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ON MINIMALITY IN PSEUDO-BCI-ALGEBRAS

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Abstract. In this paper we consider pseudo-BCK/BCI-algebras. In particular, we consider properties of minimal elements ($x \leq a$ implies $x = a$) in terms of the binary relation $\leq$ which is reflexive and anti-symmetric along with several more complicated conditions. Some of the properties of minimal elements obtained bear resemblance to properties of B-algebras in case the algebraic operations $*$ and $\circ$ are identical, including the property $0 \circ (0 * a) = a$. The condition $0 * (0 \circ x) = 0 \circ (0 * x) = x$ for all $x \in X$ defines the class of p-semisimple pseudo-BCK/BCI-algebras ($0 \leq x$ implies $x = 0$) as an interesting subclass whose further properties are also investigated below.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([6, 7]). We refer useful textbooks for BCK/BCI-algebra to [5, 10, 11]. G. Georgescu and A. Iorgulescu ([3]) introduced the notion of a pseudo BCK-algebra as an extension of BCK-algebra, and Y. B. Jun ([8]) characterized pseudo BCK-algebras. He found conditions for a pseudo BCK-algebras to be $\wedge$-semilattice ordered. S. S. Ahn et al. ([1]) fuzzified the notion of pseudo-BCI-ideals, and Y. B. Jun et al. ([9]) discussed pseudo-BCI ideals in pseudo-BCI-algebras. A. Gilani and B. N. Waphare ([4]) studied pseudo a-ideals in pseudo-BCI-algebras. Recently, G. Dymek ([2]) introduced the notion of p-semisimple pseudo-BCI-algebras, and discussed the set $L_p(X)$ of pseudo-atoms of a pseudo-BCI-algebra X. He showed that $L_p(X)$ is a p-semisimple pseudo-BCI-algebra and showed that a pseudo-BCI-algebra X is p-semisimple if and only if $X = L_p(X)$.

In this paper we deal with a class of algebras which shows similarity to the class of companion d-algebras but which in addition is equipped with a reflexive and antisymmetric relation subject to certain constraints imposed by the binary relations defined for these algebras. Introducing the notion of minimality in a rather natural way permits us to consider minimal elements either singly or

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collectively and so characterize them and the algebra they belong to in a variety of ways. In particular, we introduce the notion of p-semisimplicity below and we develop alternative descriptions of p-semisimple pseudo-BCK/BCI-algebras as a consequence. In several instances one may note some similarity with B-algebras, especially when the two algebraic operations are identical, while other aspects compare to defining identities for BCK/BCI-algebras, thus justifying the terminology which has been used.

2. Preliminaries

A *pseudo-BCI-algebra* is an algebraic structure $X = (X, \leq, *, \circ, 0)$ where "$\leq$" is a binary relation on a set X, "$*$" and "$\circ$" are binary operations on X and "0" is an element of X satisfying the following axioms: for any $x, y, z \in X$,

(a1) $(x * y) \circ (x * z) \leq z * y, (x \circ y) * (x \circ z) \leq z \circ y$;
(a2) $x * (x \circ y) \leq y, x \circ (x * y) \leq y$;
(a3) $x \leq x$;
(a4) $x \leq y, y \leq x$ imply $x = y$;
(a5) $x \leq y \Longleftrightarrow x * y = 0 \Longleftrightarrow x \circ y = 0$.

Note that every pseudo-BCI-algebra satisfying $x * y = x \circ y$ for any $x, y \in X$ is a BCI-algebra, and every pseudo-BCI-algebra satisfying $0 \leq x$ for all $x \in X$ is called a *pseudo-BCK-algebra*.

Proposition 2.1 ([2]). *Let X be a pseudo-BCI-algebra. Then the following holds: for any $x, y, z \in X$,*

(b1) $x \leq 0 \Longrightarrow x = 0$;
(b2) $x \leq y \Longrightarrow z * y \leq z * x, z \circ y \leq z \circ x$;
(b3) $x \leq y, y \leq z \Longrightarrow x \leq z$;
(b4) $(x * y) \circ z = (x \circ z) * y$;
(b5) $x * y \leq z \Longleftrightarrow x \circ z \leq y$;
(b6) $x \leq y \Longrightarrow x * z \leq y * z, x \circ z \leq y \circ z$;
(b7) $x * (x \circ (x * y)) = x * y, x \circ (x * (x \circ y)) = x \circ y$;
(b8) $0 \circ (x \circ y) = (0 * x) * (0 \circ y)$;
(b9) $0 * x = 0 \circ x$;
(b10) $x * 0 = x = x \circ 0$.

Proposition 2.2 ([2]). *An algebraic structure $X = (X, \leq, *, \circ, 0)$ is a pseudo-BCI-algebra if and only if it satisfies* (a1), (a4), (a5) *and* (b9).

Example 2.3 ([9]). Let $X := [0, \infty)$ and let "$\leq$" be the usual order on X. If we define binary operations "$*$" and "$\circ$" on X by

$$x * y = \begin{cases} 0 & \text{if } x \leq y, \\ \frac{2x}{\pi} \tan^{-1}(\ln(\frac{x}{y})) & \text{otherwise,} \end{cases}$$

$$x \circ y = \begin{cases} 0 & \text{if } x \leq y, \\ xe^{-\tan(\frac{\pi y}{2x})} & \text{otherwise} \end{cases}$$

for any $x, y \in X$, then $X = (X, \leq, *, \circ, 0)$ is a pseudo-BCK-algebra, and hence it is a pseudo-BCI-algebra.

Example 2.4 ([2])**.** Let $Y = \mathbb{R}^2$. If we define two binary operations "$*$" and "$\circ$" and a binary relation "$\leq$" on Y by

$$(x_1, y_1) * (x_2, y_2) = (x_1 - x_2, (y_1 - y_2)e^{-x_2}),$$
$$(x_1, y_1) \circ (x_2, y_2) = (x_1 - x_2, y_1 - y_2 e^{x_1 - x_2}),$$

$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow (x_1, y_1) * (x_2, y_2) = (0, 0) = (x_1, y_1) \circ (x_2, y_2)$ for any $(x_1, y_1), (x_2, y_2) \in Y$, then $Y = (Y, \leq, *, \circ, 0)$ is a pseudo-BCI-algebra.

Example 2.5 ([2])**.** Let Z be the set of all bijective mappings $f : A \to A$, where $A \neq \emptyset$. Define two binary operations "$*$" and "$\circ$" and a binary relation "$\leq$" on Z by

$$f * g = fg^{-1},$$
$$f \circ g = g^{-1} f,$$
$$f \leq g \Leftrightarrow f * g = I_A = f \circ g$$

for all $f, g \in Z$, where I_A is the identity map on A. Then $Z = (Z, \leq, *, \circ, I_A)$ is a pseudo-BCI-algebra.

A pseudo-BCI-algebra X is said to be *p-semisimple* if for any $x \in X$,

$$0 \leq x \Rightarrow x = 0.$$

Theorem 2.6 ([2])**.** *Let X be a pseudo-BCI-algebra. Then the following are equivalent: for all $x, y, a, b \in X$,*

(1) *X is p-semisimple;*
(2) $x \leq y \Rightarrow x = y$;
(3) $x * (x \circ y) = y = x \circ (x * y)$;
(4) $0 * (0 \circ x) = x = 0 \circ (0 * x)$;
(5) $x * a = x * b \Rightarrow a = b$;
(6) $x \circ a = x \circ b \Rightarrow a = b$.

3. Minimality of pseudo-BCI-algebras

Let X be a pseudo-BCI-algebra. An element $a \in X$ is said to be *minimal* if $x \leq a \Rightarrow x = a$.

Theorem 3.1. *Let $X = (X, \leq, *, \circ, 0)$ be a pseudo-BCI-algebra and let $a \in X$. Then the following are equivalent:*

(1) *a is minimal;*
(2) $0 \circ (0 * a) = a$;
(3) *there exists $x \in X$ such that $a = 0 * x$.*

Proof. (1)⇒(2): By Proposition 2.6-(b4), $(0 \circ (0 * a)) * a = (0 * a) \circ (0 * a) = 0$ and hence $0 \circ (0 * a) \leq a$. Since a is minimal, we obtain $a = 0 \circ (0 * a)$.

(2)⇒(3): If we let $x := 0 * a$, then $a = 0 \circ (0 * a) = 0 \circ x = 0 * x$.

(3)⇒(1): Let $a := 0 * x$ for some $x \in X$. If $y \leq a$, then $0 = y \circ a = y \circ (0 * x)$ and hence

$$\begin{aligned} a \circ y =&(0 * x) \circ y \\ =&[0 * (0 \circ (0 * x))] \circ y && \text{[by (b7)]} \\ =&(0 \circ y) * (0 \circ (0 * x)) && \text{[by (b4)]} \\ =&(0 * y) * (0 \circ (0 * x)) && \text{[by (b9)]} \\ =&0 \circ (y \circ (0 * x)) && \text{[by (b8)]} \\ =&0, \end{aligned}$$

i.e., $a \leq y$, proving that $a = y$, i.e., a is minimal. □

Example 3.2. (i) Consider a pseudo-BCI-algebra $Y = (Y, \leq, *, \circ, 0)$ in Example 2.4. Assume $a := (a_1, a_2)$ is any element of Y. Then $0 * a = (0,0) * (a_1, a_2) = (-a_1, -a_2 e^{-a_1})$ and $0 \circ (0 * a) = (0,0) \circ (-a_1, -a_2 e^{-a_1}) = (a_1, a_2) = a$. By Proposition 3.1, a is a minimal element of Y. (ii) It is easy to show that every element of Z in Example 2.5 is a minimal element of Z, since $I_A \circ (I_A * f) = f$ for any $f \in Z$.

Example 3.3. Consider a pseudo-BCK-algebra X in Example 2.3. Since $0 \circ (0 * x) = 0 \circ 0 = 0 \neq x$ for any $x \neq 0$ in X, every non-zero element of X is not a minimal element of X.

Proposition 3.4. *Let $X = (X, \leq, *, \circ, 0)$ be a pseudo-BCI-algebra and let $a \in X$. Then the following are equivalent*:

(1) *a is minimal*;
(2) *$a * x = (0 * x) \circ (0 * a)$ for any $x \in X$*;
(3) *$a * x = 0 \circ (x * a)$ for any $x \in X$.*

Proof. (1)⇒(2): If a is minimal, then, by Theorem 3.1 and (b4), $a * x = (0 \circ (0 * a)) * x = (0 * x) \circ (0 * a)$.

(2)⇒(3): Assume that $a * x = (0 * x) \circ (0 * a)$ for any $x \in X$. Then $0 \circ (x * a) = (0 \circ x) \circ (0 * a) = (0 * x) \circ (0 * a) = a * x$. (3)⇒(1): Let $y \leq a$. Then $y * a = y \circ a = 0$. Hence $a * y = 0 \circ (y * a) = 0 \circ 0 = 0$, i.e., $a \leq y$ and hence $y = a$, proving the proposition. □

Proposition 3.4′. *Let $X = (X, \leq, *, \circ, 0)$ be a pseudo-BCI-algebra and let $a \in X$. Then the following are equivalent*:

(1) *a is minimal*;
(2) *$a \circ x = (0 \circ x) * (0 \circ a)$ for any $x \in X$*;
(3) *$a \circ x = 0 * (x \circ a)$ for any $x \in X$.*

Proposition 3.5. *Let $X = (X, \leq, *, \circ, 0)$ be a pseudo-BCI-algebra and $x, y \in X$. Then*

(1) $0 * x$ *is minimal*;
(2) *if* $y \le x$, *then* $0 \circ x = 0 * x = 0 * y = 0 \circ y$.

Proof. (1). Since $0 \circ (0 * (0 \circ x)) = 0 \circ x$, if we take $a := 0 \circ x$, then $0 \circ (0 * a) = a$. By Theorem 3.1, $a = 0 * x = 0 \circ x$ is minimal.

(2). If $y \le x$, then by (b2) $0 * x \le 0 * y$. Since $0 * x, 0 * y$ are minimal, we obtain $0 * x = 0 * y$. □

Proposition 3.6. *Let* $X = (X, \le, *, \circ, 0)$ *be a p-semisimple pseudo-BCI-algebra. Then* $(X \setminus \{0\}, \le)$ *is an anti-chain.*

Proof. Let $x, y \in X \setminus \{0\}$ with $x \lneqq y$. Then by Proposition 3.5, we have $0 * x = 0 * y$. Since X is p-semisimple, by Theorem 2.6, we obtain $x = y$, a contradiction. □

Theorem 3.7. *Let* $X = (X, \le, *, \circ, 0)$ *be a pseudo-BCI-algebra and* $a, x \in X$ *satisfying*

$$(q) \qquad a * (a \circ x) = x.$$

Then $a * (a \circ (x * y)) = x * y$ *for any* $y \in X$.

Proof. Given $y \in X$, we claim that $[(a \circ (x * y)) * (a \circ x)] * y = 0$. In fact, by (a1), $(a \circ (x * y)) * (a \circ x) \le x \circ (x * y)$ and hence $[(a \circ (x * y)) * (a \circ x)] * y \le [x \circ (x * y)] * y = 0$. Using (b1), we obtain the result. Using the claim and the condition (q) we obtain

$$\begin{aligned}(x * y) \circ [a * (a \circ (x * y)] &= [\{a * (a \circ x)\} * y] \circ [a * (a \circ (x * y)] \\ &= [\{a * (a \circ x)\} \circ [a * (a \circ (x * y)]] * y \\ &= [\{a \circ [a * (a \circ (x * y)]\} * (a \circ x)] * y \\ &= [(a \circ (x * y)) * (a \circ x)] * y \\ &= 0,\end{aligned}$$

which proves $x * y \le a * (a \circ (x * y)$. By applying (a2), we proves the proposition. □

Theorem 3.7′. *Let* $X = (X, \le, *, \circ, 0)$ *be a pseudo-BCI-algebra and* $a, x \in X$ *satisfying*

$$(q') \qquad a \circ (a * x) = x.$$

Then $a \circ (a * (x \circ y)) = x \circ y$ *for any* $y \in X$.

Let $X = (X, \le, *, \circ, 0)$ be a pseudo-BCI-algebra and let $u \in X$. We denote $u * X, u * [X], u \circ X$ and $u \circ [X]$ as follows:

$$\begin{aligned} u * X &= \{u * x | x \in X\}, \\ u * [X] &= \{x \in X | u * (u \circ x) = x\}, \\ u \circ X &= \{u \circ x | x \in X\}, \\ u \circ [X] &= \{x \in X | u \circ (u * x) = x\}. \end{aligned}$$

By Theorems 3.7 and 3.7′, we obtain $(a*[X])*X \subseteq a*[X]$ and $(a\circ[X])\circ X \subseteq a\circ[X]$ for any $a \in X$.

Theorem 3.8. *Let $X = (X, \leq, *, \circ, 0)$ be a pseudo-BCI-algebra and let $u \in X$. Then*

(1) $u * X = u * [X]$;
(2) $u \circ X = u \circ [X]$;
(3) $(u * [X], *)$ *is a subalgebra of* $(X, *)$;
(4) $(u \circ [X], \circ)$ *is a subalgebra of* $(X, \circ)$;
(5) *if* $v \in u * X$, *then* $v * X \subseteq u * X$;
(6) *if* $v \in u \circ X$, *then* $v \circ X \subseteq u \circ X$.

Proof. (1) If $\alpha \in u*[X]$, then $\alpha = u*(u\circ\alpha)$. Since $u\circ\alpha \in X$, we have $\alpha \in u*X$. Conversely, if $\alpha \in u * X$, then there exists $x_0 \in X$ such that $\alpha = u * x_0$ and hence $u * (u \circ \alpha) = u * (u \circ (u * x_0)) = u * x_0 = \alpha$. Hence $\alpha \in u * [X]$.

(2) Similar to (1).

(3) Since $u * (u \circ u) = u * 0 = u$, $u \in u * [X]$, i.e., $u * [X] \neq \emptyset$. For any $x, y \in u * [X]$, we have $u * (u \circ x) = x, u * (u \circ y) = y$. By applying Theorem 3.7, we obtain $u * (u \circ (x * y)) = x * y$, i.e., $x * y \in u * [X]$.

(4) Using Theorem 3.7′, it is similar to (3).

(5) Since $u*X = u*[X]$, if $v \in u*X$, then $v = u*(u\circ v)$. By Theorem 3.7, $v*x = u*(u\circ(v*x))$ for any $x \in X$. This means that $v*x \in u*[X] = u*X$ for any $x \in X$. Thus $v * X \subseteq u * X$.

(6) If we apply Theorem 3.7′, then it is similar to (5). □

Theorem 3.9. *Let $X = (X, \leq, *, \circ, 0)$ be a pseudo-BCI-algebra and let $P := \{x \in X \mid x \text{ is minimal}\}$. Then $(P, \leq, *, \circ, 0)$ is a subalgebra of $X = (X, \leq, *, \circ, 0)$.*

Proof. Since 0 is minimal element, $P \neq \emptyset$. Given $a, b \in P$, let $x \in X$ such that $x \leq a * b$.

$$\begin{aligned} x \circ a &\leq (a * b) \circ a && \text{[by (b6)]} \\ &= (a \circ a) * b && \text{[by (b4)]} \\ &= 0 * b, && \text{[by (a3)]} \end{aligned}$$

i.e., $x \circ a \leq 0 * b$. It follows that $x * (0 * b) \leq a$ by (b5). Since a is minimal, we obtain $a = x * (0 * b)$. Hence $a \circ x = (x * (0 * b)) \circ x = (x \circ x) * (0 * b) = 0*(0*b) = 0*(0\circ b) \leq b$, i.e., $a\circ x \leq b$. By (b5), we have $a*b \leq x$. This proves that $x = a * b$, i.e., $a * b$ is minimal. On the other hand, given $a, b \in P$, let $x \in X$ such that $x \leq a \circ b$. Using the same method, we can see that $x = a \circ b$, i.e., $a \circ b$ is minimal. This completes the proof. □

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