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ON MINIMIZING THE RUIN PROBABILITY BY INVESTMENT AND REINSURANCE

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We consider a classical risk model and allow investment into a risky asset modelled as a Black–Scholes model as well as (proportional) reinsurance. Via the Hamilton–Jacobi–Bellman approach we find a candidate for the optimal strategy and develop a numerical procedure to solve the HJB equation. We prove a verification theorem in order to show that any increasing solution to the HJB equation is bounded and solves the optimisation problem. We prove that an increasing solution to the HJB equation exists. Finally two numerical examples are discussed.

1. Introduction. Consider a classical risk process

$$X_t^{01} = u + ct - \sum_{i=1}^{N_t} Y_i$$

where $c > 0$, $u \geq 0$, $\{N_t\}$ is a Poisson process with rate $\lambda > 0$ and $\{Y_i\}$ are iid with distribution function $G(x)$ where $G(0) = 0$. We denote the claim times by $\{T_i\}$. It is assumed that $\{N_t\}$ and $\{Y_i\}$ are independent. We interpret here the process $\{X_t^{01}\}$ as an approximation to a collective risk (or a large portfolio of similar contracts) after discounting by inflation. That is, the premium as well as the claim sizes increase with inflation. In the rest of the paper all monetary quantities are discounted by inflation.

We suppose here that the premium rate c has been fixed already. We need not to assume a positive safety loading because the positive drift will come from investment as soon as the surplus becomes large enough.

The insurance company has several decisions to take. The company has the possibilities to invest the capital and to take reinsurance. If reinsurance is not allowed the problem was solved by Hipp and Plum (2000) and Hipp (2000). If the surplus cannot be invested the problem was solved by Schmidli (2001). In this paper we consider the case where investment and reinsurance are possible. In order to avoid technical problems we consider proportional reinsurance and assume that $G(x)$ is continuous. Other types of reinsurance can be treated similarly, see also Hipp and Vogt (2001).

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The risky asset is described by a geometric Brownian motion

$$Z_t = \exp\{\sigma W_t + (\mu - \tfrac{1}{2}\sigma^2)t\},$$

where $\{W_t\}$ is a standard Brownian motion. The parameters σ, μ are assumed to be strictly positive. Then Z_t is the (discounted) value at time t of one unit invested at time 0. Let $\{A_t\}$ denote the amount invested into the risky asset at time t. We allow that the company invests more than its current surplus into the risky asset. In this case money has to be borrowed for such a strategy. For large capital, however, investing more than the surplus into the risky asset cannot be optimal, see also Kalashnikov and Norberg (1999). Recall that the ruin probability of a Brownian motion with drift decreases exponentially fast; see Rolski et al. (1999), page 416.

The retention level for reinsurance is $b_t \in [0, 1]$, which means that the insurer pays $b_t Y$ of a claim occurring at time t, the reinsurer $(1 - b_t)Y$. For this reinsurance, the premium rate $c(b_t)$ has to be paid. The strategies $\{A_t\}$ and $\{b_t\}$ have to be measurable and predictable with respect to the smallest right continuous filtration $\{\mathcal{F}_t\}$ such that $\{(X_t^{01}, W_t)\}$ is measurable. Then the surplus process satisfies the stochastic differential equation

$$dX_t^{Ab} = \bigl(c - c(b_t) + \mu A_t\bigr)\,dt + \sigma A_t\,dW_t - b_t\,dS_t, \qquad X_0^{Ab} = u,$$

where $S_t = u + ct - X_t^{01}$ is the aggregate claims process. In order that $\{X_t^{Ab}\}$ is well-defined we assume that $\{A_t\}$ is locally bounded. The ruin time is defined as

$$\tau^{Ab} = \inf\{t \ge 0 : X_t^{Ab} < 0\}.$$

This is a stopping time with respect to $\{\mathcal{F}_t\}$. The quantity to optimise is the probability of ultimate ruin

$$\psi^{Ab}(u) = \mathbb{P}[\tau^{Ab} < \infty],$$

that is, we want to maximise the survival probability $\delta^{Ab}(u) = 1 - \psi^{Ab}(u)$. The value function is then $\delta(u) = \sup_{A,b} \delta^{Ab}(u)$. A similar problem has been considered by Højgaard (2000). He maximised the discounted future dividend payout.

We make the following assumption on the function $c(b)$. The function is decreasing, continuous, $c(1) = 0$, $\underline{\lim}_{b\uparrow 1} c(b)/(1 - b) > 0$ and there is a value $\underline{b} > 0$ such that $c(b) > c$ for $b < \underline{b}$ and $c(b) \le c$ for $b \ge \underline{b}$. If $\mathbb{E}[Y] = \infty$ we assume that $c(0) < \infty$. The existence of $\underline{b}$ is needed in order to prevent that the whole portfolio can be reinsured, yielding $\delta(u) = 1$ for all u. That $c(b)$ is continuous is not necessary, but it will simplify the presentation below. That $c(b)$ is decreasing is natural, otherwise more reinsurance would be cheaper. If $c(b)$ is not strictly decreasing near 1 one would always choose at least the maximal reinsurance with zero cost, and the problem could be reformulated with claims already reinsured at zero costs. That $c(b)$ is decreasing close to one at at least a linear rate is a technical condition that will imply that the optimal reinsurance strategy for small capital is not to reinsure. For most of the premium principles used these conditions will be fulfilled because $G(x)$ is continuous. If $\mathbb{E}[Y] = \infty$ the condition on $c(0)$ is needed in order to prevent almost sure ruin (the whole portfolio may be reinsured). However, such claims are called "not-insurable," but there are insurance companies claiming to insure everything. In order to simplify the notation we will often drop the superscript Ab.

2. The Hamilton–Jacobi–Bellman equation. Suppose the function $\delta(u)$ is twice continuously differentiable, stochastic integrals with respect to Brownian motion are martingales and all limits and expectations can be interchanged. Then one can show that the function $\delta(u)$ will satisfy the Hamilton-Jacobi-Bellman equation [see also Schmidli (2001)]

$$\begin{aligned}\sup_{b\in[0,1]}\sup_{A\geq 0}\Big[\tfrac{1}{2}\sigma^2A^2\delta''(u) + \big(c - c(b) + \mu A\big)\delta'(u) \\ + \lambda\big(\mathbb{E}[\delta(u-bY)] - \delta(u)\big)\Big] = 0,\end{aligned} \tag{1}$$

where we use $\delta(u) = 0$ for $u < 0$. One boundary condition is $\delta(\infty) = 1$. The other boundary condition will be determined below. It follows immediately that $f(u) = k\delta(u)$ solves (1) with boundary condition $f(\infty) = k$. Let us therefore look for a solution with $f(0) = 1$ instead. That $\delta(0) > 0$ follows from the case of no investment and no reinsurance if the safety loading is strictly positive. If the safety loading is negative there is a strictly positive probability that no claim occurs until the capital is large enough in order that a positive safety loading is achieved from investment. At this level ruin is not certain anymore.

We first show that either ruin occurs or X_t^{Ab} tends to infinity as $t \to \infty$. Even though the result is intuitively clear a result of this type has not been proven before. Hipp and Plum (2000) have circumvented the problem by considering a family of strategies close to the strategy under consideration.

LEMMA 1. *Let $\{(A_t, b_t)\}$ an arbitrary strategy. Then with probability one either ruin occurs or X_t^{Ab} diverges to infinity as $t \to \infty$.*

PROOF. We will just describe the argument. A formal argument should always mean with probability one. Let $\varepsilon < 1$ be small. Suppose $\tau = \infty$ and $\underline{\lim}_{t\to\infty} X_t \in [M, M + \varepsilon/2)$ for some $M < \infty$. Then there are infinitely many points t_k for which $X_{t_k} < M + \varepsilon$. We can assume that $t_1 < t_1 + 1 < t_2 < t_2 + 1 < \cdots$. By the strong law of large numbers, there are infinitely many points t_k for which $\sup\{Z_t/Z_{t_k} : t \in [t_k, t_k + 1]\} < 1 + \varepsilon$. Suppose therefore that the latter holds for all points t_k. The possible income from investment in an interval $[t_k, t_k + 1]$ can be taken into the premium rate c, and therefore we do not consider investment in the remaining part of the proof.

Let $\delta = \varepsilon/c$. Suppose there are infinitely many k for which the Lebesgue measure of $\{t \in [t_k, t_k + 1] : b_t \geq \underline{b}/2\}$ is larger than δ. Note that $\{S_t\}$ has stationary and independent increments and its increment $S_{t+s} - S_t$ is independent of the strategy $\{b_v : 0 \le v \le t\}$. Then for any Borel set I with Lebesgue measure $|I|$ one has $\int_I dS_t \stackrel{\mathrm{d}}{=} S_{|I|}$. Because $\mathbb{P}[\underline{b}S_\delta/2 > M + c + 1] > 0$ ruin has to occur by the strong law of large numbers. Thus the Lebesgue measure of $\{t \in [t_k, t_k + 1] : b_t \ge \underline{b}/2\}$ is smaller than δ for all but a finite number of k's. We can assume that the latter holds for all k. One has

$$X_{t_k+1} < X_{t_k} + \delta c - \bigl(c(\underline{b}/2) - c\bigr)(1-\delta) < M + 2\varepsilon - \bigl(c(\underline{b}/2) - c\bigr)(1-\delta).$$

Because ε is arbitrary, this would imply $\underline{\lim}_{t\to\infty} X_t < M$, which is a contradiction. Thus M cannot be finite. □

We next show the following property of $\delta(u)$.

LEMMA 2. *Suppose that the function $\delta(u)$ is twice continuously differentiable and solves the Hamilton–Jacobi–Bellman equation* (1). *Then*:

(i) *$\delta(u)$ is strictly increasing.*
(ii) *$\delta(u)$ is strictly concave.*

PROOF. Choose initial capitals $0 \le x < y$. Let (A, b) be a strategy for initial capital x. Let us use the same strategy for initial capital y. Then ruin cannot occur for initial capital y before ruin occurs for initial capital x. This shows that $\delta(x) \le \delta(y)$. That $\delta(x) < \delta(y)$ will follow from strict concavity.

Suppose that $\delta''(u) > 0$. This will hold on some interval around u. The maximum over A in (1) is attained as $A \to \infty$ and is infinite. This is a contradiction. Suppose now $\delta''(u) = 0$ on some interval. If $\delta'(u) > 0$ again a contradiction is obtained. Thus $\delta'(u) = 0$ and $\delta(u)$ is constant on some interval. Because $\delta(u) > 0$ this implies $\delta(u)$ is constant for all $u \ge u_0$, that is, $\delta(u) = 1$. In fact, let $x < y < z$ with $\delta(x) = \delta(y) < \delta(z)$. Then applying the same strategy for initial capital x and y, the process starting in y will at some time t reach a level where $\delta(X_t^y) > \delta(X_t^x)$, provided ruin has not occurred before. If we only consider strategies, where ruin is not certain, this happens with positive probability, and $\delta(x) = \delta(y)$ is not possible. On the other hand, there is always a positive probability for ruin; see the proof of Lemma 1. Hence $\delta''(u)$ cannot be zero on an interval. Thus $\delta(u)$ is strictly concave. □

REMARK. In order to show that $\delta(u)$ is strictly increasing it is not necessary to assume that $\delta(u)$ solves (1). There is always a positive probability that, using the same strategies up to ruin, ruin will occur for initial capital x and the deficit at ruin is less than $(y - x)/2$. For initial capital y there is then a strictly positive probability that ruin does not occur.

We therefore restrict to strictly increasing and strictly concave solutions to (1).

The maximum over A in (1) is attained for

$$A^*(u) = -\frac{\mu\delta'(u)}{\sigma^2\delta''(u)} = -\frac{\mu f'(u)}{\sigma^2 f''(u)} > 0. \tag{2}$$

Note that $A^*(u)$ is measurable and locally bounded. Hence the equation to solve remains

$$\sup_{b\in[0,1]} -\frac{\mu^2 f'(u)^2}{2\sigma^2 f''(u)} + \big(c - c(b)\big)f'(u) + \lambda\big(\mathbb{E}[f(u-bY)] - f(u)\big) = 0. \tag{3}$$

Note that the equation is continuous in both u and b. This has the consequence that the argument $b^*(u)$ (not necessarily unique) for which the supremum is taken exists and that $b^*(u)$ can be chosen to be measurable. We will in the sequel write $b(u)$ if we do not interpret it as the optimal strategy.

For u small, the above equation will not vary widely as b varies. The supremum in (3) will therefore be determined by the infimum of $c(b)$ for u small. Thus one can conjecture that for u small the supremum is taken for $b^*(u) = 1$.

LEMMA 3. *Suppose there exists a solution $f(u)$ to* (3) *on some interval* $[0, \eta)$ *with* $\eta > 0$. *Then there exists* $\varepsilon > 0$ *such that* $b^*(u) = 1$ *for* $u < \varepsilon$.

PROOF. Note that $b(0) = 1$. Let

$$H(u,b) = -\frac{\mu^2 f'(u)^2}{2\sigma^2 f''(u)} + \big(c - c(b)\big)f'(u) + \lambda\big(\mathbb{E}[f(u-bY)] - f(u)\big).$$

Then

$$\frac{H(u,1) - H(u,b)}{1-b} = \frac{c(b)}{1-b}f'(u) + \lambda\mathbb{E}\left[\frac{f(u-Y) - f(u-bY)}{1-b}\mathbb{1}_{Y\le u/b}\right].$$

By Taylor's theorem,

$$f(u-bY) = f(u-Y) + (1-b)Yf'\big(\zeta(Y)\big)$$

where $\zeta(Y) \in (0,u)$. Because $f'(x)$ is bounded on $[0, \eta/2]$ we have by bounded convergence

$$\lim_{b\uparrow 1}\mathbb{E}\left[\frac{f(u-Y) - f(u-bY)}{1-b}\right] = -\mathbb{E}[Yf'(u-Y)\mathbb{1}_{Y<u}]$$

for $u < \eta/2$. The absolute value of this term can be made arbitrarily small by choosing u small enough. On the other hand, $\underline{\lim}_{b\uparrow 1} c(b)/(1-b) > 0$. Thus for u small enough, $H(u,b)$ is strictly decreasing in b for b close to one. Thus if $b(u) < 1$ for $0 < u < \varepsilon$ this is only possible if $\overline{\lim}_{u\downarrow 0} b(u) < 1$, that is $b(u)$ jumps at $u = 0$. Because $H(u,b)$ is continuous in u and b this is only possible if

$H(0,b) = H(0,1) = 0$ for some $b < 1$. Because $H(0,b)$ is strictly decreasing in b for b close to one this is not possible. □

The equation to solve for u small is therefore identical with the equation to solve in Hipp and Plum (2000).

REMARK. Note that in (3) μ and σ^2 only appear as $\alpha = \mu/\sigma$. Thus for fixed α the solution $\delta(u)$ and the optimal strategy $b^*(u)$ will not depend on the real size of σ. The σ will appear as a proportional factor in $A^*(x)$, hence a larger volatility will yield less investment.

In principle, (3) could be solved numerically and has not to be investigated further. However, we want to find an alternative representation in order to find $f'(0)$. This alternative representation will give an opportunity to show that a solution to (3) exists. Moreover, it will be possible to find a solution iteratively. This turns out to be helpful in order to find the optimal strategies $b^*(u)$ and $A^*(u)$. Indeed, in the numerical examples treated below it turns out to be necessary to iterate the solution obtained by an Euler scheme before the correct strategies are obtained. Of course, alternatively equation (3) could be iterated. This approach is chosen by Hipp and Plum (2000).

Equation (3) can be written as

$$-\frac{f''(u)}{f'(u)^2} = \frac{\mu^2}{2\sigma^2} \frac{1}{\inf_{b\in[0,1]} \lambda(f(u) - \mathbb{E}[f(u-bY)]) - (c - c(b))f'(u)}.$$

Observe that the denominator must be strictly positive for $u > 0$. Integrating from u_0 to u leads to

$$\frac{1}{f'(u)} = \frac{\mu^2}{2\sigma^2} \int_{u_0}^{u} \frac{1}{\inf_{b\in[0,1]} \lambda(f(x) - \mathbb{E}[f(x-bY)]) - (c - c(b))f'(x)}\,dx + \frac{1}{f'(u_0)}.$$

Because $f'(u_0)$ is decreasing in u_0 it is possible to let $u_0 \to 0$ yielding

$$\frac{1}{f'(u)} = \frac{\mu^2}{2\sigma^2} \int_{0}^{u} \frac{1}{\inf_{b\in[0,1]} \lambda(f(x) - \mathbb{E}[f(x-bY)]) - (c - c(b))f'(x)}\,dx + \frac{1}{f'(0)}, \tag{4}$$

where $1/\infty = 0$. From Hipp and Plum (2000) we know that $A^*(u)$ tends to zero as $u \to 0+$. Indeed, $\overline{\lim}_{u\to 0} A^*(u) > 0$ would imply $\delta(0) = 0$ because in any small interval the infimum of the return from investment is strictly negative. From $c(b^*(u)) \to 0$ and $A^*(u) \to 0$ we can conclude from (1) that $f'(0+) = \lambda/c$. Taking the derivative in (4) or letting $u \to 0$ in (2) yields $f''(0+) = -\infty$.

Letting $g(u) = f'(u)$ gives the equation

$$(5) \quad g(u) = \frac{1}{\frac{\mu^2}{2\sigma^2}\int_0^u \frac{1}{\inf_{b\in[0,1]} \lambda(1-G(x/b)+\int_0^x (1-G((x-z)/b))g(z)\,dz)-(c-c(b))g(x)}\,dx + \frac{c}{\lambda}}.$$

3. A verification theorem. We now show that the Hamilton–Jacobi–Bellman approach leads to the correct solution, provided the correct initial values are chosen. Note that, in contrast to the proof of Hipp and Plum (2000) we do not need to assume that $f(u)$ is bounded.

THEOREM 1. *Let $f(x): \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing twice continuously differentiable function solving the Hamilton–Jacobi–Bellman equation* (1) *or equivalently* (3). *This will be the case for $f(x) = 1 + \int_0^x g(z)\,dz$, provided $g(x)$ is a decreasing solution to* (5). *Then $f(x)$ is bounded and $\delta(u) = f(u)/f(\infty)$. Moreover, the optimal strategy is $A^*(X_{t-})$ and $b^*(X_{t-})$, where $A^*(x)$ is given by* (2) *and $b^*(x)$ is the argument maximising the left hand side in the Hamilton–Jacobi–Bellman equation. In particular, there is at most one strictly increasing twice continuously differentiable solution to* (1) *with $f(0) = 1$.*

PROOF. Let us start by considering the process $\{X_t^*\}$ following the optimal strategy $\{(A_t^*, b_t^*)\}$. Let $0 < \varepsilon < u < n$ and $\tau_\varepsilon^{*n} = \inf\{t \ge 0 : X_t^* \notin (\varepsilon, n)\}$. Then A_t^* is bounded on $[0, \tau_\varepsilon^{*n}]$. By Itô's formula,

$$\begin{aligned} f(X^*_{t\wedge\tau_\varepsilon^{*n}}) &= f(u) + \int_0^{t\wedge\tau_\varepsilon^{*n}} \big((c - c(b_s^*) + \mu A_s^*) f'(X_s^*) + \tfrac{1}{2}\sigma^2 A_s^{*2} f''(X_s^*)\big)\,ds \\ &\quad + \int_0^{t\wedge\tau_\varepsilon^{*n}} \sigma A_s^* f'(X_s^*)\,dW_s + \sum_{i=1}^{N_{t\wedge\tau_\varepsilon^{*n}}} \big(f(X_{T_i}^*) - f(X_{T_i-}^*)\big) \\ &= f(u) + \int_0^{t\wedge\tau_\varepsilon^{*n}} \sigma A_s^* f'(X_s^*)\,dW_s + \sum_{i=1}^{N_{t\wedge\tau_\varepsilon^{*n}}} \big(f(X_{T_i}^*) - f(X_{T_i-}^*)\big) \\ &\quad + \lambda \int_0^{t\wedge\tau_\varepsilon^{*n}} \big(f(X_s^*) - \mathbb{E}[f(X_s^* - b_s^* Y)]\big)\,ds, \end{aligned}$$

where we used (1). Because $\sigma A_s^* f'(X_s)$ is bounded the first integral is a martingale. Note that the second integral does not change if s is replaced by $s-$. Thus $\{f(X^*_{t\wedge\tau_\varepsilon^{*n}})\}$ is a martingale [see Brémaud (1981), page 27]. Taking expectations, and letting first $\varepsilon \to 0$ and then $n \to \infty$, applying the monotone convergence theorem, it follows that $\mathbb{E}[f(X^*_{\tau^*\wedge t})] = f(u)$.

Let us now consider an arbitrary predictable strategy $\{(A_t, b_t)\}$ such that the process $\{X_t^{Ab}\}$ is well defined. Let $\{\xi_{n,m}\}$ be a localization sequence of the local martingale $\int_0^{t\wedge\tau_\varepsilon^n} \sigma A_s f'(X_s)\,dW_s$, where $\tau_\varepsilon^n = \inf\{t \ge 0 : X_t \notin (\varepsilon, n)\}$. Proceeding as for the optimal strategy we find for $n, m \in \mathbb{N}$

$$\mathbb{E}[f(X_{t\wedge\tau_\varepsilon^n\wedge\xi_{n,m}})] \le f(u).$$

Because $f(x)$ is bounded on $(-\infty, n]$ the bounded convergence theorem yields as $m \to \infty$

$$\mathbb{E}[f(X_{t\wedge\tau_\varepsilon^n})] \le f(u).$$

By monotone convergence we conclude $\mathbb{E}[f(X_{t\wedge\tau})] \le f(u)$. From Lemma 1 we know that either ruin occurs or X_t tends to infinity. By Fatou's lemma we obtain

$$f(\infty)\mathbb{P}[\tau = \infty] \le f(\infty)\mathbb{P}[\tau = \infty] + f(0)\mathbb{P}[\tau < \infty, X_\tau = 0] \le f(u)$$

as $t \to \infty$. Because there exists a strategy for which $\mathbb{P}[\tau = \infty] > 0$ we must have $f(\infty) < \infty$. This yields $\delta^{Ab}(u) \le f(u)/f(\infty)$, with equality for $\delta^{A^*b^*}(u)$. Note that $X_{\tau^*}^* \ne 0$ because the claim sizes have a continuous distribution and ruin cannot occur by investment. Indeed, let $\varepsilon > 0$ such that $b^*(x) = 0$ for $x \le 2\varepsilon$ and choose the strategy $\tilde{b}_t = b_t^*$ and $\tilde{A}_t = \mathbb{1}_{\tilde{X}_t > 2\varepsilon} A^*(\tilde{X}_t)$. Then

$$f(u) = \mathbb{E}[f(\tilde{X}_{\tilde{\tau}_{2\varepsilon}})] = \mathbb{E}[f(X^*_{\tau^*_{2\varepsilon}})]$$

and

$$f(u) = \mathbb{E}[f(X^*_{\tau^*_\varepsilon})] \ge \mathbb{E}[f(\tilde{X}_{\tilde{\tau}_\varepsilon})].$$

Because $\tilde{X}_t$ can reach $[0, \varepsilon)$ only by a jump the difference between $\mathbb{E}[f(\tilde{X}_{\tilde{\tau}_{2\varepsilon}})]$ and $\mathbb{E}[f(\tilde{X}_{\tilde{\tau}_\varepsilon})]$ will vanish as $\varepsilon \to 0$. Similarly it follows that $\mathbb{E}[f(\tilde{X}_{\tilde{\tau}_\varepsilon})] \to f(\infty)\mathbb{P}[\tau^* = \infty]$. Thus $\mathbb{P}[\tau^* < \infty, X^*_{\tau^*} = 0] = 0$. Because $f(u) = f(\infty)\delta(u)$ the solution is unique. □

REMARKS. (i) The proof shows that the derivative in zero is completely determined by the requirement, that the solution has to exist on the whole half line $\mathbb{R}_+$. Thus for all other initial derivatives the solution will explode or not be strictly increasing. In fact, a similar argument as in the proof above will give that $f(x)$ cannot be bounded from below unless we choose $f'(0)$ correctly.

(ii) Hipp and Plum (2000) prove that their solution is bounded if the claim sizes have a finite mean. This follows here directly from the verification theorem. Moreover, if $f(u)$ is unbounded, then ruin is almost surely. From the proof above it follows that $-f(X_{\tau\wedge t})$ is a submartingale. By the convergence theorem [see for instance Rolski et al. (1999)] $-f(X_{\tau\wedge t})$ converges to an integrable random variable. Thus the limit cannot be $-\infty$. This shows that $\mathbb{P}[\tau = \infty] = 0$.

(iii) Note that $\mathbb{P}[\tau^* < \infty, X^*_{\tau^*} = 0] = 0$ was not verified by Hipp and Plum (2000). The same argument as used above will close this gap.

4. Existence of a solution. The approach below follows Hipp and Plum (2000). However, some additional technical problems occur in the proof of the existence of a solution because $b^*(u)$ is not constant. In order to show that a solution exists it is enough to show there is a solution to (5). The problem is to secure that the integral in the denominator is finite for small u. Hence we start by showing that a solution exists on an interval close to zero. As mentioned above, the optimal strategy will be $b^*(u) = 1$ for u small enough. Let us therefore first consider the case without reinsurance previously treated in Hipp and Plum (2000). This allows us to use their result to start with. In order to simplify the notation we choose in this section $c = \lambda = 1$, readily obtained by a change of time and monetary unit.

In the denominator of the integral the inner integral will not be important because it is of order $O(x)$. Indeed, then the outer integral would not be defined. Thus we have $1 - g(x) - G(x)$ is dominating and we would be interested in a solution where $1 - g(x)$ is the dominating factor. By Taylor's expansion one gets for $h(u) = 1 - g(u)$ and some constant K,

$$1 - h(u) \approx \frac{1}{1 + K \int_0^u \frac{1}{h(x)}\,dx} \approx 1 - K \int_0^u \frac{1}{h(x)}\,dx.$$

Thus the solution is approximatively $h(u) \approx \sqrt{2Ku}$. One conjectures therefore that the solution is of the form $g(u) = 1 - \tilde{K}\sqrt{u} + o(\sqrt{u})$ as $u \to 0$.

LEMMA 4. *Suppose that $G(x)$ has a bounded density. Then there exists an $\varepsilon > 0$ and a function $g(x)$ solving* (5) *on* $[0, \varepsilon)$. *Moreover*, $g(u) = \lambda/c - \alpha\sqrt{u} + o(\sqrt{u})$ *as $u \to 0$, where*

$$\alpha = \frac{\lambda\mu}{\sigma c^{3/2}}.$$

PROOF. From Hipp and Plum (2000) we know that the assertion is true if $b(u) = 1$ on $[0, \varepsilon]$. By Lemma 3 this holds if ε is small enough. □

REMARK. The assumption of a bounded density seems to be quite strong. However, we need that the second derivative of $\delta(u)$ exists in order that our approach works. That the assumption of a bounded density is close to a necessary condition can also be seen from the theory of perturbed risk processes, see Schmidli (1995) or Schmidli (1999) and references therein. In this case a sufficient condition that the differential equation to solve has a twice continuously differentiable solution will be a continuous density of the claim size distribution.

Note that

$$u^{-1/2}\Big(1 - g(u) - G(u) + \int_0^u \big(1 - G(u-z)\big)g(z)\,dz\Big)$$
$$= \alpha + o(1) - u^{-1/2}G(u) + \sqrt{u}\frac{1}{u}\int_0^u \big(1 - G(u-z)\big)g(z)\,dz \to \alpha$$

as $u \to 0$. Thus

$$\inf_{b\in[0,1]} \lambda\biggl(1 - G(u/b) + \int_0^u \bigl(1 - G\bigl((u-z)/b\bigr)g(z)\bigr)\,dz\biggr) - \bigl(c - c(b)\bigr)g(u) > 0 \tag{6}$$

for $u \neq 0$ small enough. As long as (6) holds it is no problem to extend the solution. Thus we need to show that (6) holds also on the right endpoint of an interval on which a solution to (5) exists.

LEMMA 5. *Assume $G(x)$ has a bounded density. Suppose there is a decreasing solution $g(u)$ to* (5) *on the interval* $[0, u_0)$ *such that* (6) *holds for all $u \in (0, u_0)$. Then the solution can be extended to* $[0, u_0]$ *and* (6) *holds also for $u = u_0$.*

PROOF. We assume again that $\lambda = c = 1$. Because $g(u)$ is decreasing we get that $g(u_0)$ can be defined as the limit of $g(u)$ as u approaches u_0. By continuity, (5) is fulfilled for $u = u_0$, where in the case $g(u_0) = 0$ the integral must be infinite. This case, however, will be excluded by verifying (6) for $u = u_0$. Let x_n be a sequence converging monotonically to u_0 and let $b_n = b(x_n)$. Because $b_n \in [0, 1]$ we can assume (possibly by restricting to a subsequence) that the limit $b_0 = \lim_{n\to\infty} b_n$ exists. By continuity, b_0 is an argument minimising the left hand side of (6) for $u = u_0$. Let us now assume that (6) does not hold for $u = u_0$, i.e.

$$1 - G(u_0/b_0) + \int_0^{u_0} \bigl(1 - G\bigl((u_0 - z)/b_0\bigr)\bigr)g(z)\,dz - \bigl(1 - c(b_0)\bigr)g(u_0) = 0.$$

Suppose first $b_0 > 0$. Taking the derivative in (5) and letting $u \to u_0$ gives $g'(u_0) = -\infty$, where here the derivative has to be interpreted as derivative from the left. Moreover, $c(b_0) \neq 1$ because otherwise (6) would hold. If $b_0 < \underline{b}$ we get from $g(u_0) \geq 0$ and $1 - c(b_0) < 0$ that $g(u_0) = 0$, implying $b_0 = 0$ which we had excluded. Thus $c(b_0) < 1$. Clearly,

$$1 - G(u/b_0) + \int_0^u \bigl(1 - G\bigl((u - z)/b_0\bigr)\bigr)g(z)\,dz - \bigl(1 - c(b_0)\bigr)g(u) > 0$$

for any $u < u_0$. Taking the difference of the above two displayed equations and dividing by $(u_0 - u)$ yields

$$\begin{aligned}
&\frac{G(u_0/b_0) - G(u/b_0)}{u_0 - u} - \frac{1}{u_0 - u}\int_u^{u_0} \bigl(1 - G\bigl((u_0 - z)/b_0\bigr)\bigr)g(z)\,dz \\
&\quad + \int_0^u \frac{G((u_0 - z)/b_0) - G((u - z)/b_0)}{u_0 - u} g(z)\,dz \\
&\quad - \bigl(1 - c(b_0)\bigr)\frac{g(u) - g(u_0)}{u_0 - u} > 0.
\end{aligned}$$

Letting $u \to u_0$ yields the contradiction $-\infty \geq 0$. Thus $g(u_0) = b_0 = 0$. In particular, $b(u) \to 0$ as $u \to u_0$. Thus $b(u) < \underline{b}/2$ for $u > u_1$. Then

$$\frac{1}{g(u)} < 1 + \gamma + \frac{\mu^2}{2\sigma^2}\int_{u_1}^u \frac{1}{(c(\underline{b}/2) - 1)g(x)}\,dx$$

for an appropriate γ. By Gronwall's inequality [see Ethier and Kurtz (1986), page 498] we find

$$\frac{1}{g(u)} < (1+\gamma)\mathrm{e}^{\kappa(u-u_1)}$$

for some κ. In particular, $g(u_0) = 0$ is not possible, which is a contradiction. This proves the lemma. □

We are now able to prove the main result of this section.

THEOREM 2. *Suppose that $G(x)$ has a bounded density. Then there exists a unique strictly decreasing solution $g(x)$ to* (5) *on* $[0, \infty)$.

PROOF. We already proved uniqueness in Theorem 1. Let $(0, u_0)$ be the largest interval such that (5) and (6) are fulfilled. From Lemma 4 we know that $u_0 > 0$. Suppose $u_0 < \infty$. Then by Lemma 5, (5) and (6) are fulfilled on $(0, u_0]$. As in Hipp and Plum (2000) we want to show that the solution can be extended to an interval $[0, u_0 + \eta)$. As before choose $\lambda = c = 1$ and let $\alpha = \mu/\sigma$. Define

$$\kappa = 1 + \frac{\alpha^2}{2}\int_0^{u_0} 1\Big/\Big(\inf_{b\in[0,1]} 1 - G(x/b) + \int_0^x \big(1 - G\big((x-z)/b\big)\big)g(z)\,dz - \big(1-c(b)\big)g(x)\Big)\,dx$$

and

$$\xi = \tfrac{1}{2}\Big(1 - G(u_0/b) + \int_0^{u_0}\big(1 - G\big((u_0-z)/b\big)\big)g(z)\,dz - \big(1-c(b)\big)g(u_0)\Big) > 0.$$

Let us now consider the operator on positive continuously decreasing functions $h(u)$ on $[u_0, \infty)$ with $h(u_0) = 1/\kappa$:

$$\mathcal{V}h(u) = \frac{1}{\kappa + \frac{\alpha^2}{2}\int_{u_0}^u \frac{1}{[\inf_{b\in[0,1]} 1-G(x/b)+\int_0^x(1-G((x-z)/b))h(z)\,dz-(1-c(b))h(x)]\vee\xi}\,dx}. \tag{7}$$

Let $h_1(u)$ and $h_2(u)$ be two positive continuously decreasing functions with $h_i(u_0) = 1/\kappa$. Because we want to iterate the operator $\mathcal{V}$ we can assume $h_i(u) \geq (\kappa + \alpha^2/(2\xi)(u-u_0))^{-1}$, which is a lower bound for $\mathcal{V}h(u)$ for any function $h(u)$. Denote the arguments for which the infimum is taken by $b_i(u)$. Denote by

$$I_i(x) = \Big[1 - G\big(x/b_i(x)\big) + \int_0^x \big(1 - G\big((x-z)/b_i(x)\big)\big)h_i(z)\,dz - \big(1 - c\big(b_i(x)\big)\big)h_i(x)\Big] \vee \xi.$$

Then

$$|\mathcal{V}h_1(u) - \mathcal{V}h_2(u)| \le \frac{\alpha^2}{2} \frac{\int_{u_0}^u |I_1^{-1}(x) - I_2^{-1}(x)|\,dx}{\left(\kappa + \frac{\alpha^2}{2}\int_{u_0}^u I_1^{-1}(x)\,dx\right)\left(\kappa + \frac{\alpha^2}{2}\int_{u_0}^u I_2^{-1}(x)\,dx\right)}.$$

The integral in the numerator can be estimated by

$$\int_{u_0}^u |I_1^{-1}(x) - I_2^{-1}(x)|\,dx \le \xi^{-2}\int_{u_0}^u |I_1(x) - I_2(x)|\,dx.$$

Let now $b(x) = b_2(x)$ if $I_1(x) \ge I_2(x)$ and $b(x) = b_1(x)$ otherwise. Then

$$\begin{aligned}|I_1(x) - I_2(x)| \le \Big|&\big(1 - c(b(x))\big)\big(h_1(x) - h_2(x)\big) \\ &- \int_{u_0}^x \big(1 - G\big((x-z)/b(x)\big)\big)\big(h_1(z) - h_2(z)\big)\,dz\Big|.\end{aligned}$$

If $I_i(x) = \xi$ then $(c(b_i(x)) - 1)h_i(x) \le \xi$ and $c(b_i(x)) \le 1 + \xi/h_i(x)$. Suppose $u - u_0 \le 1$. Then $b_i(x)$ can be chosen on (u_0, u_0+1) such that $c(b_i(x))$ remains bounded. Let ζ be the maximal possible value of $|c(b_i(x)) - 1| \ge 1$. Then for $u \le u_0 + 1$,

$$\begin{aligned}&\int_{u_0}^u |I_1(x) - I_2(x)|\,dx \\ &\qquad \le \sup_{u_0 \le x \le u_0+1} |h_1(x) - h_2(x)|\big[\zeta(u-u_0) + \tfrac{1}{2}(u-u_0)^2\big].\end{aligned}$$

We can now choose $\tilde\eta > 0$ such that $\zeta\tilde\eta + \frac{1}{2}\tilde\eta^2 = (2\xi^2)/\alpha^2$. We just have proved

$$|\mathcal{V}h_1(u) - \mathcal{V}h_2(u)| \le \kappa^{-2} \sup_{u_0 \le x \le u_0+\tilde\eta} |h_1(x) - h_2(x)|$$

for $u \in (u_0, u_0 + \tilde\eta)$. Thus the operator $\mathcal{V}$ is a contraction on $(u_0, u_0+\tilde\eta)$. In particular, there is a fixed point $h(x)$ solving $\mathcal{V}h(x) = h(x)$. Because

$$I(x) = \inf_{b \in [0,1]} \big(1 - G(x/b)\big) + \int_0^x \big(1 - G\big((x-z)/b\big)\big)h(z)\,dz - \big(1 - c(b)\big)h(x)$$

is continuous in x, there must be an $0 < \eta \le \tilde\eta$ such that $I(x) > \xi$ on $(u_0, u_0+\eta)$. Thus there is a solution $g(x)$ to (5) on $(0, u_0+\eta)$ such that (6) holds. Because $(0, u_0)$ was the largest interval on which this holds, this is a contradiction and $u_0 = \infty$ follows. $\square$

5. Examples. As an illustration we calculate the optimal strategies for two examples, a light tailed and a heavy tailed claim size distribution. Without the possibility to control the risk process the ruin probabilities in these two examples are quite different; see Rolski et al. (1999). We consider the case with no safety loading, $\lambda = \mathbb{E}[Y] = c = 1$. Without the possibility of investment the ruin probability would be one. The investment yields here a possibility to charge a low premium. The diffusion parameters are chosen as $\mu = 0.04$ and $\sigma^2 = 0.01$.

REMARK. The choice of μ and σ^2 has a big influence on the ruin probability. If μ^2/σ^2 is large the ruin probability can even for zero initial capital be almost zero. This is, because the ruin probability of the Brownian motion is exponentially decreasing with exponent $2\mu/\sigma^2$. It is therefore possible to reinsure the whole risk and to invest a certain amount already for quite small initial capital. In this way the risk can almost be removed. If μ^2/σ^2 is small it will be risky to invest. The optimal strategy will then turn out not to reinsure until a certain level is reached, such that investment can give a positive drift without increasing the risk too much. We have chosen the parameters here, such that the ruin probability is not decreased to much and such that reinsurance will take place inside the range where the function $\delta(u)$ is calculated.

The reinsurance company charges a premium obtained from an expected value principle with safety loading 0.2, that is, $c(b) = 1.2(1 - b)\lambda\mathbb{E}[Y]$. For this premium the assumptions on $c(b)$ are fulfilled, also because $c(0) = 1.2 > 1 = c$.

The numerical solution is obtained as follows. First a solution $g(u) = f'(u)$ is obtained from (3) by an Euler scheme, where of course $\mathbb{E}[f(u - bY)] - f(u)$ is replaced by the corresponding expression containing $g(u)$ only. The form of $g(u)$ for u close to zero is obtained from Lemma 4. We choose this initial function $g_0(u)$ in order to save computer time. With an arbitrary initial function the scheme could take long time to converge because on the interval $(u_0, u_0 + \eta)$ the scheme starts to converge only after $g_n(u)$ is close to $g(u)$ on the interval $(0, u_0)$. Thereafter the numerical solution is iterated using (5). Note that (5) is locally a contraction and therefore the scheme is convergent. The reason for the iterations is, because of the numerical errors (at least in the author's program), that the optimal strategies $b^*(u)$ and $A^*(u)$ turn out not to be correct even though the numerical solution to the Euler scheme is close to the function $\delta(u)$ after the iterations. It should be noted that because of the discretization of the state space only numerical approximations are obtained. The function $\delta(u)$ will then be close to the correct solution. Because we worked with the derivative $f'(u)$ also $\delta'(u)$ can be expected to be close to the correct derivative. How close the obtained optimal strategies $A_n^*(u)$ and $b_n^*(u)$ will be is an open question. We have not investigated here whether the corresponding optimal controls converge to the correct optimal controls as the discretisation interval tends to zero. The author believes that this is the case. In any case, the strategies shown in the figures below will be good enough in the sense that the

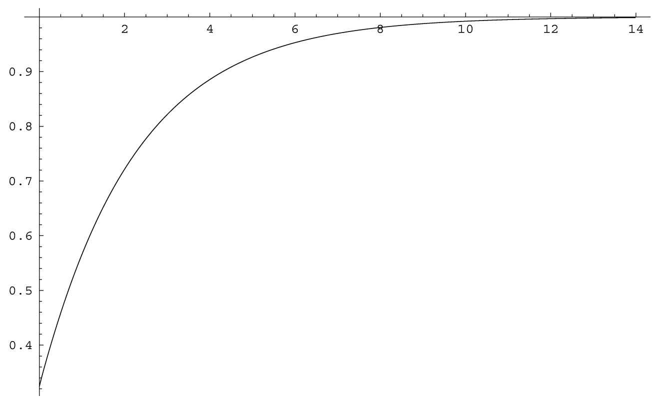

FIG. 1. $\delta(u)$ *for Exp*(1) *distributed claim sizes.*

ruin probability connected to these strategies will be close to the optimal ruin probability. The reader should keep in mind that the discussion of the strategies below are based on the author's numerical results and not on the true strategies.

5.1. *Exponentially distributed claim sizes.* A typical example for small claims are exponentially distributed claim sizes $G(x) = 1 - e^{-x}$. The survival probability is given in Figure 1. The ruin probability goes to zero exponentially fast. This was also expected because the ruin probability both for a classical model with exponentially claim sizes (and positive loading) and for the Brownian motion decrease exponentially fast. For initial capital zero the ruin probability is 0.6756, that is, it has decreased by one third due to optimisation.

The function $b^*(u)$ is given in Figure 2. The optimal reinsurance strategy is not to reinsure close to zero. Then the retention level jumps to 0.04 and thereafter increases slowly to the asymptotic value 0.20. This means, close to zero the goal

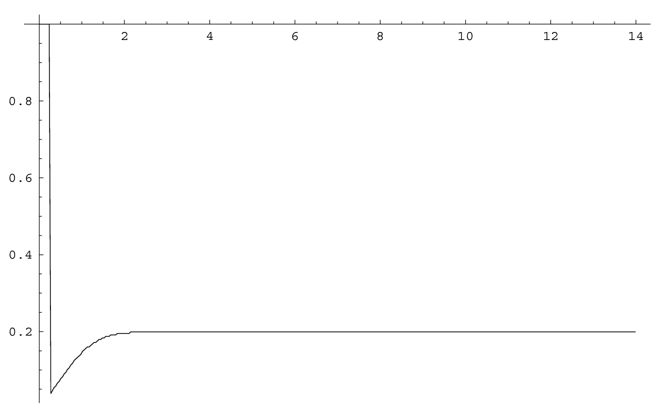

FIG. 2. *Optimal reinsurance strategy* $b^*(u)$ *for Exp*(1) *distributed claim sizes.*

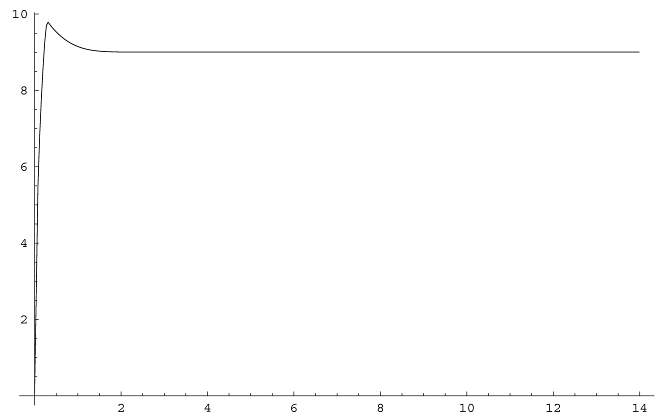

FIG. 3. *Optimal investment strategy $A^*(u)$ for Exp(1) distributed claim sizes.*

is to get away from zero as quick as possible. Then reinsurance is taken in order to make the jumps small. As capital increases less reinsurance is needed, but there is an asymptotic optimal retention level that will be approached. The optimal strategy in the case without investments looks similar; see Schmidli (2001).

The function $A^*(u)$ is given in Figure 3. The optimal investment strategy is to invest more than the surplus close to zero in order to make the drift positive. From Lemma 4 and equation (5) it follows that $A^*(u) \sim C\sqrt{u}$ as $u \to 0$. From the point on, where reinsurance is taken, the investment decreases to the asymptotic optimal value 9.00. Also here, close to zero the goal is to get away from zero as quickly as possible. Drift and volatility have to be balanced. After reinsurance is taken the drift increases because the retention level is increased. Therefore it is possible to invest less. Until reinsurance is taken, the strategy is the same as the one found in Hipp and Plum (2000). Thereafter it changes because the drift increases by taking less reinsurance.

Numerically, this case is quite simple because $1 - G(u)$ is decreasing exponentially fast. This will give a good approximation to the second derivative of $\delta(u)$ and therefore $A_n^*(u)$ should be close to the correct strategy. Also the exponential decrease in (6) will give a "nice" function in b in order to determine $b_n^*(u)$. Thus the author believes that in this case the optimal strategies are close to their numerical approximations.

5.2. *Pareto distributed claim sizes.* A typical example for large claims are Pareto distributed claims sizes $G(x) = 1 - (1 + x)^{-2}$. Let us first consider the case where the whole insurance risk is reinsured. Then the premium rate left to the insurer is -0.2. Calculating the survival probability yields $\delta^{A,0}(u) = 1 - e^{-0.4u}$ and the optimal investment is $A = 10$, where we used that ruin occurs almost surely if $u = 0$. Therefore it is not surprising that the ruin probability under optimal reinsurance and investment goes to zero exponentially fast; see Figure 4. Hence

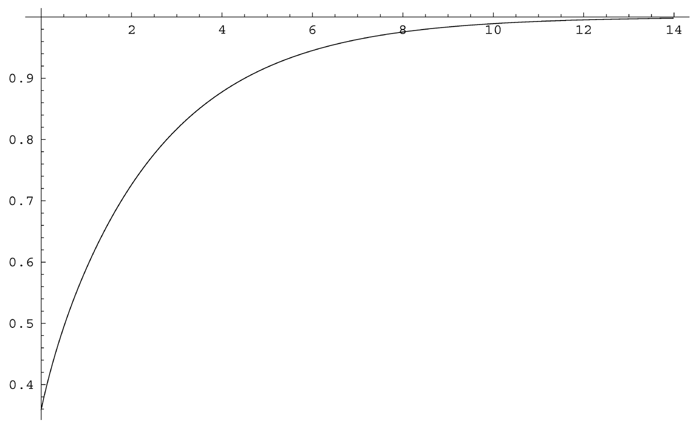

FIG. 4. *$\delta(u)$ for Par(2) distributed claim sizes.*

investment and reinsurance decrease the ruin probability considerably for larger initial capital. Without these two possibilities the ruin probability would decrease with a power tail. For zero initial capital the ruin probability is 0.6411.

The function $b^*(u)$ is given in Figure 5. As always, for small initial capital no reinsurance is taken. Then the optimal retention level jumps to 0.09. After a slight increase the retention level decreases slowly and then jumps to zero. Thus for small initial capital the goal is to get away from zero as fast as possible. Then reinsurance is taken to decrease the claim sizes. As the capital increases the retention level is only changed slightly. As soon as the capital is large enough such that the investment risk becomes small enough, the whole risk is transferred to the reinsurer and the company is only left with the investment risk. This strategy may be prohibited by law. Because the probability of a claim larger than u is

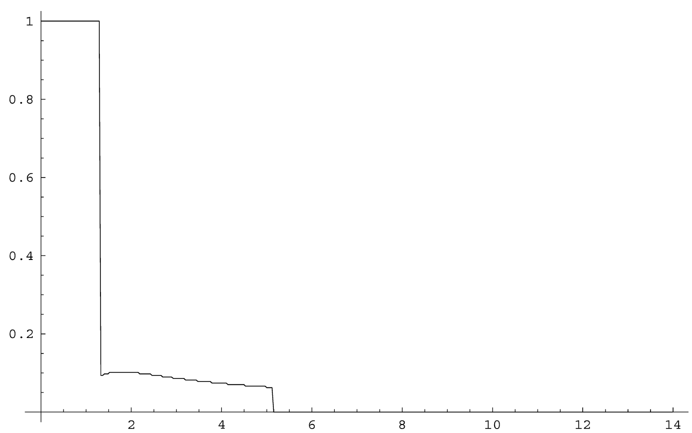

FIG. 5. *Optimal reinsurance strategy $b^*(u)$ for Par(2) distributed claim sizes.*

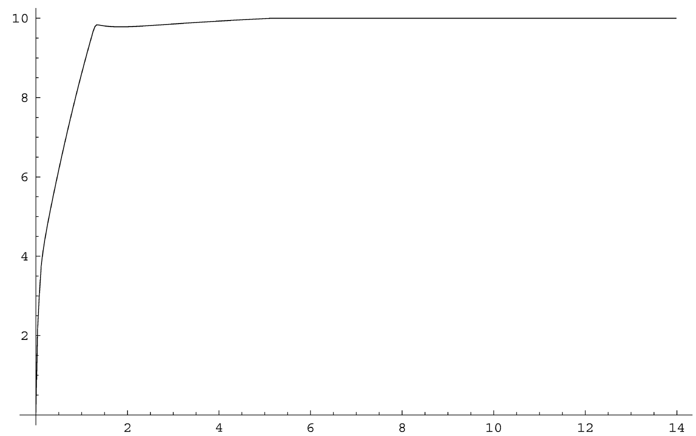

FIG. 6. *Optimal investment strategy $A^*(u)$ for Par(2) distributed claim sizes.*

$(1+u)^{-2} \sim u^{-2}$ the ruin probability cannot decrease faster than $b(u)u^{-2}$. Thus $b(u)$ has to converge to zero exponentially fast in order to obtain an exponential decrease of the ruin probability. The reinsurance strategy looks quite different from the optimal strategy without investment; see Schmidli (2001). Without an investment possibility the optimal retention level decreases slowly and converges to an optimal asymptotic value, but is far from this asymptotic value for moderate initial capital. Whether $b^*(u)$ really jumps to zero or not is not clear. Because of the slow decrease of $1 - G(u)$ the expression in (6) will not show big differences as a function of b. In the author's program the smallest b is chosen if several b yield the same expression. It is therefore possible that $b^*(u) > 0$ but numerically will yield the same value as $b = 0$.

The function $A^*(u)$ is given in Figure 6. The investment increases close to zero like $C\sqrt{u}$. At the point where reinsurance is taken the investment starts to decrease slightly and then increases slowly until the point where full reinsurance is chosen. At this point the asymptotically optimal investment 10 is reached. Also here close to zero investment risk is taken to increase the drift in order to get away from zero as fast as possible. In some interval reinsurance and investment are chosen to be balanced, until the whole risk is transferred to the investment. The strategy is quite different from the optimal strategy obtained in Hipp and Plum (2000) where reinsurance was not possible. With and without reinsurance the two strategies of course coincide until reinsurance is taken. In Hipp and Plum (2000) the optimal investment first decreases after some point and then increases slowly such that $A(u)$ converges to infinity in order to increase the drift because the heavy tailed claim sizes are a constant threat to the company.

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