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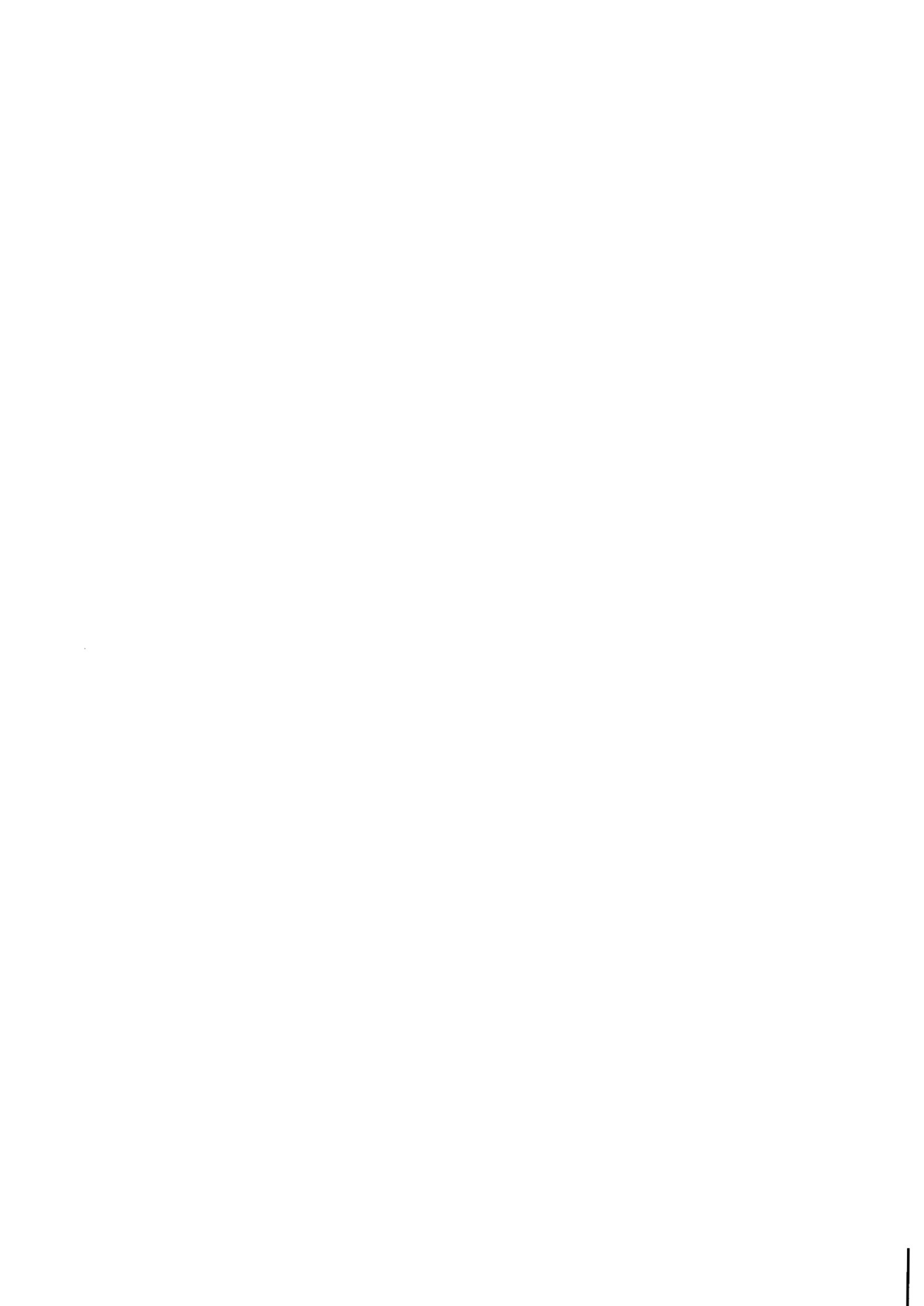
ON MINIMIZING THE SUM OF A CONVEX FUNCTION  
AND A CONCAVE FUNCTION

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## PREFACE

In this paper, the author presents an algorithm for minimizing the sum of a convex function and a concave function. The functions involved are not necessarily smooth and the resulting function is quasidifferentiable. The main property of such functions is the non-uniqueness of directions of steepest descent (and ascent), and therefore special precautions must be taken to guarantee that the algorithm converges to a stationary point.

This paper is a contribution to research on nondifferentiable optimization currently underway within the System and Decision Sciences Program.

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We consider here the problem of minimizing a particular subclass of quasidifferentiable functions: those which may be represented as the sum of a convex function and a concave function. It is shown that in an  $n$ -dimensional space this problem is equivalent to the problem of minimizing a concave function on a convex set. A successive approximations method is suggested; this makes use of some of the principles of  $\epsilon$ -steepest-descent-type approaches.

*Key words:* Quasidifferentiable Functions, Convex Functions, Concave Functions,  $\epsilon$ -Steepest-Descent Methods.

1. Introduction

The problem of minimizing nonconvex nondifferentiable functions poses a considerable challenge to specialists in mathematical programming. Most of the difficulties arise from the fact that there may be several directions of steepest descent. To solve this problem requires both a new technique and a new approach. In this paper we discuss a special subclass of nondifferentiable functions: those which can be represented in the form

$$f(x) = f_1(x) + f_2(x) ,$$

where  $f_1$  is a finite function which is convex on  $E_n$  and  $f_2$  is a finite function which is concave on  $E_n$ . Then  $f$  is continuous and quasidifferentiable on  $E_n$ , with a quasidifferential at  $x \in E_n$  which may be taken to be the pair of sets

$$Df(x) = [\underline{\partial}f(x) , \bar{\partial}f(x)] ,$$

where

$$\underline{\partial}f(x) = \partial f_1(x) = \{v \in E_n \mid f_1(z) - f_1(x) \geq (v, z-x) \quad \forall z \in E_n\} ,$$

$$\bar{\partial}f(x) = \partial f_2(x) = \{w \in E_n \mid f_2(z) - f_2(x) \leq (w, z-x) \quad \forall z \in E_n\} .$$

In other words,  $\underline{\partial}f(x)$  is the subdifferential of the convex function  $f_1$  at  $x \in E_n$  (as defined in convex analysis) and  $\bar{\partial}f(x)$  is the superdifferential of the concave function  $f_2$  at  $x \in E_n$ .

Consider the problem of calculating

$$\inf_{x \in E_n} f(x) . \tag{1}$$

Quasidifferential calculus shows that for  $x^* \in E_n$  to be a minimum point of  $f$  on  $E_n$  it is necessary that

$$-\bar{\partial}f(x^*) \subset \underline{\partial}f(x^*) . \tag{2}$$

We shall now show that the problem of minimizing  $f$  on the space  $E_n$  can be reduced to that of minimizing a concave function on a convex set.

Let  $\Omega$  denote the *epigraph* of the convex function  $f_1$ , i.e.,

$$\Omega = \text{epi } f = \{z = [x, \mu] \in E_n \times E_1 \mid h(z) \equiv f_1(x) - \mu \leq 0\},$$

and define the following function on  $E_n \times E_1$ :

$$\psi(z) = f_2(x) + \mu, \quad z = [x, \mu] \in E_n \times E_1.$$

Set  $\Omega$  is closed and convex and function  $\psi$  is quasidifferentiable at any point  $z \in E_n \times E_1$ . Take as its quasidifferential at  $z = [x, \mu]$  the pair of sets  $D\psi(z) = [\{0\}, \partial f_2(x) \times \{1\}]$ , where  $0 \in E_{n+1}$ .

Let us now consider the problem of finding

$$\inf_{z \in \Omega} \psi(z). \quad (3)$$

It is well-known (see, e.g., [3]) that if a concave function achieves its infimal value on a convex set, this value is achieved on the boundary of the set.

Theorem 1. For a point  $x^*$  to be a solution of problem (1), it is both necessary and sufficient that point  $[x^*, \mu^*]$  be a solution to problem (3), where  $\mu^* = f(x^*)$ .

Proof

*Necessity.* Let  $x^*$  be a solution of problem (1). Then

$$\mu + f_2(x) \geq f_1(x) + f_2(x) \geq f_1(x^*) + f_2(x^*) \quad \forall \mu \geq f_1(x), \quad \forall x \in E_n. \quad (4)$$

But (4) implies that

$$\psi(z) \geq f_1(x^*) + f_2(x^*) = f_2(x^*) + \mu^* ,$$

where  $\mu^* = f_1(x^*)$  . Thus there exists a  $z^* = [x^*, \mu^*] \in \Omega$  such that

$$\psi(z) \geq \psi(z^*) \quad \forall z \in \Omega . \quad (5)$$

This proves that the condition is necessary.

*Sufficiency.* That the condition is also sufficient can be proved in an analogous way by arguing backwards from inequality (5).

## 2. A numerical algorithm

Set  $\varepsilon \geq 0$  . A point  $x_0 \in E_n$  is called an  $\varepsilon$ -*inf-stationary* point of the function  $f$  on  $E_n$  if

$$-\bar{\partial}f(x_0) \subset \underline{\partial}_\varepsilon f(x_0) , \quad (6)$$

where

$$\begin{aligned} \underline{\partial}_\varepsilon f(x_0) &= \underline{\partial}_\varepsilon f_1(x_0) = \{v \in E_n \mid f_1(z) - f_1(x_0) \geq \\ &\geq (v_1, z-x_0) - \varepsilon \quad \forall x \in E_n\} , \end{aligned}$$

i.e.,  $\underline{\partial}_\varepsilon f(x_0)$  is the  $\varepsilon$ -subdifferential of the convex function  $f_1$  at  $x_0$  . Fix  $g \in E_n$  and set



$$\frac{\partial_{\varepsilon} f(x_0)}{\partial g} = \max_{v \in \bar{\partial}_{-\varepsilon} f(x_0)} (v, g) + \min_{w \in \bar{\partial} f(x_0)} (w, g) . \quad (7)$$

Theorem 2. For a point  $x_0$  to be an  $\varepsilon$ -inf-stationary point of the function  $f$  on  $E_n$ , it is both necessary and sufficient that

$$\min_{\|g\|=1} \frac{\partial_{\varepsilon} f(x_0)}{\partial g} \geq 0 . \quad (8)$$

Proof

*Necessity.* Let  $x_0$  be an  $\varepsilon$ -inf-stationary point of  $f$  on  $E_n$ .

Then from (6) it follows that

$$0 \in w + \bar{\partial}_{-\varepsilon} f(x_0) \quad \forall w \in \bar{\partial} f(x_0) .$$

Hence

$$\min_{\|g\|=1} \max_{z \in w + \bar{\partial}_{-\varepsilon} f(x_0)} (z, g) \geq 0 \quad \forall w \in \bar{\partial} f(x_0) ,$$

and thus for every  $g \in E_n$ ,  $\|g\|=1$ , we have

$$\min_{w \in \bar{\partial} f(x_0)} \max_{v \in \bar{\partial}_{-\varepsilon} f(x_0)} (z, g) \geq 0 .$$

However, this means that

$$\min_{\|g\|=1} \frac{\partial_{\varepsilon} f(x_0)}{\partial g} \geq 0 \quad (9)$$

proving that the condition is necessary. That it is also sufficient can be demonstrated in an analogous way, arguing backwards from the inequality (9).

Note that since the mapping

$$\partial_{-\varepsilon} f : E_n \times [0, +\infty) \longrightarrow 2^{E_n}$$

is Hausdorff-continuous if  $\varepsilon > 0$  (see, e.g., [1]), then the following theorem holds.

Theorem 3. *If  $\varepsilon > 0$  then the function  $\max_{v \in \partial_{-\varepsilon} f(x)} (v, g)$  is continuous in  $x$  on  $E_n$  for any fixed  $g \in E_n$ .*

Assume that  $x_0$  is not an  $\varepsilon$ -inf-stationary point. Then we can describe the vector

$$g_{\varepsilon}(x_0) = \arg \min_{\|g\|=1} \frac{\partial_{\varepsilon} f(x_0)}{\partial g}$$

as a direction of  $\varepsilon$ -steepest-descent of function  $f$  at point  $x_0$ .

It is not difficult to show that the direction

$$g_{\varepsilon} = - \left( \frac{v_{0\varepsilon} + w_0}{\|v_{0\varepsilon} + w_0\|} \right),$$

where  $v_{0\varepsilon} \in \partial_{-\varepsilon} f(x_0)$ ,  $w_0 \in \bar{\partial} f(x_0)$  and

$$- \max_{w \in \bar{\partial} f(x_0)} \min_{v \in \partial_{-\varepsilon} f(x_0)} \|v+w\| = -\|v_{0\varepsilon} + w_0\| = a_{\varepsilon}(x_0),$$

is a direction of  $\varepsilon$ -steepest-descent of function  $f$  at point  $x_0$ .

Now let us consider the following method of successive approximations.

Fix  $\varepsilon > 0$  and choose an arbitrary initial approximation  $x_0 \in E_n$ . Suppose that the Lebesgue set

$$D(x_0) = \{x_0 \in E_n \mid f(x) \leq f(x_0)\}$$

is bounded. Assume that a point  $x_k \in E_n$  has already been found. If  $-\bar{\partial}f(x_k) \subset \partial_{-\varepsilon}f(x_k)$ , then  $x_k$  is an  $\varepsilon$ -inf-stationary point of  $f$  on  $E_n$ ; if not, take

$$x_{k+1} = x_k + \alpha_k g_{\varepsilon k}, \quad \alpha_k = \arg \min_{\alpha \geq 0} f(x_k + \alpha g_{\varepsilon k}),$$

where  $g_{\varepsilon k} = g_{\varepsilon}(x_k)$  is an  $\varepsilon$ -steepest-descent direction of  $f$  at  $x_k$ .

Theorem 4. *The following relation holds:*

$$\lim_{k \rightarrow \infty} a_{\varepsilon}(x_k) = 0.$$

Proof. We shall prove the theorem by contradiction. Assume that a subsequence  $\{x_{k_s}\}$  of sequence  $\{x_k\}$  and a number  $a > 0$  exist such that

$$a_{\varepsilon}(x_{k_s}) \leq -a \quad \forall s.$$

(The required subsequence must exist since  $D(x_0)$  is compact.) Without loss of generality, we can assume that  $x_{k_s} \longrightarrow x^*$  (clearly,  $x^* \in D(x_0)$ ). Then

$$f(x_{k_s} + \alpha g_{\epsilon k_s}) = f(x_{k_s}) + \int_0^\alpha \left( \frac{\partial f_1(x_{k_s} + \tau g_{\epsilon k_s})}{\partial g_{\epsilon k_s}} \right) d\tau + \\ + \alpha \left( \frac{\partial f_2(x_{k_s})}{\partial g_{\epsilon k_s}} \right) + o(\alpha, g_{\epsilon k_s}),$$

where

$$\frac{o(\alpha, g_{\epsilon k_s})}{\alpha} \xrightarrow{\alpha \rightarrow +0} 0.$$

The term  $o(\alpha, g_{\epsilon k_s})$  appears in the above equation due to the concavity of  $f_2$ . The fact that function  $f_2$  is concave implies that

$$o(\alpha, g_{\epsilon k_s}) \leq 0 \quad \forall \alpha > 0, \quad \forall g_{\epsilon k_s} \in E_n,$$

and therefore

$$f(x_{k_s} + \alpha g_{\epsilon k_s}) \leq f(x_{k_s}) + \int_0^\alpha \max_{v \in \partial f_1(x_{k_s} + \tau g_{\epsilon k_s})} (v, g_{\epsilon k_s}) d\tau + \\ + \alpha \min_{w \in \partial f_2(x_{k_s})} (w, g_{\epsilon k_s}).$$

Since  $\partial_\epsilon f_1(x) \supset \partial f_1(x)$  for every  $x \in E_n$ , we have

$$\max_{v \in \partial_\epsilon f_1(x_{k_s} + \tau g_{\epsilon k_s})} (v, g_{\epsilon k_s}) \geq \max_{v \in \partial f_1(x_{k_s} + \tau g_{\epsilon k_s})} (v, g_{\epsilon k_s}),$$

and thus

$$f(x_{k_s} + \alpha g_{\epsilon k_s}) \leq f(x_{k_s}) + \int_0^\alpha \max_{v \in \partial_\epsilon f_1(x_{k_s} + \tau g_{\epsilon k_s})} (v, g_{\epsilon k_s}) d\tau + \\ + \alpha \min_{w \in \partial f_2(x_{k_s})} (w, g_{\epsilon k_s}) .$$

Since the mapping  $\partial_\epsilon f_1$  is Hausdorff-continuous at the point  $x^*$ , there exists a  $\delta > 0$  such that

$$\partial_\epsilon f_1(x) \subset \partial_\epsilon f_1(y) + \frac{a}{2} S_1(0) \quad \forall x, y \in S_\delta(x^*) ,$$

where  $S_r(z) = \{x \in E_n \mid \|x-z\| \leq r\}$ . Also, there exists a number  $K > 0$  such that

$$x_{k_s} \in S_{\delta/2}(x^*) \quad \forall k_s > K ,$$

and hence

$$f(x_{k_s} + \alpha g_{\epsilon k_s}) \leq f(x_{k_s}) + \alpha(a_\epsilon(x_{k_s}) + \frac{a}{2}) \\ \forall \alpha \in (0, \frac{\delta}{2}] , \quad \forall k_s > K .$$

Therefore

$$f(x_{k_s+1}) = \min_{\alpha \geq 0} f(x_{k_s} + \alpha g_{\epsilon k_s}) \leq f(x_{k_s} + \frac{\delta}{2} g_{\epsilon k_s}) \leq \\ \leq f(x_{k_s}) - \frac{\delta a}{4} . \quad (10)$$

Inequality (10) contradicts the fact that sequence  $\{f(x_k)\}$  is bounded, thus proving the theorem.

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