where $0 < \beta < 1, \lambda > 0$. We find that

$$a_n = \frac{\lambda + n - 1}{n!} \beta(1 + \lambda\beta) \cdots (n - 1 + [\lambda + n - 2]\beta), \qquad n = 1, 2, \cdots,$$

with $a_0 = 1$. In particular, when $\lambda = 1$, we have

$$a_n = (1 + \beta)^{n-1}\beta, \qquad n = 1, 2, \cdots,$$

with $a_0 = 1$, and each row of **A** is a truncated modified geometric distribution.

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ON MINIMUM VARIANCE ESTIMATORS[1]

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Chapman and Robbins [1] have given a simple improvement on the Cramér-Rao inequality without postulating the regularity assumptions under which the latter is usually proved. The purpose of this note is to show by examples how a similarly derived stronger inequality (see equation (2)) may be used to verify that certain estimators are uniform minimum variance unbiased estimators. This stronger inequality is that which (under additional restrictions) was shown in [2] to be the best possible, but is in a more useful form for applications than the form given in [2]. For simplicity we consider only an inequality on the variance of unbiased estimators, but inequalities on other moments than the second (see [2]), or for biased estimators, may be found similarly. The two examples considered here are ones where the regularity conditions of [2] are not satisfied, where the method of [1] does not give the best bound, and where the method of this note is used to find the best bound and thus to verify that certain estimators are uniform minimum variance unbiased. (For the examples considered this also follows from completeness of the sufficient statistic; the method used here applies, of course, more generally.)

Let X be a chance variable with density $f(x; \theta)$ with respect to some fixed σ-finite measure μ. ($\theta \, \varepsilon \, \Omega$, $x \, \varepsilon \, \mathfrak{X}$). We suppose suitable Borel fields to be given and $f(x; \theta)$ to be measurable in its arguments. Ω is a subset of the real line. For each θ, let $\Omega_\theta = \{h \mid (\theta + h) \, \varepsilon \, \Omega\}$. For fixed θ, let λ_1 and λ_2 be any two probability measures on Ω_θ such that $E_i h = \int_{\Omega_\theta} h d\lambda_i(h)$ exists for $i = 1, 2$. Then, for any

[1] Research sponsored by the Office of Naval Research.

$t(x)$ for which $E_\theta t = \theta$, we have

$$\int_{\mathfrak{X}} (t-\theta)\sqrt{f(x;\theta)}\left\{\frac{\int_{\Omega_\theta} f(x;\theta+h)\, d[\lambda_1(h)-\lambda_2(h)]}{f(x;\theta)}\right\}\sqrt{f(x;\theta)}\, d\mu = E_1 h - E_2 h. \tag{1}$$

Applying Schwarz's inequality, we have after some obvious manipulations,

$$E_\theta(t-\theta)^2 \geqq \sup\left\{\frac{(E_1h - E_2h)^2}{\int_{\mathfrak{X}} \frac{\left\{\int_{\Omega_\theta} f(x;\theta+h)\, d[\lambda_1(h)-\lambda_2(h)]\right\}^2}{f(x;\theta)}\, d\mu}\right\}, \tag{2}$$

where for each θ the supremum is taken over all λ_1 and λ_2 for which $\lambda_1 \neq \lambda_2$ and for which the integrand of the integral over $\mathfrak{X}$ is defined a.e. (μ).

We remark that the supremum of (2) is easily seen to be unimproved if λ_i and E_ih are multiplied by real numbers c_i ($i = 1, 2$) with respect to which the supremum is also taken. From this fact it is easy to verify that the right side of (2) must coincide with the expression given in Theorem 4 of [2] (for $s = 2$ there), and which Barankin shows (under the assumption that $f(x;\theta+h)/f(x;\theta)$ is defined a.e. (μ) and (for our case) belongs to L_2 with respect to the measure $\nu(A) = \int_A f(x;\theta)\, d\mu$ for all $h \varepsilon \Omega_\theta$) to be the best possible bound. However, the form of equation (2) is more useful for applications, since one can sometimes find λ_i for which the bound is attained but where no discrete λ_i (essentially what are used in the form of [2]) actually give this bound.

It will often suffice in applications to let λ_2 give measure one to the point $h = 0$. This gives

$$E_\theta(t-\theta)^2 \geqq \sup_{\lambda_1}\left\{\frac{(E_1h)^2}{\int_{\mathfrak{X}} \frac{\left[\int_{\Omega_\theta} f(x;\theta+h)\, d\,\lambda_1(h)\right]^2}{f(x;\theta)}\, d\mu - 1}\right\}. \tag{3}$$

If we consider only those λ_1 which give measure one to a single h, we obtain

$$E_\theta(t-\theta)^2 \geqq \frac{1}{\inf_h \frac{1}{h^2}\left\{\int_{\mathfrak{X}} \frac{[f(x;\theta+h)]^2}{f(x;\theta)}\, d\mu - 1\right\}}, \tag{4}$$

where the infimum is over all $h \neq 0$ for which $h \varepsilon \Omega_\theta$ and for which $f(x;\theta) = 0$ implies $f(x;\theta+h) = 0$ a.e. (μ). The latter is precisely the condition of equation (2) of [1], the result of which thus coincides with (4).

We now give two examples where the right side of (3) suffices to give the best bound, where the right side of (4) does not give the best bound, and where the previously mentioned restrictions of [2] are not satisfied. In both examples μ is Lebesgue measure on the real line.

EXAMPLE 1. We have n observations from a rectangular distribution from 0 to θ ($\Omega = \{\theta \mid \theta > 0\}$). It suffices to consider the maximum Y of the observations, whose density is ny^{n-1}/θ^n for $0 \leqq y \leqq \theta$, and 0 elsewhere. For $n = 1$, the denominator of the right side of (4) becomes $\inf_{-\theta<h<0}\{-1/[h(\theta + h])\}$, so that (4) gives the bound $\theta^2/4$. It would be too tedious to carry this calculation out for each n, but it can be shown that, as $n \to \infty$, (4) asymptotically gives the bound $.648\theta^2/n^2$. On the other hand, if we put $d\lambda_1(h) = [(n + 1)/\theta]\,(h/\theta + 1)^n\,dh$ for $-\theta < h < 0$, the term in braces on the right side of (3) becomes $\theta^2/[n(n + 2)]$, which is in fact attained as the variance of the unbiased estimator $[(n + 1)/n]Y$.

EXAMPLE 2. We have m observations from the distribution with density $e^{-(x-\theta)}$ for $x \geqq \theta$ and 0 elsewhere (Ω is the real line). Here the minimum Z of the observations is sufficient and has density $me^{-m(z-\theta)}$, $z \geqq \theta$. The denominator of (4) is $\inf_{h>0}\,([e^{mh} - 1]/h^2)$. The infimum is attained for $mh = 1.5936$, and yields $.648/m^2$ as the bound given by (4). On the other hand, putting $d\lambda_1(h) = me^{-mh}\,dh$ for $0 < h < \infty$ and 0 otherwise, the expression in braces of (3) becomes $1/m^2$, which is actually attained as the variance of the unbiased estimator $Z - 1/m$.

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BHATTACHARYYA BOUNDS WITHOUT REGULARITY ASSUMPTIONS

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1. Summary. In [1] a method for removing the regularity conditions from the Cramér-Rao Inequality was given and applied to the estimation of a single real parameter. It was noted there that the method would extend to problems more general than estimating a single real parameter. However, the method extends also for the estimation of a single real parameter and produces analogues of the Bhattacharyya bounds with and without nuisance parameters.

2. Introduction. Let $\mu(x)$ be a σ-finite measure defined over an additive class $\mathcal{A}$ of subsets of a space $\mathfrak{X}$, and let X be a random variable with density

$$f(x; \theta_1, \cdots, \theta_k)$$

with respect to $\mu(x)$. $\theta_1, \cdots, \theta_k$ are real with $(\theta_1, \cdots, \theta_k) = \Theta \,\varepsilon\, A \subset R^k$. The carrier $S(\theta_1, \cdots, \theta_k)$ of the distribution is defined by

$$S(\theta_1, \cdots, \theta_k) = \{x \mid f(x; \theta_1, \cdots, \theta_k) > 0\}.$$