots\dots \\
 DE_t &= -\frac{2k_t}{h} E_{t-1} dz + \frac{t+1}{h} E_t(\bar{z}dz - zd\bar{z}) + \frac{2k_{t+1}}{h} E_{t+1} dz, \\
 &\dots\dots\dots \\
 DE_m &= -\frac{2k_m}{h} E_{m-1} dz + \frac{m+1}{h} E_m(\bar{z}dz - zd\bar{z}),
 \end{aligned}$$

where D denotes the covariant differentiation of \hat{M} and $h=1+z\bar{z}$. Now, let \hat{M} be a $(2m + 2)$ -dimensional sphere $S^{2m+2}(R)$ with radius $R = 1/\sqrt{c} = \sqrt{2/(m+1)(m+2)}$. We may consider $S^{2m+2}(R) \subset E^{2m+3}$ and put $x = Re_{2m+3}$. By an computation analogous to the one in [7], we can verify that M is a generalized Veronese surface. Thus, we have proved

Theorem 4. *If the assumptions of Theorem 3 are satisfied and $(\nu + 2)(\nu + 4)K - 8c$ does not change its sign, then K is a positive constant on \hat{M} . Let $K = 1$ and \hat{M} be a $(\nu + 2)$ -dimensional sphere of constant curvature $(m + 1)(m + 2)/2$. Then M is a generalized Veronese surface.*

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