REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 12, Pages 83–130 (March 5, 2008)
S 1088-4165(08)00322-1

ON MIRKOVIĆ-VILONEN CYCLES AND CRYSTAL COMBINATORICS

PIERRE BAUMANN AND STÉPHANE GAUSSENT

Abstract. Let G be a complex connected reductive group and let $G^\vee$ be its Langlands dual. Let us choose a triangular decomposition $\mathfrak{n}^{-,\vee} \oplus \mathfrak{h}^\vee \oplus \mathfrak{n}^{+,\vee}$ of the Lie algebra of $G^\vee$. Braverman, Finkelberg and Gaitsgory show that the set of all Mirković-Vilonen cycles in the affine Grassmannian $G\big(\mathbb{C}((t))\big)/G\big(\mathbb{C}[[t]]\big)$ is a crystal isomorphic to the crystal of the canonical basis of $U(\mathfrak{n}^{+,\vee})$. Starting from the string parameter of an element of the canonical basis, we give an explicit description of a dense subset of the associated MV cycle. As a corollary, we show that the varieties involved in Lusztig's algebraic-geometric parametrization of the canonical basis are closely related to MV cycles. In addition, we prove that the bijection between LS paths and MV cycles constructed by Gaussent and Littelmann is an isomorphism of crystals.

1. Introduction

Let G be a complex connected reductive group, $G^\vee$ its Langlands dual, and $\mathscr{G}$ its affine Grassmannian. The geometric Satake correspondence of Lusztig [22], Beilinson and Drinfeld [3] and Ginzburg [12] relates rational representations of $G^\vee$ to the geometry of $\mathscr{G}$. More precisely, let us fix a pair of opposite Borel subgroups in G, to enable us to speak of weights and dominance. Each dominant weight λ for $G^\vee$ determines a $G(\mathbb{C}[[t]])$-orbit $\mathscr{G}_\lambda$ in $\mathscr{G}$. Then the geometric Satake correspondence identifies the underlying space of the irreducible rational $G^\vee$-module $L(\lambda)$ with highest weight λ with the intersection cohomology of $\overline{\mathscr{G}_\lambda}$.

In [27], Mirković and Vilonen present a proof of the geometric Satake correspondence valid in any characteristic. Their main tool is a class $\mathscr{Z}(\lambda)$ of subvarieties of $\overline{\mathscr{G}_\lambda}$, the so-called MV cycles, which affords a basis of the intersection cohomology of $\overline{\mathscr{G}_\lambda}$. It is tempting to try to compare this construction with standard bases in $L(\lambda)$, for instance with the canonical basis of Lusztig [23] (also known as the global crystal basis of Kashiwara [15]).

Several works achieve such a comparison on a combinatorial level. More precisely, let us recall that the combinatorial object that indexes naturally the canonical basis of $L(\lambda)$ is the crystal $\mathbf{B}(\lambda)$. In [9], Braverman and Gaitsgory endow the set $\mathscr{Z}(\lambda)$ with the structure of a crystal and show the existence of an isomorphism of crystals $\Xi(\lambda) : \mathbf{B}(\lambda) \xrightarrow{\simeq} \mathscr{Z}(\lambda)$. In [11], Gaussent and Littelmann introduce a set $\Gamma^+_{\mathrm{LS}}(\gamma_\lambda)$

Received by the editors April 26, 2007 and, in revised form, October 17, 2007.
2000 *Mathematics Subject Classification.* Primary 20G05; Secondary 05E15 14M15 17B10 22E67.
Key words and phrases. Affine Grassmannian, Mirković-Vilonen cycle, crystal.
Both authors are members of the European Research Training Network "LieGrits", contract no. MRTN-CT 2003-505078.

of "LS galleries". They endow it with the structure of a crystal and they associate an MV cycle $Z(\delta) \in \mathscr{Z}(\lambda)$ to each LS gallery $\delta \in \Gamma^+_{\mathrm{LS}}(\gamma_\lambda)$. Finally they show the existence of an isomorphism of crystals $\chi : \mathbf{B}(\lambda) \xrightarrow{\simeq} \Gamma^+_{\mathrm{LS}}(\gamma_\lambda)$ and they prove that the map $Z : \Gamma^+_{\mathrm{LS}}(\gamma_\lambda) \to \mathscr{Z}(\lambda)$ is a bijection. One of the results of the present paper (Theorem 5.8) says that Gaussent and Littelmann's map Z is the composition $\Xi(\lambda) \circ \chi^{-1}$; in particular, Z is an isomorphism of crystals.

Let Λ be the lattice of weights of $G^\vee$, let $\mathfrak{n}^{-,\vee} \oplus \mathfrak{h} \oplus \mathfrak{n}^{+,\vee}$ be the triangular decomposition of the Lie algebra of $G^\vee$ afforded by the pinning of G, and let $\mathbf{B}(-\infty)$ be the crystal of the canonical basis of $U(\mathfrak{n}^{+,\vee})$. Then for each dominant weight λ, the crystal $\mathbf{B}(\lambda)$ can be embedded into a shifted version $\mathbf{T}_{w_0\lambda} \otimes \mathbf{B}(-\infty)$ of $\mathbf{B}(-\infty)$, where $w_0\lambda$ is the smallest weight of $\mathbf{B}(\lambda)$. It is thus natural to consider a big crystal $\widetilde{\mathbf{B}(-\infty)} = \bigoplus_{\lambda\in\Lambda} \mathbf{T}_\lambda \otimes \mathbf{B}(-\infty)$ in order to deal with all the $\mathbf{B}(\lambda)$ simultaneously. The isomorphisms $\Xi(\lambda) : \mathbf{B}(\lambda) \xrightarrow{\simeq} \mathscr{Z}(\lambda)$ then assemble in a big bijection $\Xi : \widetilde{\mathbf{B}(-\infty)} \xrightarrow{\simeq} \mathscr{Z}$. The set $\mathscr{Z}$ here collects subvarieties of $\mathscr{G}$ that have been introduced by Anderson in [1]. These varieties are a slight generalization of the usual MV cycles; indeed $\mathscr{Z} \supseteq \mathscr{Z}(\lambda)$ for each dominant weight λ. Kamnitzer [13] calls the elements of $\mathscr{Z}$ "stable MV cycles", but we will simply call them MV cycles. The existence of Ξ and of a crystal structure on $\mathscr{Z}$, and the fact that Ξ is an isomorphism of crystals are due to Braverman, Finkelberg and Gaitsgory [8].

The crystal $\mathbf{B}(-\infty)$ can be parametrized in several ways. Two families of parametrizations, usually called the Lusztig parametrizations and the string parametrizations (see [6]), depend on the choice of a reduced decomposition of the longest element in the Weyl group of G; they establish a bijection between $\mathbf{B}(-\infty)$ and tuples of natural integers. On the contrary, Lusztig's algebraic-geometric parametrization [25] is intrinsic and describes $\mathbf{B}(-\infty)$ in terms of closed subvarieties in $U^-\big(\mathbb{C}[[t]]\big)$, where U^- is the unipotent radical of the negative Borel subgroup of G.

One of the main results of the present paper is Theorem 4.6, which describes very explicitly the MV cycle $\Xi(t_0 \otimes b)$ starting from the string parameter of $b \in \mathbf{B}(-\infty)$. In the course of his work on MV polytopes [13], Kamnitzer obtains a similar result, this time starting from the Lusztig parameter of b. Though both results are related (see Section 4.5), our approach is foreign to Kamnitzer's methods. Our main ingredient indeed is a concrete algebraic formula for Braverman, Finkelberg and Gaitsgory's crystal operations on $\mathscr{Z}$ that translates the original geometric definition (Proposition 4.5). Moreover, our result implies that Lusztig's algebraic-geometric parametrization is closely related to MV cycles (Proposition 4.9).

The paper consists of four sections (plus the introduction). Section 2 fixes some notation and gathers facts and terminology from the theory of crystals bases. Section 3 recalls several standard constructions in the affine Grassmannian and presents the known results concerning MV cycles. Section 4 defines Braverman, Finkelberg and Gaitsgory's crystal operations on $\mathscr{Z}$ and presents our results concerning string parametrizations. Section 5 establishes that Gaussent and Littelmann's bijection $Z : \Gamma^+_{\mathrm{LS}}(\gamma_\lambda) \to \mathscr{Z}(\lambda)$ is a crystal isomorphism. Each section opens with a short summary which gives a more detailed account of its contents.

Acknowledgments. We wish to thank M. Ehrig, J. Kamnitzer, P. Littelmann, I. Mirković, S. Morier-Genoud and G. Rousseau for fruitful conversations, vital information and/or useful indications. We are also grateful to the referee for his attentive reading and his skillful suggestions.

2. Preliminaries

The task devoted to Section 2.1 is to fix the notation concerning the pinned group G. In Section 2.2, we fix the notation concerning crystal bases for $G^\vee$-modules.

2.1. **Notations for pinned groups.**

In the entire paper, G will be a complex connected reductive algebraic group. We assume that a Borel subgroup B^+ and a maximal torus $T \subseteq B^+$ are fixed. We let B^- be the opposite Borel subgroup to B^+ relatively to T. We denote the unipotent radical of $B^\pm$ by $U^\pm$.

We denote the character group of T by $X = X^*(T)$; we denote the lattice of all one-parameter subgroups of T by $\Lambda = X_*(T)$. A point $\lambda \in \Lambda$ is a morphism of algebraic groups $\mathbb{C}^\times \to T,\ a \mapsto a^\lambda$. We denote the root system and the coroot system of (G,T) by Φ and $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$, respectively. The datum of B^+ splits Φ into the subset Φ_+ of positive roots and the subset Φ_- of negative roots. We set $\Phi_+^\vee = \{\alpha^\vee \mid \alpha \in \Phi_+\}$. We denote by $X_{++} = \{\eta \in X \mid \forall \alpha^\vee \in \Phi_+^\vee,\ \langle \eta, \alpha^\vee\rangle \geqslant 0\}$ and $\Lambda_{++} = \{\lambda \in \Lambda \mid \forall \alpha \in \Phi_+,\ \langle \alpha, \lambda\rangle \geqslant 0\}$ the cones of dominant weights and coweights. We index the simple roots as $(\alpha_i)_{i\in I}$. The coroot lattice is the subgroup $\mathbb{Z}\Phi^\vee$ generated by the coroots in Λ; the height of an element $\lambda = \sum_{i\in I} n_i\alpha_i^\vee$ in $\mathbb{Z}\Phi^\vee$ is defined as $\mathrm{ht}(\lambda) = \sum_{i\in I} n_i$. The dominance order on X is the partial order $\leqslant$ defined by

$$\eta \geqslant \theta \iff \eta - \theta \in \mathbb{N}\Phi_+.$$

The dominance order on Λ is the partial order $\leqslant$ defined by

$$\lambda \geqslant \mu \iff \lambda - \mu \in \mathbb{N}\Phi_+^\vee.$$

For each simple root α_i, we choose a non-trivial additive subgroup x_i of U^+ such that $a^\lambda x_i(b) a^{-\lambda} = x_i\big(a^{\langle \alpha_i, \lambda\rangle} b\big)$ holds for all $\lambda \in \Lambda$, $a \in \mathbb{C}^\times$, $b \in \mathbb{C}$. Then there is a unique morphism $\varphi_i : \mathbf{SL}_2 \to G$ such that

$$\varphi_i \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = x_i(b) \quad \text{and} \quad \varphi_i \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a^{\alpha_i^\vee}$$

for all $a \in \mathbb{C}^\times$, $b \in \mathbb{C}$. We set

$$y_i(b) = \varphi_i \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \quad \text{and} \quad \overline{s_i} = \varphi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $N_G(T)$ be the normalizer of T in G and let $W = N_G(T)/T$ be the Weyl group of (G,T). Each element $\overline{s_i}$ normalizes T; its class s_i modulo T is called a simple reflection. Endowed with the set of simple reflections, the Weyl group becomes a Coxeter system. Since the elements $\overline{s_i}$ satisfy the braid relations, we may lift each element $w \in W$ to an element $\overline{w} \in G$ so that $\overline{w} = \overline{s_{i_1}} \cdots \overline{s_{i_l}}$ for any reduced decomposition $s_{i_1} \cdots s_{i_l}$ of w. For any two elements w and w' in W, there exists an element $\lambda \in \mathbb{Z}\Phi^\vee$ such that $\overline{ww'} = (-1)^\lambda\, \overline{w}\, \overline{w'}$. We denote the longest element of W by w_0.

Let α be a positive root. We make the choice of a simple root α_i and of an element $w \in W$ such that $\alpha = w\alpha_i$. Then we define the one-parameter additive subgroups

$$x_\alpha : b \mapsto \overline{w}\, x_i(b)\, \overline{w}^{-1} \quad \text{and} \quad x_{-\alpha} : b \mapsto \overline{w}\, y_i(b)\, \overline{w}^{-1} \tag{2.1}$$

and the element $\overline{s_\alpha} = \overline{w}\, \overline{s_i}\, \overline{w}^{-1}$.

Products in G may then be computed using several commutation rules:

- For any $\lambda \in \Lambda$, any root α, any $a \in \mathbb{C}^\times$ and any $b \in \mathbb{C}$,

$$a^\lambda x_\alpha(b) = x_\alpha\big(a^{\langle\alpha,\lambda\rangle} b\big) a^\lambda. \tag{2.2}$$

- For any root α and any $a, b \in \mathbb{C}$ such that $1 + ab \neq 0$,

$$x_\alpha(a) x_{-\alpha}(b) = x_{-\alpha}\big(b/(1+ab)\big)\big(1+ab\big)^{\alpha^\vee} x_\alpha\big(a/(1+ab)\big). \tag{2.3}$$

- For any positive root α and any $a \in \mathbb{C}^\times$,

$$\begin{aligned} x_\alpha(a)\, x_{-\alpha}(-a^{-1})\, x_\alpha(a) &= x_{-\alpha}(-a^{-1})\, x_\alpha(a)\, x_{-\alpha}(-a^{-1}) \\ &= a^{\alpha^\vee}\, \overline{s_\alpha} \\ &= \overline{s_\alpha}\, a^{-\alpha^\vee}. \end{aligned} \tag{2.4}$$

- (Chevalley's commutator formula) If α and β are two linearly independent roots, then there are numbers $C_{i,j,\alpha,\beta} \in \{\pm 1, \pm 2, \pm 3\}$ such that

$$x_\beta(b)^{-1} x_\alpha(a)^{-1} x_\beta(b)\, x_\alpha(a) = \prod_{i,j>0} x_{i\alpha+j\beta}\big(C_{i,j,\alpha,\beta}(-a)^i b^j\big) \tag{2.5}$$

  for all a and b in $\mathbb{C}$. The product on the right-hand side is taken over all pairs of positive integers i, j for which $i\alpha + j\beta$ is a root, in order of increasing $i + j$.

2.2. **Crystals.** Let $G^\vee$ be the Langlands dual of G. This connected reductive group is equipped with a Borel subgroup $B^{+,\vee}$ and a maximal torus $T^\vee \subseteq B^{+,\vee}$ so that Λ is the weight lattice of $T^\vee$ and $\Phi^\vee$ is the root system of $(G^\vee, T^\vee)$, the set of positive roots being $\Phi_+^\vee$. The Lie algebra $\mathfrak{g}^\vee$ of $G^\vee$ has a triangular decomposition $\mathfrak{g}^\vee = \mathfrak{n}^{-,\vee} \oplus \mathfrak{h}^\vee \oplus \mathfrak{n}^{+,\vee}$.

A crystal for $G^\vee$ (in the sense of Kashiwara [18]) is a set $\mathbf{B}$ endowed with maps

$$\tilde{e}_i, \tilde{f}_i : \mathbf{B} \to \mathbf{B} \sqcup \{0\}, \quad \varepsilon_i, \varphi_i : \mathbf{B} \to \mathbb{Z} \sqcup \{-\infty\}, \quad \text{and} \quad \operatorname{wt} : \mathbf{B} \to \Lambda,$$

where 0 is a ghost element added to $\mathbf{B}$ in order that $\tilde{e}_i$ and $\tilde{f}_i$ may be everywhere defined. These maps are required to satisfy certain axioms, which the reader may find in Section 7.2 of [18]. The map wt is called the weight.

A morphism from a crystal $\mathbf{B}$ to a crystal $\mathbf{B}'$ is a map $\psi : \mathbf{B} \sqcup \{0\} \to \mathbf{B}' \sqcup \{0\}$ satisfying $\psi(0) = 0$ and compatible with the structure maps $\tilde{e}_i$, $\tilde{f}_i$, ε_i, φ_i and wt. The conditions are written in full detail in [18].

Given a crystal $\mathbf{B}$, one defines a crystal $\mathbf{B}^\vee$ whose elements are written $b^\vee$, where $b \in \mathbf{B}$, and whose structure maps are given by

$$\begin{aligned} \varepsilon_i(b^\vee) &= \varphi_i(b), & \tilde{e}_i(b^\vee) &= (\tilde{f}_i b)^\vee, \\ \varphi_i(b^\vee) &= \varepsilon_i(b), & \tilde{f}_i(b^\vee) &= (\tilde{e}_i b)^\vee, \\ \operatorname{wt}(b^\vee) &= -\operatorname{wt}(b), \end{aligned}$$

where one sets $0^\vee = 0$. The correspondence $\mathbf{B} \rightsquigarrow \mathbf{B}^\vee$ is a covariant functor. (Caution: Usually in this paper, the symbol $\vee$ is used to adorn coroots or objects related to the Langlands dual. Here and in Section 4.4, however, it will also be used to denote contragredient duality for crystals.)

The most important crystals for our work are the crystal $\mathbf{B}(\infty)$ of the canonical basis of $U(\mathfrak{n}^{-,\vee})$ and the crystal $\mathbf{B}(-\infty)$ of the canonical basis of $U(\mathfrak{n}^{+,\vee})$. The crystal $\mathbf{B}(\infty)$ is a highest weight crystal; this means that it has an element annihilated by all operators $\tilde{e}_i$ and from which any other element of $\mathbf{B}(\infty)$ can be obtained by applying the operators $\tilde{f}_i$. This element is unique and its weight is 0; we denote it by 1. Likewise, the crystal $\mathbf{B}(-\infty)$ is a lowest weight crystal; its lowest weight element has weight 0 and is also denoted by 1.

The antiautomorphism of the algebra $U(\mathfrak{n}^{-,\vee})$ that fixes the Chevalley generators leaves its canonical basis stable; it therefore induces an involution $b \mapsto b^*$ of the set $\mathbf{B}(\infty)$. This involution $*$ preserves the weight. The operators $\tilde{f}_i$ and $b \mapsto (\tilde{f}_i b^*)^*$ correspond roughly to the left and right multiplication in $U(\mathfrak{n}^{-,\vee})$ by the Chevalley generator with index i (see Proposition 5.3.1 in [16] for a more precise statement). One can therefore expect that $\tilde{f}_i$ and $b \mapsto (\tilde{f}_j b^*)^*$ commute for all $i, j \in I$. This actually holds if and only if $i \neq j$; and when $i = j$, one can analyze precisely the mutual behavior of these operators. In return, one obtains a characterization of $\mathbf{B}(\infty)$ as the unique highest weight crystal generated by a highest weight element of weight 0 and endowed with an involution $*$ with specific properties (see Section 2 in [17], Proposition 3.2.3 in [19], and Section 12 in [8] for more details).

For any weight $\lambda \in \Lambda$, we consider the crystal $\mathbf{T}_\lambda$ with unique element t_λ, whose structure maps are given by

$$\mathrm{wt}(t_\lambda) = \lambda, \quad \tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0 \quad \text{and} \quad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$$

(see Example 7.3 in [18]). There are two operators $\oplus$ and $\otimes$ on crystals (see Section 7.3 in [18]). We set $\widetilde{\mathbf{B}(-\infty)} = \bigoplus_{\lambda \in \Lambda} \mathbf{T}_\lambda \otimes \mathbf{B}(-\infty)$. Thus for any $\lambda \in \Lambda$, any $b \in \mathbf{B}(-\infty)$, and any $i \in I$,

$$\begin{aligned} &\varepsilon_i(t_\lambda \otimes b) = \varepsilon_i(b) - \langle \alpha_i, \lambda \rangle, && \tilde{e}_i(t_\lambda \otimes b) = t_\lambda \otimes \tilde{e}_i(b), \\ &\varphi_i(t_\lambda \otimes b) = \varphi_i(b), && \tilde{f}_i(t_\lambda \otimes b) = t_\lambda \otimes \tilde{f}_i(b), \\ &\mathrm{wt}(t_\lambda \otimes b) = \mathrm{wt}(b) + \lambda. \end{aligned}$$

We transport the involution $*$ from $\mathbf{B}(\infty)$ to $\mathbf{B}(-\infty)$ by using the isomorphism $\mathbf{B}(-\infty) \cong \mathbf{B}(\infty)^\vee$ and by setting $(b^\vee)^* = (b^*)^\vee$ for each $b \in \mathbf{B}(\infty)$. Then we extend it to $\widetilde{\mathbf{B}(-\infty)}$ by setting

$$(t_\lambda \otimes b)^* = t_{-\lambda - \mathrm{wt}(b)} \otimes b^*.$$

For $\lambda \in \Lambda$, we denote by $L(\lambda)$ the irreducible rational representation of $G^\vee$ whose highest weight is the unique dominant weight in the orbit $W\lambda$. We denote the crystal of the canonical basis of $L(\lambda)$ by $\mathbf{B}(\lambda)$. It has a unique highest weight element b_{high} and a unique lowest weight element b_{low}, which satisfy $\tilde{e}_i b_{\mathrm{high}} = \tilde{f}_i b_{\mathrm{low}} = 0$ for any $i \in I$. If λ is dominant, there is a unique embedding of crystals $\kappa_\lambda : \mathbf{B}(\lambda) \hookrightarrow \mathbf{B}(\infty) \otimes \mathbf{T}_\lambda$; it maps the element b_{high} to $1 \otimes t_\lambda$ and its image is

$$\{b \otimes t_\lambda \mid b \in \mathbf{B}(\infty) \text{ such that } \forall i \in I,\ \varepsilon_i(b^*) \leqslant \langle \alpha_i, \lambda \rangle\}$$

(see Proposition 8.2 in [18]). If λ is antidominant, then the sequence

$$\mathbf{B}(\lambda) \cong \mathbf{B}(-\lambda)^\vee \xrightarrow{(\kappa_{-\lambda})^\vee} \big(\mathbf{B}(\infty) \otimes \mathbf{T}_{-\lambda}\big)^\vee \cong \mathbf{T}_\lambda \otimes \mathbf{B}(-\infty)$$

defines an embedding of crystals $\iota_\lambda : \mathbf{B}(\lambda) \hookrightarrow \mathbf{T}_\lambda \otimes \mathbf{B}(-\infty)$; it maps the element b_{low} to $t_\lambda \otimes 1$ and its image is

$$\{t_\lambda \otimes b \mid b \in \mathbf{B}(-\infty) \text{ such that } \forall i \in I,\ \varphi_i(b^*) \leqslant -\langle \alpha_i, \lambda \rangle\}.$$

3. The affine Grassmannian

In Section 3.1, we recall the definition of an affine Grassmannian. In Section 3.2, we present several properties of orbits in the affine Grassmannian of G under the action of the groups $G\big(\mathbb{C}[[t]]\big)$ and $U^{\pm}\big(\mathbb{C}((t))\big)$. Section 3.3 recalls the notion of MV cycle, in the original version of Mirković and Vilonen and in the somewhat generalized version of Anderson. Finally, Section 3.4 introduces a map from the affine Grassmannian of G to the affine Grassmannian of a Levi subgroup of G.

An easy but possibly new result in this section is Proposition 3.6 (iii). Joint with Mirković and Vilonen's work, it implies the expected Proposition 3.9, which provides the dimension estimates that Anderson needs for his generalization of MV cycles.

3.1. **Definitions.** We denote the ring of formal power series by $\mathscr{O} = \mathbb{C}[[t]]$ and we denote its field of fractions by $\mathscr{K} = \mathbb{C}((t))$. We denote the valuation of a non-zero Laurent series $f \in \mathscr{K}^\times$ by $\operatorname{val}(f)$. Given a complex linear algebraic group H, we define the affine Grassmannian of H as the space $\mathscr{H} = H(\mathscr{K})/H(\mathscr{O})$. The class in $\mathscr{H}$ of an element $h \in H(\mathscr{K})$ will be denoted by $[h]$.

Example 3.1. If H is the multiplicative group $\mathbf{G}_m$, then the valuation map yields a bijection from $\mathscr{H} = \mathscr{K}^\times/\mathscr{O}^\times$ onto $\mathbb{Z}$. More generally, if H is a torus, then the map $\lambda \mapsto [t^\lambda]$ is a bijection from the lattice $X_*(H)$ of one-parameter subgroups in H onto the affine Grassmannian $\mathscr{H}$.

The affine Grassmannian $\mathscr{H}$ is the set of $\mathbb{C}$-points of an ind-scheme defined over $\mathbb{C}$ (see [2] for $H = \mathbf{GL}_n$ or $\mathbf{SL}_n$ and Chapter 13 of [20] for H simple). This means, in particular, that $\mathscr{H}$ is the direct limit of a system

$$\mathscr{H}_0 \hookrightarrow \mathscr{H}_1 \hookrightarrow \mathscr{H}_2 \hookrightarrow \cdots$$

of complex algebraic varieties and of closed embeddings. We endow $\mathscr{H}$ with the direct limit of the Zariski topologies on the varieties $\mathscr{H}_n$. A noetherian subspace Z of $\mathscr{H}$ thus enjoys the specific topological properties of a subset of a complex algebraic variety; for instance, if Z is locally closed, then $\dim Z = \dim \overline{Z}$.

The affine Grassmannian of the groups G and T considered in Section 2.1 will be denoted by $\mathscr{G}$ and $\mathscr{T}$, respectively. The inclusion $T \subseteq G$ gives rise to a closed embedding $\mathscr{T} \hookrightarrow \mathscr{G}$.

3.2. **Orbits.** We first look at the action of the group $G(\mathscr{O})$ on $\mathscr{G}$ by left multiplication. The orbit $G(\mathscr{O})[t^\lambda]$ depends only on the W-orbit of λ in Λ, and the Cartan decomposition of $G(\mathscr{K})$ says that

$$\mathscr{G} = \bigsqcup_{W\lambda \in \Lambda/W} G(\mathscr{O})[t^\lambda].$$

For each coweight $\lambda \in \Lambda$, the orbit $\mathscr{G}_\lambda = G(\mathscr{O})[t^\lambda]$ is a noetherian subspace of $\mathscr{G}$. If λ is dominant, then the dimension of $\mathscr{G}_\lambda$ is $\operatorname{ht}(\lambda - w_0\lambda)$ and its closure is

$$\overline{\mathscr{G}_\lambda} = \bigsqcup_{\substack{\mu \in \Lambda_{++} \\ \lambda \geqslant \mu}} \mathscr{G}_\mu. \tag{3.1}$$

From this, one can quickly deduce that it is often possible to truncate power series when dealing with the action of $G(\mathscr{O})$ on $\mathscr{G}$. Given a positive integer s, let $G_{(s)}$ denote the s-th congruence subgroup of $G(\mathscr{O})$, that is, the kernel of the reduction map $G(\mathscr{O}) \to G(\mathscr{O}/t^s\mathscr{O})$.

Proposition 3.2. *For each noetherian subset Z of $\mathscr{G}$, there exists a level s such that $G_{(s)}$ fixes Z pointwise.*

Proof. Let $\big(\Lambda_{++}^{(n)}\big)_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of Λ_{++} such that

$$\big\{\nu \in \Lambda_{++} \bigm| \nu \leqslant \mu\big\} \subseteq \Lambda_{++}^{(n)} \quad \text{for each } \mu \in \Lambda_{++}^{(n)} \quad \text{and that} \quad \bigcup_{n\in\mathbb{N}} \Lambda_{++}^{(n)} = \Lambda_{++}.$$

Set $\mathscr{G}_n = \bigsqcup_{\mu\in\Lambda_{++}^{(n)}} \mathscr{G}_\mu$. The Cartan decomposition shows that $(\mathscr{G}_n)_{n\geqslant 0}$ is an increasing and exhaustive filtration of $\mathscr{G}$, and Equation (3.1) shows that each $\mathscr{G}_n$ is closed. Therefore each noetherian subset Z of $\mathscr{G}$ is contained in $\mathscr{G}_n$ for n sufficiently large. To prove the proposition, it is thus enough to show that for each integer n, there is an $s \geqslant 1$ such that $G_{(s)}$ fixes $\mathscr{G}_n$ pointwise.

Let $\lambda \in \Lambda$, and choose $s \geqslant 1$ larger than $\langle \alpha, \lambda\rangle$ for all $\alpha \in \Phi$. Using that $G_{(s)}$ is generated by elements $(1 + t^s p)^\lambda$ and $x_\alpha(t^s p)$ with $\lambda \in \Lambda$, $\alpha \in \Phi$ and $p \in \mathscr{O}$, one readily checks that $G_{(s)}$ fixes the point $[t^\lambda]$. Since $G_{(s)}$ is normal in $G(\mathscr{O})$, it pointwise fixes the orbit $\mathscr{G}_\lambda$. The proposition then follows from the fact that each $\mathscr{G}_n$ is a finite union of $G(\mathscr{O})$-orbits. □

We now look at the action of the unipotent group $U^\pm(\mathscr{K})$ on $\mathscr{G}$. It can be described by the Iwasawa decomposition

$$\mathscr{G} = \bigsqcup_{\lambda\in\Lambda} U^\pm(\mathscr{K})[t^\lambda].$$

We will denote the orbit $U^\pm(\mathscr{K})[t^\lambda]$ by $S_\lambda^\pm$. Proposition 3.1 (a) in [27] asserts that the closure of a stratum $S_\lambda^\pm$ is the union

$$\overline{S_\lambda^\pm} = \bigsqcup_{\substack{\mu\in\Lambda\\ \pm(\lambda-\mu)\geqslant 0}} S_\mu^\pm. \tag{3.2}$$

This equation implies, in particular,

$$S_\lambda^\pm = \overline{S_\lambda^\pm} \setminus \left(\bigcup_{i\in I} \overline{S_{\lambda\mp\alpha_i^\vee}^\pm}\right),$$

which shows that each stratum $S_\lambda^\pm$ is locally closed.

As pointed out by Mirković and Vilonen (Equation (3.5) in [27]), these strata $S_\lambda^\pm$ can be understood in terms of a Białynicki-Birula decomposition: indeed, the choice of a dominant and regular coweight $\xi \in \Lambda$ defines an action of $\mathbb{C}^\times$ on $\mathscr{G}$, and

$$S_\lambda^\pm = \{x \in \mathscr{G} \mid \lim_{\substack{a\to 0\\ a\in\mathbb{C}^\times}} a^{\pm\xi} \cdot x = [t^\lambda]\}$$

for each $\lambda \in \Lambda$. We will generalize this result in Remark 3.11. For now, we record the following two (known and obvious) consequences:

- The set of points in $\mathscr{G}$ fixed by the action of T is $\mathscr{G}^T = \big\{[t^\lambda] \bigm| \lambda \in \Lambda\big\}$; in other words, $\mathscr{G}^T$ is the image of the embedding $\mathscr{T} \hookrightarrow \mathscr{G}$.
- If Z is a closed and T-invariant subset of $\mathscr{G}$, then Z meets a stratum $S_\lambda^\pm$ if and only if $[t^\lambda] \in Z$.

The following proposition is in essence due to Kamnitzer (see Section 3.3 in [13]).

Proposition 3.3. *Let Z be an irreducible and noetherian subset of $\mathscr{G}$.*

(i) *The set $\{\lambda \in \Lambda \mid Z \cap S_\lambda^+ \neq \varnothing\}$ is finite and has a largest element. Denoting the latter by μ_+, the intersection $Z \cap S_{\mu_+}^+$ is open and dense in Z.*

(ii) *The set $\{\lambda \in \Lambda \mid Z \cap S_\lambda^- \neq \varnothing\}$ is finite and has a smallest element. Denoting the latter by μ_-, the intersection $Z \cap S_{\mu_-}^-$ is open and dense in Z.*

Given an irreducible and noetherian subset Z in $\mathscr{G}$, we indicate the coweights $\mu_\pm$ exhibited in Proposition 3.3 by the notation $\mu_\pm(Z)$.

Proof of Proposition 3.3. The Cartan decomposition and the equality $\mathscr{G}^T = \{[t^\lambda] \mid \lambda \in \Lambda\}$ imply that the obvious inclusion $(\mathscr{G}_\nu)^T \supseteq \{[t^{w\nu}] \mid w \in W\}$ is indeed an equality for each coweight $\nu \in \Lambda$. Therefore X^T is finite for each subset $X \subseteq \mathscr{G}$ that is a finite union of $G(\mathscr{O})$-orbits. This is, in particular, the case for each of the subsets $\mathscr{G}_n$ used in the proof of Proposition 3.2. Since $\mathscr{G}_n$ is, moreover, closed and T-invariant, this means that it meets only finitely many strata S_λ^+. Thus a noetherian subset of $\mathscr{G}$ meets only finitely many strata S_λ^+, for it is contained in $\mathscr{G}_n$ for n large enough.

Assume now that Z is an irreducible and noetherian subset of $\mathscr{G}$. Each intersection $Z \cap S_\lambda^+$ is locally closed in Z and Z is covered by finitely many such intersections, so there exists a coweight μ_+ for which the intersection $Z \cap S_{\mu_+}^+$ is dense in Z. Then $Z \subseteq \overline{S_{\mu_+}^+}$; by Equation (3.2), this means that μ_+ is the largest element in $\{\lambda \in \Lambda \mid Z \cap S_\lambda^+ \neq \varnothing\}$. Moreover, $Z \cap S_{\mu_+}^+$ is locally closed in Z; it is therefore open in its closure in Z, which is Z.

The arguments above prove Assertion (i). The proof of Assertion (ii) is entirely similar. $\square$

Examples 3.4. (i) If Z is an irreducible and noetherian subset of $\mathscr{G}$, then $Z \cap S_{\mu_+(Z)}^+ \cap S_{\mu_-(Z)}^-$ is dense in Z. Thus Z and $\overline{Z}$ are contained in $\overline{S_{\mu_+(Z)}^+ \cap S_{\mu_-(Z)}^-}$. One deduces from this the equality $\mu_\pm(\overline{Z}) = \mu_\pm(Z)$.

(ii) For any coweight $\lambda \in \Lambda$, $\mu_+(\mathscr{G}_\lambda) = \mu_+\big(\overline{\mathscr{G}_\lambda}\big)$ and $\mu_-(\mathscr{G}_\lambda) = \mu_-\big(\overline{\mathscr{G}_\lambda}\big)$ are the largest and the smallest element in the orbit $W\lambda$, respectively.

We now present a method that allows us to find the parameter λ of an orbit $\mathscr{G}_\lambda$ or $S_\lambda^\pm$ to which a given point of $\mathscr{G}$ belongs. Given a $\mathbb{C}$-vector space V, we may form the $\mathscr{K}$-vector space $V \otimes_\mathbb{C} \mathscr{K}$ by extending the base field and regard V as a subspace of it. In this situation, we define the valuation $\operatorname{val}(v)$ of a non-zero vector $v \in V \otimes_\mathbb{C} \mathscr{K}$ as the largest $n \in \mathbb{Z}$ such that $v \in V \otimes t^n\mathscr{O}$; thus the valuation of a non-zero element $v \in V$ is zero. We define the valuation $\operatorname{val}(f)$ of a non-zero endomorphism $f \in \operatorname{End}_\mathscr{K}(V \otimes_\mathbb{C} \mathscr{K})$ as the largest $n \in \mathbb{Z}$ such that $f(V \otimes_\mathbb{C} \mathscr{O}) \subseteq V \otimes t^n\mathscr{O}$; equivalently, $\operatorname{val}(f)$ is the valuation of f viewed as an element in $\operatorname{End}_\mathbb{C}(V) \otimes_\mathbb{C} \mathscr{K}$.

For each weight $\eta \in X$, we denote by $V(\eta)$ the simple rational representation of G whose highest weight is the dominant weight in the orbit $W\eta$, and we choose an extremal weight vector $v_\eta \in V(\eta)$ of weight η. The structure map $g \mapsto g_{V(\eta)}$ from G to $\operatorname{End}_\mathbb{C}(V(\eta))$ of this representation extends to a map from $G(\mathscr{K})$ to $\operatorname{End}_\mathscr{K}(V(\eta) \otimes_\mathbb{C} \mathscr{K})$; we denote the latter also by $g \mapsto g_{V(\eta)}$, or simply by $g \mapsto g\cdot?$ if there is no risk of confusion.

Proposition 3.5. *Let $g \in \mathscr{G}(\mathscr{K})$.*

(i) *The antidominant coweight $\lambda \in \Lambda$ such that $[g] \in \mathscr{G}_\lambda$ is characterized by the equations*

$$\forall \eta \in X_{++}, \quad \langle \eta, \lambda \rangle = \operatorname{val}\bigl(g_{V(\eta)}\bigr).$$

(ii) *The coweight $\lambda \in \Lambda$ such that $[g] \in S_\lambda^\pm$ is characterized by the equations*

$$\forall \eta \in X_{++}, \quad \pm\langle \eta, \lambda \rangle = -\operatorname{val}(g^{-1} \cdot v_{\pm\eta}).$$

Proof. Assertion (ii) is due to Kamnitzer (this is Lemma 2.4 in [13]), so we only have to prove Assertion (i). Let $\lambda \in \Lambda$ be antidominant and let $\eta \in X_{++}$. Then for each weight θ of $V(\eta)$, the element t^λ acts by $t^{\langle \lambda, \theta \rangle}$ on the θ-weight subspace of $V(\eta)$, with here $\langle \lambda, \theta \rangle \geqslant \langle \lambda, \eta \rangle$ since $\theta \leqslant \eta$. It follows that $\operatorname{val}\bigl((t^\lambda)_{V(\eta)}\bigr) = \langle \lambda, \eta \rangle$. Thus the proposed formula holds for $g = t^\lambda$. To conclude the proof, it suffices to observe that $\operatorname{val}\bigl(g_{V(\eta)}\bigr)$ depends only of the double coset $G(\mathscr{O})gG(\mathscr{O})$, for the action of $G(\mathscr{O})$ leaves $V(\eta) \otimes_{\mathbb{C}} \mathscr{O}$ invariant. □

We end this section with a proposition that provides some information concerning intersections of orbits. We agree to say that an assertion $A(\lambda)$ depending on a coweight $\lambda \in \Lambda$ holds when λ is antidominant enough if

$$(\exists N \in \mathbb{Z}) \quad (\forall \lambda \in \Lambda) \quad (\forall i \in I,\ \langle \alpha_i, \lambda \rangle \leqslant N) \Longrightarrow A(\lambda).$$

Proposition 3.6. (i) *Let $\lambda, \nu \in \Lambda$. If $S_\lambda^+ \cap S_\nu^- \neq \varnothing$, then $\lambda \geqslant \nu$.*

(ii) *Let $\lambda \in \Lambda$. Then $S_\lambda^+ \cap S_\lambda^- = \bigl\{[t^\lambda]\bigr\}$.*

(iii) *Let $\nu \in \Lambda$ such that $\nu \geqslant 0$. If $\lambda \in \Lambda$ is antidominant enough, then $S_{\lambda+\nu}^+ \cap S_\lambda^- = S_{\lambda+\nu}^+ \cap \mathscr{G}_\lambda$.*

The proof of this proposition requires a lemma.

Lemma 3.7. *Let $\nu \in \Lambda$ such that $\nu \geqslant 0$. If $\lambda \in \Lambda$ is antidominant enough, then $S_{\lambda+\nu}^+ \cap S_\lambda^- \subseteq \mathscr{G}_\lambda$.*

Proof. For the whole proof, we fix $\nu \in \Lambda$ such that $\nu \geqslant 0$.

For each $\eta \in X_{++}$, we make the following construction. We form the list $(\theta_1, \theta_2, \ldots, \theta_N)$ of all the weights of $V(\eta)$, repeated according to their multiplicities and ordered in such a way that $(\theta_i > \theta_j \Longrightarrow i < j)$ for all indices i, j. Thus $N = \dim V(\eta)$, $\theta_1 = \eta > \theta_i$ for all $i > 1$, and $\theta_1 + \theta_2 + \cdots + \theta_N$ is W-invariant hence orthogonal to $\mathbb{Z}\Phi^\vee$. We say then that a coweight $\lambda \in \Lambda$ satisfies Condition $A_\eta(\lambda)$ if

$$\forall j \in \{1, \ldots, N\}, \quad \langle \theta_1 - \theta_j, \lambda \rangle \leqslant \langle \theta_j + \theta_{j+1} + \cdots + \theta_N, \nu \rangle.$$

Certainly Condition $A_\eta(\lambda)$ holds if λ is antidominant enough.

Now we choose a finite subset $Y \subseteq X_{++}$ that spans the lattice X up to torsion. To prove the lemma, it is enough to show that $S_{\lambda+\nu}^+ \cap S_\lambda^- \subseteq \mathscr{G}_\lambda$ for all antidominant λ satisfying Condition $A_\eta(\lambda)$ for each $\eta \in Y$.

Suppose that λ satisfies these requirements and let $g \in U^-(\mathscr{K})t^\lambda$ be such that $[g] \in S_{\lambda+\nu}^+$. We use Proposition 3.5 (i) to show that $[g] \in \mathscr{G}_\lambda$. Let $\eta \in Y$. Let $(v_1, v_2, \ldots, v_N)$ be a basis of $V(\eta)$ such that for each i, v_i is a vector of weight θ_i. We denote the dual basis in $V(\eta)^*$ by $(v_1^*, v_2^*, \ldots, v_N^*)$; thus v_i^* is of weight $-\theta_i$. Then

$$\operatorname{val}\bigl(g_{V(\eta)}\bigr) = \min\bigl\{\operatorname{val}(\langle v_j^*, g \cdot v_i \rangle) \bigm| 1 \leqslant i, j \leqslant N\bigr\}.$$

The choice $g \in U^-(\mathscr{K})t^\lambda$ implies that the matrix of $g_{V(\eta)}$ in the basis $(v_i)_{1\leqslant i\leqslant N}$ is lower triangular, with diagonal entries $\bigl(t^{\langle\theta_i,\lambda\rangle}\bigr)_{1\leqslant i\leqslant N}$. Let $i \leqslant j$ be two indices. Then

$$g\cdot(v_i\wedge v_{j+1}\wedge v_{j+2}\wedge\cdots\wedge v_N) = t^{\langle\theta_{j+1}+\theta_{j+2}+\cdots+\theta_N,\lambda\rangle}(g\cdot v_i)\wedge v_{j+1}\wedge v_{j+2}\wedge\cdots\wedge v_N.$$

Therefore

$$\begin{aligned}
&\mathrm{val}(\langle v_j^*, g\cdot v_i\rangle) + \langle\theta_{j+1}+\theta_{j+2}+\cdots+\theta_N,\lambda\rangle\\
&\qquad = \mathrm{val}(\langle v_j^*\wedge v_{j+1}^*\wedge v_{j+2}^*\wedge\cdots v_N^*, g\cdot(v_i\wedge v_{j+1}\wedge v_{j+2}\wedge\cdots\wedge v_N)\rangle)\\
&\qquad = \mathrm{val}(\langle g^{-1}\cdot(v_j^*\wedge v_{j+1}^*\wedge v_{j+2}^*\wedge\cdots v_N^*), v_i\wedge v_{j+1}\wedge v_{j+2}\wedge\cdots\wedge v_N\rangle)\\
&\qquad \geqslant \mathrm{val}(g^{-1}\cdot(v_j^*\wedge v_{j+1}^*\wedge\cdots\wedge v_N^*))\\
&\qquad = \langle\theta_j+\theta_{j+1}+\cdots+\theta_N,\lambda+\nu\rangle;
\end{aligned}$$

the last equality here comes from Proposition 3.5 (ii), taking into account that $[g] \in S_{\lambda+\nu}^+$ and that $v_j^*\wedge v_{j+1}^*\wedge\cdots\wedge v_N^*$ is a highest weight vector of weight $-(\theta_j+\theta_{j+1}+\cdots+\theta_N)$ in $\bigwedge^{N-j+1}V(\eta)^*$. By Condition $A_\eta(\lambda)$, this implies

$$\mathrm{val}(\langle v_j^*, g\cdot v_i\rangle) \geqslant \langle\theta_j,\lambda\rangle + \langle\theta_j+\theta_{j+1}+\cdots+\theta_N,\nu\rangle \geqslant \langle\eta,\lambda\rangle.$$

Therefore $\mathrm{val}\bigl(g_{V(\eta)}\bigr) \geqslant \langle\eta,\lambda\rangle$. On the other hand, $\mathrm{val}\bigl(g_{V(\eta)}\bigr) \leqslant \mathrm{val}(\langle v_1^*, g\cdot v_1\rangle) = \langle\eta,\lambda\rangle$. Thus the equality $\mathrm{val}(g_{V(\eta)}) = \langle\eta,\lambda\rangle$ holds for each $\eta\in Y$, and we conclude by Proposition 3.5 (i) that $[g]\in\mathscr{G}_\lambda$. □

Proof of Proposition 3.6. We first prove Assertion (i). We let $\mathbb{C}^\times$ act on $\mathscr{G}$ through a dominant and regular coweight $\xi\in\Lambda$. Let $\lambda,\nu\in\Lambda$ and assume there exists an element $x\in S_\lambda^+\cap S_\nu^-$. Then

$$[t^\nu] = \lim_{a\to 0} a^{-\xi}\cdot x \quad \text{belongs to} \quad \overline{S_\lambda^+} = \bigcup_{\substack{\mu\in\Lambda\\ \lambda\geqslant\mu}} S_\mu^+.$$

This shows that $\lambda\geqslant\nu$.

If $\mu\in\Lambda$ is antidominant enough, then

$$S_\mu^+\cap S_\mu^- \subseteq S_\mu^+\cap\mathscr{G}_\mu = \bigl\{[t^\mu]\bigr\}$$

by Lemma 3.7 and Formula (3.6) in [27]. Thus $S_\mu^+\cap S_\mu^- = \bigl\{[t^\mu]\bigr\}$ if μ is antidominant enough. It follows that for each $\lambda\in\Lambda$,

$$S_\lambda^+\cap S_\lambda^- = t^{\lambda-\mu}\cdot\bigl(S_\mu^+\cap S_\mu^-\bigr) = t^{\lambda-\mu}\cdot\bigl\{[t^\mu]\bigr\} = \bigl\{[t^\lambda]\bigr\}.$$

Assertion (ii) is proved.

Now let $\nu\in\Lambda$ such that $\nu\geqslant 0$. By Lemma 3.7, the property

$$\forall\sigma,\tau\in\Lambda,\quad (0\leqslant\tau\leqslant\nu \text{ and } \lambda\leqslant\sigma\leqslant\lambda+\nu) \Longrightarrow (S_{\sigma+\tau}^+\cap S_\sigma^-\subseteq\mathscr{G}_\sigma) \tag{3.3}$$

holds if λ is antidominant enough. We assume that this is the case and, moreover, that

$$W\lambda\cap\{\sigma\in\Lambda\mid\sigma\leqslant\lambda+\nu\} = \{\lambda\}.$$

We now show the equality $S_{\lambda+\nu}^+\cap S_\lambda^- = S_{\lambda+\nu}^+\cap\mathscr{G}_\lambda$. Let us take $x\in S_{\lambda+\nu}^+\cap\mathscr{G}_\lambda$. Calling σ the coweight such that $x\in S_\sigma^-$, we necessarily have $\lambda\leqslant\sigma\leqslant\lambda+\nu$ (using Example 3.4 (ii) for the first inequality). Setting $\tau=\lambda+\nu-\sigma$, we have $0\leqslant\tau\leqslant\nu$ and $x\in S_{\sigma+\tau}^+\cap S_\sigma^-$, whence $x\in\mathscr{G}_\sigma$ by our assumption (3.3). This entails $\sigma\in W\lambda$, then $\sigma=\lambda$, and thus $x\in S_\lambda^-$. This reasoning shows $S_{\lambda+\nu}^+\cap\mathscr{G}_\lambda\subseteq S_{\lambda+\nu}^+\cap S_\lambda^-$.

The converse inclusion also holds (set $\tau = \nu$ and $\sigma = \lambda$ in (3.3)). Assertion (iii) is proved. □

Remark 3.8. Assertion (ii) of Proposition 3.6 can also be proved in the following way. Let K be the maximal compact subgroup of the torus T. The Lie algebra of K is $\mathfrak{k} = i(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$. The affine Grassmannian $\mathscr{G}$ is a Kähler manifold and the action of K on $\mathscr{G}$ is hamiltonian. Let $\mu : \mathscr{G} \to \mathfrak{k}^*$ be the moment map. Fix a dominant and regular coweight $\xi \in \Lambda$. Then $\mathbb{R}_+^\times$ acts on $\mathscr{G}$ through the map $\mathbb{R}_+^\times \hookrightarrow \mathbb{C}^\times \xrightarrow{\xi} T$. The map $\langle \mu, i\xi \rangle$ from $\mathscr{G}$ to $\mathbb{R}$ strictly increases along any non-constant orbit for the $\mathbb{R}_+^\times$-action. Now take $\lambda \in \Lambda$ and $x \in S_\lambda^+ \cap S_\lambda^-$. Then $\lim_{a\to 0} a^\xi \cdot x = \lim_{a\to\infty} a^\xi \cdot x = [t^\lambda]$. Thus $\langle \mu, i\xi \rangle$ cannot increase strictly along the orbit $\mathbb{R}_+^\times \cdot x$. This implies that this orbit is constant; in other words, $x = [t^\lambda]$.

3.3. **Mirković-Vilonen cycles.**

Let $\lambda, \nu \in \Lambda$. In order that $S_\nu^+ \cap \mathscr{G}_\lambda \neq \varnothing$, it is necessary that $[t^\nu] \in \overline{\mathscr{G}_\lambda}^T$, hence that $\nu - \lambda \in \mathbb{Z}\Phi^\vee$ and that ν belongs to the convex hull of $W\lambda$ in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.

Assume that λ is antidominant and denote by $L(w_0\lambda)$ the irreducible rational representation of $G^\vee$ with lowest weight λ. Mirković and Vilonen proved that the intersection $S_\nu^+ \cap \mathscr{G}_\lambda$ is of pure dimension $\operatorname{ht}(\nu - \lambda)$ and has as many irreducible components as the dimension of the ν-weight subspace of $L(w_0\lambda)$ (Theorem 3.2 and Corollary 7.4 in [27]). From this result and from Proposition 3.6 (iii), one readily deduces the following fact.

Proposition 3.9. *Let $\lambda, \nu \in \Lambda$ with $\nu \geqslant 0$. Then the intersection $S_{\lambda+\nu}^+ \cap S_\lambda^-$ (viewed as a topological subspace of $\mathscr{G}$) is noetherian of pure dimension* $\operatorname{ht}(\nu)$ *and has as many irreducible components as the dimension of the ν-weight subspace of $U(\mathfrak{n}^{+,\vee})$.*

Proof. As an abstract topological space, $S_{\lambda+\nu}^+ \cap S_\lambda^-$ does not depend on λ, because the action of t^μ on $\mathscr{G}$ maps $S_{\lambda+\nu}^+ \cap S_\lambda^-$ onto $S_{\lambda+\mu+\nu}^+ \cap S_{\lambda+\mu}^-$, for any $\mu \in \Lambda$. We may therefore assume that λ is antidominant enough so that the conclusion of Proposition 3.6 (iii) holds and that the $(\lambda + \nu)$-weight space of $L(w_0\lambda)$ has the same dimension as the ν-weight subspace of $U(\mathfrak{n}^{+,\vee})$. The proposition then follows from Mirković and Vilonen results. □

If X is a topological space, we denote the set of irreducible components of X by $\operatorname{Irr}(X)$. For $\lambda, \nu \in \Lambda$, we set

$$\mathscr{Z}(\lambda)_\nu = \operatorname{Irr}\Big(\overline{S_\nu^+ \cap \mathscr{G}_\lambda}\Big).$$

An element Z in a set $\mathscr{Z}(\lambda)_\nu$ is called an MV cycle. Such a Z is necessarily a closed, irreducible and noetherian subset of $\mathscr{G}$. It is also T-invariant, for the action of the connected group T on $\overline{S_\nu^+ \cap \mathscr{G}_\lambda}$ does not permute the irreducible components of this intersection closure. The coweight ν can be recovered from Z by the rule $\mu_+(Z) = \nu$; indeed, Z is the closure of an irreducible component Y of $S_\nu^+ \cap \mathscr{G}_\lambda$, so that $\mu_+(Z) = \mu_+(Y) = \nu$. The union

$$\mathscr{Z}(\lambda) = \bigsqcup_{\nu \in \Lambda} \mathscr{Z}(\lambda)_\nu$$

is therefore disjoint.

We finally set

$$\mathscr{Z} = \bigsqcup_{\substack{\lambda,\nu\in\Lambda\\ \lambda\geqslant\nu}} \operatorname{Irr}\Big(\overline{S_\lambda^+ \cap S_\nu^-}\Big).$$

Arguing as above, one sees that if Z is an irreducible component of $\overline{S_\lambda^+ \cap S_\nu^-}$, then λ and ν are determined by Z through the equations $\mu_+(Z) = \lambda$ and $\mu_-(Z) = \nu$. Using Example 3.4 (i), one checks without difficulty that for any irreducible and noetherian subset Z of $\mathscr{G}$,

$$\begin{aligned} \overline{Z} \in \mathscr{Z} &\iff \overline{Z} \text{ is an irreducible component of } \overline{S^+_{\mu_+(Z)} \cap S^-_{\mu_-(Z)}} \\ &\iff \dim Z = \operatorname{ht}(\mu_+(Z) - \mu_-(Z)). \end{aligned} \tag{3.4}$$

A result of Anderson (Proposition 3 in [1]) asserts that for any $\lambda, \nu \in \Lambda$ with λ antidominant,

$$\mathscr{Z}(\lambda)_\nu = \big\{ Z \in \mathscr{Z} \big| \mu_+(Z) = \nu, \mu_-(Z) = \lambda \text{ and } Z \subseteq \overline{\mathscr{G}_\lambda} \big\}.$$

This fact implies that if λ and μ are two antidominant coweights such that $\mu - \lambda \in \Lambda_{++}$ and if $Z \in \mathscr{Z}(\mu)$, then $t^{\lambda-\mu} \cdot Z \in \mathscr{Z}(\lambda)$. The set $\mathscr{Z}$ appears thus as the right way to stabilize the situation, namely

$$\mathscr{Z} = \left\{ t^\nu \cdot Z \;\middle|\; \nu \in \Lambda, Z \in \bigsqcup_{\lambda\in\Lambda_{++}} \mathscr{Z}(\lambda) \right\}.$$

It seems therefore legitimate to call MV cycles the elements of $\mathscr{Z}$.

From now on, our main aim will be to describe MV cycles as precisely as possible. We treat here the case where G has semisimple rank 1. We set $\mathbb{C}[t^{-1}]_0^+ = \mathbb{C}[t^{-1}]_0^* = \{0\}$. For each positive integer n, we consider the subsets

$$\mathbb{C}[t^{-1}]_n^+ = \big\{ a_{-n}t^{-n} + \cdots + a_{-1}t^{-1} \;\big|\; (a_{-n}, \ldots, a_{-1}) \in \mathbb{C}^n \big\}$$

and

$$\mathbb{C}[t^{-1}]_n^* = \big\{ a_{-n}t^{-n} + \cdots + a_{-1}t^{-1} \;\big|\; (a_{-n}, \ldots, a_{-1}) \in \mathbb{C}^n,\ a_{-n} \neq 0 \big\}$$

of $\mathscr{K}$; these are affine complex varieties. Finally, we set $\mathbb{C}[t^{-1}]^+ = t^{-1}\mathbb{C}[t^{-1}] = \bigcup_{n\in\mathbb{N}} \mathbb{C}[t^{-1}]_n^+$ and endow it with the inductive limit of the Zariski topologies on the subspaces $\mathbb{C}[t^{-1}]_n^+$.

Proposition 3.10. *Assume that G has semisimple rank* 1. *Let $\nu \in \Lambda$ and denote the unique simple root by α. Then the map $f : p \mapsto x_{-\alpha}\big(pt^{-\langle\alpha,\nu\rangle}\big)[t^\nu]$ from $\mathbb{C}[t^{-1}]^+$ onto S_ν^- is a homeomorphism. Moreover, for each $n \in \mathbb{N}$, the map f induces homeomorphisms*

$$\mathbb{C}[t^{-1}]_n^+ \xrightarrow{\simeq} \overline{S^+_{\nu+n\alpha^\vee}} \cap S_\nu^- \quad \textit{and} \quad \mathbb{C}[t^{-1}]_n^* \xrightarrow{\simeq} S^+_{\nu+n\alpha^\vee} \cap S_\nu^-.$$

This proposition implies that if G has semisimple rank 1, then each intersection $S_\lambda^+ \cap S_\nu^-$ is either empty or irreducible. In this case thus, the map $Z \mapsto (\mu_+(Z), \mu_-(Z))$ is a bijection from $\mathscr{Z}$ onto $\{(\lambda, \nu) \mid \lambda \geqslant \nu\}$, with inverse bijection $(\lambda, \nu) \mapsto \overline{S_\lambda^+ \cap S_\nu^-}$.

Proof of Proposition 3.10. Let G, α, ν and f be as in the statement of the proposition. The additive group $\mathscr{K}$ acts transitively on S_ν^- through the map $(p, z) \mapsto x_{-\alpha}\big(pt^{-\langle\alpha,\nu\rangle}\big)z$, where $p \in \mathscr{K}$ and $z \in S_\nu^-$. The stabilizer in $\mathscr{K}$ of $[t^\nu]$ is $\mathscr{O}$. Since $\mathscr{K}/\mathscr{O} \cong \mathbb{C}[t^{-1}]^+$, the map f is bijective. It is also continuous.

Now let $n \in \mathbb{N}$. Set $\lambda = \nu + n\alpha^\vee$; then $n = \langle \alpha, \lambda - \nu \rangle/2$. Specializing the equality

$$x_{-\alpha}(-a^{-1}) = x_\alpha(-a)\, a^{\alpha^\vee} \overline{s_\alpha}\, x_\alpha(-a)$$

to the value $a = -qt^n$, where $q \in \mathscr{O}^\times$, multiplying it on the left by t^ν and noticing that $(-q)^{\alpha^\vee} \overline{s_\alpha}\, x_\alpha(qt^n) \in G(\mathscr{O})$, we get

$$\bigl[x_{-\alpha}\bigl(q^{-1}t^{-\langle \alpha, \lambda+\nu \rangle/2}\bigr)\, t^\nu\bigr] = \bigl[x_\alpha\bigl(qt^{\langle \alpha, \lambda+\nu \rangle/2}\bigr)\, t^\lambda\bigr].$$

This equality immediately implies that $f\bigl(\mathbb{C}[t^{-1}]^*_n\bigr) \subseteq S^+_{\nu+n\alpha^\vee} \cap S^-_\nu$. Since

$$\mathbb{C}[t^{-1}]^+ = \bigsqcup_{n \in \mathbb{N}} \mathbb{C}[t^{-1}]^*_n \quad \text{and} \quad S^-_\nu = \bigsqcup_{n \in \mathbb{N}} \bigl(S^+_{\nu+n\alpha^\vee} \cap S^-_\nu\bigr),$$

we deduce that $f\bigl(\mathbb{C}[t^{-1}]^*_n\bigr) = S^+_{\nu+n\alpha^\vee} \cap S^-_\nu$, and then, using (3.2), that $f\bigl(\mathbb{C}[t^{-1}]^+_n\bigr) = \overline{S^+_{\nu+n\alpha^\vee}} \cap S^-_\nu$. The map f thus yields a continuous bijection from $\mathbb{C}[t^{-1}]^+_n$ onto $\overline{S^+_{\nu+n\alpha^\vee}} \cap S^-_\nu$.

It remains to show the continuity of f^{-1}. We may assume, without loss of generality, that $\nu = 0$. We first look at the particular case $G = \mathbf{SL}_2$ with its usual pinning. Given an element $p \in \mathscr{K}$, we write $p = \{p\}_{<0} + \{p\}_{\geqslant 0}$ according to the decomposition $\mathscr{K} = \mathbb{C}[t^{-1}]^+ \oplus \mathscr{O}$, and we denote by p_0 the coefficient of t^0 in p. We consider the subsets

$$\Omega' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a_0 \neq 0 \right\} \quad \text{and} \quad \Omega'' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, b_0 \neq 0 \right\}$$

of $G(\mathscr{K})$, and we define maps

$$h' : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \bigl\{c/\{a\}_{\geqslant 0}\bigr\}_{<0} \quad \text{and} \quad h'' : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \bigl\{d/\{b\}_{\geqslant 0}\bigr\}_{<0}$$

from Ω' and Ω'', respectively, to $\mathbb{C}[t^{-1}]^+$. Certainly, Ω' and Ω'' are open subsets of $G(\mathscr{K})$, and h' and h'' are continuous (see Proposition 1.2 in [2] for details on the inductive system that defines the topology on $G(\mathscr{K})$). We now observe that $U^-(\mathscr{K})G(\mathscr{O}) \subseteq \Omega' \cup \Omega''$ and that the map $h : g \mapsto f^{-1}([g])$ from $U^-(\mathscr{K})G(\mathscr{O})$ to $\mathbb{C}[t^{-1}]^+$ is given on $\Omega' \cap U^-(\mathscr{K})G(\mathscr{O})$ by the restriction of h', and on $\Omega'' \cap U^-(\mathscr{K})G(\mathscr{O})$ by the restriction of h''. The map h is thus continuous, and we conclude that f^{-1} is continuous in our particular case $G = \mathbf{SL}_2$.

The continuity of f^{-1} is then guaranteed whenever G is the product of $\mathbf{SL}_2$ with a torus. Now any connected reductive group of semisimple rank 1 is isogenous to such a product; the general case follows, because an isogeny between two connected reductive groups induces a homeomorphism between the neutral connected components of their respective Grassmannians (see for instance Section 2 of [11]). □

3.4. **Parabolic retractions.** In Section (5.3.28) of [3], Beilinson and Drinfeld describe a way to relate $\mathscr{G}$ with the affine Grassmannians of Levi subgroups of G. We rephrase their construction in a slightly less general context.

Let P be a parabolic subgroup of G which contains T, let M be the Levi factor of P that contains T, and let $\mathscr{P}$ and $\mathscr{M}$ be the affine Grassmannians of P and M. The diagram $G \hookleftarrow P \twoheadrightarrow M$ yields similar diagrams $G(\mathscr{K}) \hookleftarrow P(\mathscr{K}) \twoheadrightarrow M(\mathscr{K})$ and $\mathscr{G} \xleftarrow{i} \mathscr{P} \xrightarrow{\pi} \mathscr{M}$. The continuous map i is bijective but is not a homeomorphism in general ($\mathscr{P}$ has usually more connected components than $\mathscr{G}$). We may, however, define the (non-continuous) map $r_P = \pi \circ i^{-1}$ from $\mathscr{G}$ to $\mathscr{M}$.

The group $P(\mathscr{K})$ acts on $\mathscr{M}$ via the projection $P(\mathscr{K}) \twoheadrightarrow M(\mathscr{K})$ and acts on $\mathscr{G}$ via the embedding $P(\mathscr{K}) \hookrightarrow G(\mathscr{K})$. The map r_P can then be characterized as the unique $P(\mathscr{K})$-equivariant section of the embedding $\mathscr{M} \hookrightarrow \mathscr{G}$ that arises from the inclusion $M \subseteq G$.

For instance, consider the case where P is the Borel subgroup $B^\pm$; then the Levi factor M is the torus T and the group $P(\mathscr{K})$ contains the group $U^\pm(\mathscr{K})$. The map $r_{B^\pm} : \mathscr{G} \to \mathscr{T}$, being a $U^\pm(\mathscr{K})$-equivariant section of the embedding $\mathscr{T} \hookrightarrow \mathscr{G}$, sends the whole stratum $S_\lambda^\pm$ to the point $[t^\lambda]$, for each $\lambda \in \Lambda$.

Remark 3.11. The map r_P can also be understood in terms of a Białynicki-Birula decomposition. Indeed, let $\mathfrak{g}$, $\mathfrak{p}$ and $\mathfrak{t}$ be the Lie algebras of G, P and T. We write $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}^\alpha$ for the root decomposition of $\mathfrak{g}$ and put $\Phi_P = \{\alpha \in \Phi \mid \mathfrak{g}^\alpha \subseteq \mathfrak{p}\}$. Choosing now $\xi \in \Lambda$ such that

$$\forall\alpha \in \Phi_P,\ \langle\alpha, \lambda\rangle \geqslant 0 \quad \text{and} \quad \forall\alpha \in \Phi \setminus \Phi_P,\ \langle\alpha, \lambda\rangle < 0,$$

one may check that $r_P(x) = \lim_{\substack{a\to 0\\ a\in\mathbb{C}^\times}} a^\xi \cdot x$ for each $x \in \mathscr{G}$. This construction justifies the name of parabolic retraction we give to the map r_P.

As noted by Beilinson and Drinfeld (see the proof of Proposition 5.3.29 in [3]), parabolic retractions enjoy a transitivity property. Namely considering a pair (P, M) inside G as above and a pair (Q, N) inside M, we get maps $\mathscr{G} \xrightarrow{r_P} \mathscr{M} \xrightarrow{r_Q} \mathscr{N}$. The preimage R of Q by the quotient map $P \twoheadrightarrow M$ is a parabolic subgroup of G, and N is the Levi factor of R that contains T. The composition $r_Q \circ r_P$ is a $R(\mathscr{K})$-equivariant section of the embedding $\mathscr{N} \hookrightarrow \mathscr{G}$; it thus coincides with r_R.

We will mainly apply these constructions to the case of standard parabolic subgroups. Let us fix the relevant terminology. For each subset $J \subseteq I$, we denote by $U_J^\pm$ the subgroup of G generated by the images of the morphisms $x_{\pm\alpha_j}$ for $j \in J$. We denote the subgroup generated by $T \cup U_J^+ \cup U_J^-$ by M_J and we denote the subgroup generated by $B^+ \cup M_J$ by P_J. Thus M_J is the Levi factor of P_J that contains T. We shorten the notation and denote the parabolic retraction r_{P_J} simply by r_J. The Weyl group of M_J can be identified with the parabolic subgroup W_J of W generated by the simple reflections s_j with $j \in J$; we denote the longest element of W_J by $w_{0,J}$.

The Iwasawa decomposition for M_J gives

$$\mathscr{M}_J = \bigsqcup_{\lambda\in\Lambda} U_J^\pm(\mathscr{K})[t^\lambda].$$

For $\lambda \in \Lambda$, we denote the $U_J^\pm(\mathscr{K})$-orbit of $[t^\lambda]$ by $S_{\lambda,J}^\pm$.

Lemma 3.12. *For each $\lambda \in \Lambda$, we have*

$$S_\lambda^+ = (r_J)^{-1}\big(S_{\lambda,J}^+\big) \quad \textit{and} \quad \overline{w_{0,J}}\, S^+_{w_{0,J}^{-1}\lambda} = (r_J)^{-1}\big(S_{\lambda,J}^-\big).$$

Proof. Consider the transitivity property $r_R = r_Q \circ r_P$ of parabolic retractions written above for $P = P_J$, $M = M_J$ and $N = T$. For the first formula, one chooses, moreover, $Q = TU_J^+$, so that $R = B^+$. Recalling the equality $(r_{B^+})^{-1}\big([t^\lambda]\big) = S_\lambda^+$ and its analogue $(r_Q)^{-1}\big([t^\lambda]\big) = S_{\lambda,J}^+$ for $\mathscr{M}_J$, we see that the desired formula simply computes the preimage of $[t^\lambda]$ by the map $r_R = r_Q \circ r_P$.

For the second formula, one chooses $Q = TU_J^-$, whence $R = \overline{w_{0,J}}\, B^+\, \overline{w_{0,J}}^{-1}$. Here we have

$$(r_R)^{-1}\Big(\big[t^\lambda\big]\Big) = \overline{w_{0,J}}\ (r_{B^+})^{-1}\Big(\big[t^{w_{0,J}^{-1}\lambda}\big]\Big) = \overline{w_{0,J}}\, S^+_{w_{0,J}^{-1}\lambda}$$

and $(r_Q)^{-1}\big([t^\lambda]\big) = S^-_{\lambda,J}$. Again the desired formula simply computes the preimage of $[t^\lambda]$ by the map $r_R = r_Q \circ r_P$. □

To conclude this section, we note that for any $\mathscr{K}$-point h of the unipotent radical of P_J, any $g \in P_J(\mathscr{K})$, and any $x \in \mathscr{G}$,

$$r_J(gh\cdot x) = (ghg^{-1})\cdot r_J(gx) = r_J(gx), \tag{3.5}$$

because ghg^{-1} is a $\mathscr{K}$-point of the unipotent radical of P_J and thus acts trivially on $\mathscr{M}_J$.

4. Crystal structure and string parametrizations

For each dominant coweight λ, the set $\mathscr{Z}(\lambda)$ yields a basis of the rational $G^\vee$-module $L(\lambda)$. One may therefore expect that $\mathscr{Z}(\lambda)$ can be turned in a natural way into a crystal isomorphic to $\mathbf{B}(\lambda)$. Braverman and Gaitsgory made this idea precise in [9]. Later in [8], these two authors and Finkelberg extended this result by endowing $\mathscr{Z}$ with the structure of a crystal isomorphic to $\widetilde{\mathbf{B}(-\infty)}$. We recall this crucial result in Section 4.1 and characterize the crystal operations on $\mathscr{Z}$ in a suitable way for comparisons (Proposition 4.2).

We begin Section 4.2 by translating the geometric definition of Braverman, Finkelberg and Gaitsgory's crystal structure on $\mathscr{Z}$ in more algebraic terms (Proposition 4.5). From there, we deduce a quite explicit description of MV cycles. More precisely, let $b \in \mathbf{B}(-\infty)$ and let $\Xi(t_0 \otimes b)$ be the MV cycle that corresponds to $t_0 \otimes b \in \widetilde{\mathbf{B}(-\infty)}$. Theorem 4.6 exhibits a parametrization of an open and dense subset of $\Xi(t_0 \otimes b)$ by a variety of the form $(\mathbb{C}^\times)^m \times \mathbb{C}^n$; this parametrization generalizes the description in semisimple rank 1 given in Proposition 3.10.

Then Section 4.3 introduces subsets $\tilde{Y}_{\mathbf{i},\mathbf{c}}$ of the affine Grassmannian $\mathscr{G}$, where $\mathbf{i} \in I^l$ and $\mathbf{c} \in \mathbb{Z}^l$. When $\mathbf{c}$ is the string parameter in direction $\mathbf{i}$ of an element $b \in \mathbf{B}(-\infty)$, the definition of $\tilde{Y}_{\mathbf{i},\mathbf{c}}$ reflects the construction in the statement of Theorem 4.6, so that $\Xi(t_0 \otimes b) = \overline{\tilde{Y}_{\mathbf{i},\mathbf{c}}}$. It turns out that the closure $\overline{\tilde{Y}_{\mathbf{i},\mathbf{c}}}$ is always an MV cycle, even when $\mathbf{c}$ does not belong to the string cone in direction $\mathbf{i}$. Proposition 4.7 presents a necessary and sufficient condition on $\overline{\tilde{Y}_{\mathbf{i},\mathbf{c}}}$ in order that $\mathbf{c}$ may belong to the string cone; its proof relies on Berenstein and Zelevinsky's characterization of the string cone in terms of $\mathbf{i}$-trails [6].

The introduction of the subsets $\tilde{Y}_{\mathbf{i},\mathbf{c}}$ finds its justification in Section 4.4. Here we use them to explain how the algebraic-geometric parametrization of $\mathbf{B}(-\infty)$ devised by Lusztig in [25] is related to MV cycles.

In the course of his work on MV polytopes [13, 14], Kamnitzer was led to a description of MV cycles similar to the equality $\Xi(t_0 \otimes b) = \overline{\tilde{Y}_{\mathbf{i},\mathbf{c}}}$, but starting from the Lusztig parameter of b instead of the string parameter. In Section 4.5, we show that the equality and Kamnitzer's description are in fact equivalent results.

4.1. **Braverman, Finkelberg and Gaitsgory's crystal structure.** In Section 13 of [8], Braverman, Finkelberg and Gaitsgory endow $\mathscr{Z}$ with the structure of a crystal with an involution $*$. The main step of their construction is an analysis of the behavior of MV cycles with respect to the standard parabolic retractions. For a subset $J \subseteq I$, we denote the analogues of the maps $\mu_\pm$ for the affine Grassmannian $\mathscr{M}_J$ by $\mu_{\pm,J}$. The following theorem is due to Braverman, Finkelberg and Gaitsgory; nevertheless, we quickly recall its proof since we ground the proofs of Propositions 4.2 and 4.5 on it.

Theorem 4.1. *Let J be a subset of I and let $Z \in \mathscr{Z}$ be an MV cycle. Set*

$$Z_J = \overline{r_J\big(Z \cap S_\nu^-\big) \cap S_{\rho,J}^-} \quad \text{and} \quad Z^J = \overline{Z \cap S_\nu^- \cap (r_J)^{-1}([t^\rho])},$$

where $\nu = \mu_-(Z)$ and $\rho = w_{0,J}\, \mu_+(\overline{w_{0,J}}^{-1} Z)$. Then the map $Z \mapsto (Z_J, Z^J)$ is a bijection from $\mathscr{Z}$ onto the set of all pairs (Z', Z''), where Z' is an MV cycle in $\mathscr{M}_J$ and Z'' is an MV cycle in $\mathscr{G}$ which satisfy

$$\mu_{-,J}(Z') = \mu_+(Z'') = w_{0,J}\, \mu_+(\overline{w_{0,J}}^{-1} Z''). \tag{4.1}$$

Under this correspondence, one has

$$\begin{aligned} \mu_+(Z) &= \mu_{+,J}(Z_J), \\ \mu_-(Z) &= \mu_-(Z^J), \\ w_{0,J}\, \mu_+(\overline{w_{0,J}}^{-1} Z) &= \mu_{-,J}(Z_J) = \mu_+(Z^J) = w_{0,J}\, \mu_+(\overline{w_{0,J}}^{-1} Z^J). \end{aligned}$$

Proof. Let us consider three coweights $\lambda, \nu, \rho \in \Lambda$, in the same coset modulo $\mathbb{Z}\Phi^\vee$, and unrelated to the MV cycle Z for the moment. The group $H = U_J^-(\mathscr{K})$ acts on $\mathscr{G}$, leaving S_ν^- stable. On the other hand, $S_{\rho,J}^-$ is the H-orbit of $[t^\rho]$; we denote by K the stabilizer of $[t^\rho]$ in H, so that $S_{\rho,J}^- \cong H/K$. Since the map r_J is H-equivariant, the action of H leaves stable the intersection $S_\nu^- \cap (r_J)^{-1}\big(S_{\rho,J}^-\big)$, the action of K leaves stable the intersection $F = S_\nu^- \cap (r_J)^{-1}([t^\rho])$, and we have a commutative diagram

$$\begin{tikzcd} F \arrow[r, hook] & H \times_K F \arrow[r, "\simeq"] \arrow[d] & S_\nu^- \cap (r_J)^{-1}\big(S_{\rho,J}^-\big) \arrow[d, "r_J"] \\ & H/K \arrow[r, "\simeq"] & S_{\rho,J}^-. \end{tikzcd}$$

In this diagram, the two leftmost arrows define a fiber bundle.

By Lemma 3.12, $F \subseteq S_\rho^+ \cap S_\nu^-$; therefore the dimension of F is at most $\operatorname{ht}(\rho-\nu)$. The group K is connected — indeed $K = U_J^-(\mathscr{K}) \cap t^\rho G(\mathscr{O}) t^{-\rho}$, so it leaves invariant each irreducible component of F. We thus have a canonical bijection $C \mapsto \tilde{C} = H \times_K C$ from $\operatorname{Irr}(F)$ onto $\operatorname{Irr}(H \times_K F)$. If, moreover, X is a subspace of $H/K = S_{\rho,J}^-$, then the assignment $(C, D) \mapsto \tilde{C} \cap (r_J)^{-1}(D)$ is a bijection from $\operatorname{Irr}(F) \times \operatorname{Irr}(X)$ onto $\operatorname{Irr}(S_\nu^- \cap (r_J)^{-1}(X))$. We will apply this fact to $X = S_{\rho,J}^- \cap \overline{S_{\lambda,J}^+}$; using (3.2) and Proposition 3.9, one sees easily that X then has dimension at most $\operatorname{ht}(\lambda - \rho)$. Since $\tilde{C} \cap (r_J)^{-1}(D)$ is a fiber bundle with fiber C and base D, its dimension is

$$\dim C + \dim D \leqslant \operatorname{ht}(\rho - \nu) + \operatorname{ht}(\lambda - \rho) = \operatorname{ht}(\lambda - \nu).$$

Now let Z be an MV cycle and set $\lambda = \mu_+(Z)$, $\rho = w_{0,J}\,\mu_+(\overline{w_{0,J}}^{-1}Z)$ and $\nu = \mu_-(Z)$ in the previous setting. By Proposition 3.3 and Lemma 3.12,

$$Z \cap S_\nu^- \quad \text{and} \quad \overline{w_{0,J}}\Big(\overline{w_{0,J}}^{-1}Z \cap S^+_{w_{0,J}^{-1}\rho}\Big) = Z \cap (r_J)^{-1}\big(S^-_{\rho,J}\big)$$

are open and dense subsets in Z. Thus $\dot Z = Z \cap S_\nu^- \cap (r_J)^{-1}\big(S^-_{\rho,J}\big)$ is a closed irreducible subset of $S_\nu^- \cap (r_J)^{-1}\big(S^-_{\rho,J}\big)$ of dimension $\dim Z = \mathrm{ht}(\lambda-\nu)$; this subset $\dot Z$ is actually contained in $S_\nu^- \cap (r_J)^{-1}(x)$, because $\dot Z \subseteq Z \subseteq \overline{S_\lambda^+}$. It is therefore an irreducible component $\tilde C \cap (r_J)^{-1}(D)$ with, moreover, $\dim C = \mathrm{ht}(\rho-\nu)$ and $\dim D = \mathrm{ht}(\lambda-\rho)$.

One observes then that $[t^\rho] \in D$, because D is a closed and T-invariant subset of $S^-_{\rho,J}$. Then

$$C = \tilde C \cap (r_J)^{-1}([t^\rho]) = \dot Z \cap (r_J)^{-1}([t^\rho]) = Z \cap S_\nu^- \cap (r_J)^{-1}([t^\rho]),$$

and thus, by Lemma 3.12, $C \subseteq S_\nu^- \cap S_\rho^+ \cap \overline{w_{0,J}}\,S^+_{w_{0,J}^{-1}\rho}$. Therefore $\mu_-(C) = \nu$ and $\mu_+(C) = w_{0,J}\,\mu_+(\overline{w_{0,J}}^{-1}C) = \rho$; Equivalence (3.4) and the estimate $\dim C = \mathrm{ht}(\rho-\nu)$ then imply that $\overline{C}$ is an MV cycle. On the other hand, the relations $\mu_{+,J}(D) \leqslant \lambda$, $\mu_{-,J}(D) = \rho$ and $\dim D = \mathrm{ht}(\lambda-\rho)$ imply altogether that $\overline{D}$ is an MV cycle in $\mathscr{M}_J$ and that $\mu_{+,J}(D) = \lambda$. Moreover,

$$D = r_J(\dot Z) = r_J\big(Z \cap S_\nu^-\big) \cap S^-_{\rho,J}.$$

Thus $Z_J = \overline{D}$ and $Z^J = \overline{C}$ satisfy the conditions stated in the theorem.

Conversely, given Z' and Z'' as in the statement of the theorem, we take $\lambda = \mu_{+,J}(Z')$, $\nu = \mu_-(Z'')$ and $\rho = \mu_{-,J}(Z')$ in the construction above, and we set $C = Z'' \cap F$, $D = Z' \cap S^-_{\rho,J}$ and $\dot Z = \tilde C \cap (r_J)^{-1}(D)$. Then C is an open and dense subset in Z''; it is therefore irreducible with the same dimension as Z'', namely $\mathrm{ht}(\rho-\nu)$. Since it is a closed subset of F, C is an irreducible component of F. Likewise D has dimension $\mathrm{ht}(\lambda-\rho)$ and is an irreducible component of $X = S^-_{\rho,J} \cap \overline{S^+_{\lambda,J}}$. The first part of the reasoning above implies thus that $\dot Z$ is irreducible of dimension $\dim C + \dim D = \mathrm{ht}(\lambda-\nu)$. Since $\mu_+(\dot Z) = \lambda$ and $\mu_-(\dot Z) = \nu$, it follows from Equivalence (3.4) that $Z = \overline{\dot Z}$ is an MV cycle.

It is then routine to check that the two maps $Z \mapsto (Z_J, Z^J)$ and $(Z', Z'') \mapsto Z$ are mutually inverse bijections. □

We are now ready to define Braverman, Finkelberg and Gaitsgory's crystal structure on $\mathscr{Z}$. Let Z be an MV cycle. We set

$$\mathrm{wt}(Z) = \mu_+(Z).$$

Given $i \in I$, we apply Theorem 4.1 to Z and $J = \{i\}$. We set $\rho = s_i\,\mu_+(\overline{s_i}^{-1}Z)$ and get a decomposition $\big(Z_{\{i\}}, Z^{\{i\}}\big)$ of Z. Then we set

$$\varepsilon_i(Z) = \left\langle \alpha_i, \frac{-\mu_+(Z)-\rho}{2} \right\rangle \quad \text{and} \quad \varphi_i(Z) = \left\langle \alpha_i, \frac{\mu_+(Z)-\rho}{2} \right\rangle.$$

Since $\mu_+(Z) - \rho = \mu_{+,\{i\}}\big(Z_{\{i\}}\big) - \mu_{-,\{i\}}\big(Z_{\{i\}}\big)$ belongs to $\mathbb{N}\alpha_i^\vee$, the definition for $\varphi_i(Z)$ is equivalent to the equation

$$\mu_+(Z) - \rho = \varphi_i(Z)\,\alpha_i^\vee. \tag{4.2}$$

The MV cycles $\tilde e_i Z$ and $\tilde f_i Z$ are defined by the following requirements:

$$\begin{aligned}\mu_+(\tilde e_i Z)&=\mu_+(Z)+\alpha_i^\vee,\\ \mu_+(\tilde f_i Z)&=\mu_+(Z)-\alpha_i^\vee,\\ (\tilde e_i Z)^{\{i\}}&=(\tilde f_i Z)^{\{i\}}=Z^{\{i\}};\end{aligned}$$

if $\mu_+(Z)=\rho$, that is, if $\varphi_i(Z)=0$, then we set $\tilde f_i Z=0$.

These conditions do define the MV cycles $\tilde e_i Z$ and $\tilde f_i Z$. Indeed they prescribe the components $(\tilde e_i Z)^{\{i\}}$ and $(\tilde f_i Z)^{\{i\}}$ and require

$$\begin{aligned}\mu_{+,\{i\}}\big((\tilde e_i Z)_{\{i\}}\big)&=\mu_+(\tilde e_i Z)=\mu_+(Z)+\alpha_i^\vee=\mu_{+,\{i\}}\big(Z_{\{i\}}\big)+\alpha_i^\vee,\\ \mu_{-,\{i\}}\big((\tilde e_i Z)_{\{i\}}\big)&=\mu_+\big((\tilde e_i Z)^{\{i\}}\big)=\mu_+\big(Z^{\{i\}}\big)=\mu_{-,\{i\}}\big(Z_{\{i\}}\big)\end{aligned}$$

and

$$\begin{aligned}\mu_{+,\{i\}}\big((\tilde f_i Z)_{\{i\}}\big)&=\mu_+(\tilde f_i Z)=\mu_+(Z)-\alpha_i^\vee=\mu_{+,\{i\}}\big(Z_{\{i\}}\big)-\alpha_i^\vee,\\ \mu_{-,\{i\}}\big((\tilde f_i Z)_{\{i\}}\big)&=\mu_+\big((\tilde f_i Z)^{\{i\}}\big)=\mu_+\big(Z^{\{i\}}\big)=\mu_{-,\{i\}}\big(Z_{\{i\}}\big).\end{aligned}$$

These latter equations fully determine the components $(\tilde e_i Z)_{\{i\}}$ and $(\tilde f_i Z)_{\{i\}}$ because $M_{\{i\}}$ has semisimple rank 1 (see the comment after the statement of Proposition 3.10).

One checks without difficulty that $\mathscr{Z}$, endowed with these maps wt, ε_i, φ_i, $\tilde e_i$ and $\tilde f_i$, satisfies Kashiwara's axioms of a crystal. On the other hand, let $g\mapsto g^t$ be the antiautomorphism of G that fixes T pointwise and that maps $x_{\pm\alpha}(a)$ to $x_{\mp\alpha}(a)$ for each simple root α and each $a\in\mathbb{C}$. Then the involutive automorphism $g\mapsto (g^t)^{-1}$ of G extends to $G(\mathscr{K})$ and induces an involution on $\mathscr{G}$, which we denote by $x\mapsto x^*$. The image of an MV cycle Z under this involution is an MV cycle Z^*. The properties of this involution $Z\mapsto Z^*$ with respect to the crystal operations allow Braverman, Finkelberg and Gaitsgory [8] to establish the existence of an isomorphism of crystals $\Xi:\widetilde{\mathbf{B}(-\infty)}\xrightarrow{\simeq}\mathscr{Z}$. This isomorphism is unique and is compatible with the involutions $*$ on $\widetilde{\mathbf{B}(-\infty)}$ and $\mathscr{Z}$. One checks that

$$\begin{aligned}&\Xi(t_\lambda\otimes 1)=\big\{[t^\lambda]\big\}, &&\mu_-\big(\Xi(t_\lambda\otimes b)\big)=\lambda,\\ &\Xi(t_\lambda\otimes b)=t^\lambda\cdot\Xi(t_0\otimes b), &&\dim\Xi(t_\lambda\otimes b)=\mathrm{ht}(\mathrm{wt}(b)),\end{aligned}\tag{4.3}$$

for all $\lambda\in\Lambda$ and $b\in\mathbf{B}(-\infty)$.

The following proposition gives a useful criterion which says when two MV cycles are related by an operator $\tilde e_i$.

Proposition 4.2. *Let Z and Z' be two MV cycles in $\mathscr{G}$ and let $i\in I$. Then $Z'=\tilde e_i Z$ if and only if the four following conditions hold:*

$$\begin{aligned}\mu_-(Z')&=\mu_-(Z),\\ s_i\,\mu_+(\overline{s_i}^{-1}Z')&=s_i\,\mu_+(\overline{s_i}^{-1}Z),\\ \mu_+(Z')&=\mu_+(Z)+\alpha_i^\vee,\\ Z'&\supseteq Z.\end{aligned}$$

Proof. We first prove that the conditions in the statement of the proposition are sufficient to ensure that $Z'=\tilde e_i Z$. We assume that the two MV cycles Z and Z'

enjoy the conditions above and we set

$$\begin{aligned}\rho &= s_i\,\mu_+(\overline{s_i}^{-1}Z) = s_i\,\mu_+(\overline{s_i}^{-1}Z'),\\ \nu &= \mu_-(Z) = \mu_-(Z'),\\ F &= S^-_\nu \cap (r_{\{i\}})^{-1}([t^\rho]).\end{aligned}$$

The proof of Theorem 4.1 tells us that $C = Z \cap F$ and $C' = Z' \cap F$ are two irreducible components of F. The condition $Z' \supseteq Z$ then entails $C' \supseteq C$, and thus $C' = C$. It follows that

$$Z^{\{i\}} = \overline{C} = \overline{C'} = Z'^{\{i\}}.$$

This being known, the assumption $\mu_+(Z') = \mu_+(Z) + \alpha_i^\vee$ implies $Z' = \tilde{e}_i Z$.

Conversely, assume that $Z' = \tilde{e}_i Z$. Routine arguments then show that the three first conditions in the statement of the proposition hold. Setting ρ, ν, F, C and C' as in the first part of the proof, we get

$$C = \overline{C} \cap F = Z^{\{i\}} \cap F = Z'^{\{i\}} \cap F = \overline{C'} \cap F = C'.$$

On the other hand, set $D = Z_{\{i\}} \cap S^-_{\rho,\{i\}}$ and $D' = Z'_{\{i\}} \cap S^-_{\rho,\{i\}}$. Using Proposition 3.10, we see that

$$D = \overline{S^+_{\mu_+(Z),\{i\}} \cap S^-_{\rho,\{i\}}} \cap S^-_{\rho,\{i\}} = \overline{S^+_{\mu_+(Z),\{i\}}} \cap S^-_{\rho,\{i\}}$$

is contained in

$$D' = \overline{S^+_{\mu_+(Z'),\{i\}} \cap S^-_{\rho,\{i\}}} \cap S^-_{\rho,\{i\}} = \overline{S^+_{\mu_+(Z'),\{i\}}} \cap S^-_{\rho,\{i\}}.$$

Adopting the notation $\tilde{C}$ from the proof of Theorem 4.1, we deduce that $\tilde{C} \cap (r_{\{i\}})^{-1}(D)$ is contained in $\tilde{C} \cap (r_{\{i\}})^{-1}(D')$. The closure Z of the first set is thus contained in the closure Z' of the second set. $\square$

For each dominant coweight $\lambda \in \Lambda_{++}$, the two sets $\mathbf{B}(\lambda)$ and $\mathscr{Z}(\lambda)$ have the same cardinality; indeed they both index bases of two isomorphic vector spaces, namely the rational irreducible $G^\vee$-module with highest weight λ and the intersection cohomology of $\overline{\mathscr{G}_\lambda}$, respectively. More is true: in [9], Braverman and Gaitsgory endow $\mathscr{Z}(\lambda)$ with the structure of a crystal and show the existence of an isomorphism of crystals $\Xi(\lambda) : \mathbf{B}(\lambda) \xrightarrow{\simeq} \mathscr{Z}(\lambda)$ (see [9], p. 569).

Proposition 4.3. *The following diagram commutes:*

$$\begin{tikzcd} \mathbf{B}(\lambda) \arrow[r, "\Xi(\lambda)"] \arrow[d, hook, "\iota_{w_0\lambda}"'] & \mathscr{Z}(\lambda) \arrow[d, hook] \\ \mathbf{T}_{w_0\lambda} \otimes \mathbf{B}(-\infty) \arrow[r, "\Xi"'] & \mathscr{Z}. \end{tikzcd}$$

Proof. Let $Z, Z' \in \mathscr{Z}(\lambda)$ and assume that Z' is the image of Z by the crystal operator defined in Section 3.3 of [9]. The definition of this operator is so similar to the definition of our (in fact, Braverman, Finkelberg and Gaitsgory's) crystal operator $\tilde{e}_i$ that a slight modification of the proof of Proposition 4.2 yields

$$\begin{aligned} \mu_-(Z') &= \mu_-(Z), \\ s_i\,\mu_+(\overline{s_i}^{-1}Z') &= s_i\,\mu_+(\overline{s_i}^{-1}Z), \\ \mu_+(Z') &= \mu_+(Z) + \alpha_i^\vee, \\ Z' &\supseteq Z. \end{aligned}$$

By Proposition 4.2, this implies that Z' is the image of Z by our crystal operator $\tilde{e}_i$. In other words, the inclusion $\mathscr{Z}(\lambda) \hookrightarrow \mathscr{Z}$ is an embedding of crystals when $\mathscr{Z}(\lambda)$ is endowed with the crystal structure from [9].

Thus both maps $\Xi \circ \iota_{w_0\lambda}$ and $\Xi(\lambda)$ are crystal embeddings of $\mathbf{B}(\lambda)$ into $\mathscr{Z}$. Also both maps send the lowest weight element b_{low} of $\mathbf{B}(\lambda)$ onto the MV cycle $\big\{[t^{w_0\lambda}]\big\}$. The proposition then follows from the fact that each element of $\mathbf{B}(\lambda)$ can be obtained by applying a sequence of crystal operators to b_{low}. □

Remark 4.4. One can establish the equality $\Xi \circ \iota_{w_0\lambda}(\mathbf{B}(\lambda)) = \mathscr{Z}(\lambda)$ without using Braverman and Gaitsgory's isomorphism $\Xi(\lambda)$ by the following direct argument. Let $Z \in \mathscr{Z}(\lambda)$. Certainly $\mu_-(Z) = w_0\lambda$, so by Equation (4.3), $\Xi^{-1}(Z)$ may be written $t_{w_0\lambda} \otimes b$ with $b \in \mathbf{B}(-\infty)$. Take $i \in I$ and set $\rho = s_i\,\mu_-(\overline{s_i}^{-1}Z)$. Then $\overline{s_i}^{-1}Z$ meets $S^-_{s_i^{-1}\rho}$, and thus $\left[t^{s_i^{-1}\rho}\right]$ belongs to $\overline{s_i}^{-1}Z$, for $\overline{s_i}^{-1}Z$ is closed and T-stable. From the inclusion $Z \subseteq \overline{\mathscr{G}_\lambda}$, we then deduce that $[t^\rho] \in \overline{\mathscr{G}_\lambda}$. Using Equation (3.1) and the description $(\mathscr{G}_\mu)^T = \big\{[t^{w\mu}] \bigm| w \in W\big\}$ (see the proof of Proposition 3.3), this yields

$$\rho \in \big\{ w\mu \bigm| w \in W,\ \mu \in \Lambda_{++} \text{ such that } \lambda \geqslant \mu \big\}.$$

On the other side,

$$\rho - w_0\lambda = s_i\,\mu_-\big(\overline{s_i}^{-1}Z\big) - \mu_-(Z) = \mu_+(Z^*) - s_i\,\mu_+\big(\overline{s_i}^{-1}Z^*\big) = \varphi_i(Z^*)\alpha_i^\vee.$$

These two facts together entail $\varphi_i(Z^*) \leqslant \langle \alpha_i, -w_0\lambda\rangle$. Since

$$\varphi_i(Z^*) = \varphi_i(\Xi^{-1}(Z^*)) = \varphi_i((t_{w_0\lambda} \otimes b)^*) = \varphi_i(t_{-w_0\lambda - \mathrm{wt}(b)} \otimes b^*) = \varphi_i(b^*),$$

we obtain $\varphi_i(b^*) \leqslant \langle \alpha_i, -w_0\lambda\rangle$. This inequality holds for each $i \in I$, therefore the element $t_{w_0\lambda} \otimes b$ belongs to $\iota_{w_0\lambda}(\mathbf{B}(\lambda))$. We have thus established the inclusion $\Xi^{-1}(\mathscr{Z}(\lambda)) \subseteq \iota_{w_0\lambda}(\mathbf{B}(\lambda))$. Since $\mathbf{B}(\lambda)$ and $\mathscr{Z}(\lambda)$ have the same cardinality, this inclusion is an equality.

4.2. **Description of an MV cycle from the string parameter.** We begin this section with a proposition that translates Braverman, Finkelberg and Gaitsgory's geometrical definition for the crystal operation $\tilde{e}_i$ into a more algebraic language. This proposition comes in two flavors: Statement (i) is terse, whereas Statement (ii) is verbose but yields more refined information. We recall that the notations $\mathbb{C}[t^{-1}]_k^+$ and $\mathbb{C}[t^{-1}]_k^*$ have been defined in Section 3.3.

Proposition 4.5. *Let Z be an MV cycle, let $i \in I$, let $k \in \mathbb{N}$, and set $Z' = \tilde{e}_i^k(Z)$.*

(i) *For each $p \in \mathscr{O}$, the action of $y_i\big(pt^{\varepsilon_i(Z)}\big)$ stabilizes Z. The MV cycle Z' is the closure of*

$$\big\{ y_i(p)\,z \bigm| z \in Z \text{ and } p \in \mathscr{K}^\times \text{ such that } \mathrm{val}(p) = -k + \varepsilon_i(Z) \big\}.$$

(ii) *Set* $\nu = \mu^-(Z)$, $\rho = s_i\mu_+(\overline{s_i}^{-1}Z)$, $\dot{Z} = Z \cap S^-_\nu \cap \left(\overline{s_i}\, S^+_{s_i^{-1}\rho}\right)$ *and* $\dot{Z}' = Z' \cap S^-_\nu \cap \left(\overline{s_i}\, S^+_{s_i^{-1}\rho}\right)$. *Then the map* $f : (p,z) \mapsto y_i\big(pt^{\varepsilon_i(Z)}\big)z$ *is a homeomorphism from* $\mathbb{C}[t^{-1}]^+_k \times \dot{Z}$ *onto* $\dot{Z}'$. *If, moreover,* $\rho = \mu_+(Z)$, *then* $\dot{Z} = Z \cap S^-_\nu \cap S^+_{\mu_+(Z)}$ *and* f *induces a homeomorphism from* $\mathbb{C}[t^{-1}]^*_k \times \dot{Z}$ *onto an open and dense subset of* $Z' \cap S^-_\nu \cap S^+_{\mu_+(Z')}$.

Proof. We begin with the proof of Statement (ii). Let Z be an MV cycle and let $i \in I$. We adopt the notation used in the proof of Theorem 4.1, with here $J = \{i\}$. We set $\lambda = \mu_+(Z)$, $\nu = \mu_-(Z)$, $\rho = s_i\mu_+(\overline{s_i}^{-1}Z)$, $n = \varphi_i(Z) = \langle \alpha_i, \lambda - \rho\rangle/2$, $F = S^-_\nu \cap (r_{\{i\}})^{-1}([t^\rho])$ and $X = S^-_{\rho,\{i\}} \cap \overline{S^+_{\lambda,\{i\}}}$. Then

$$C = Z \cap S^-_\nu \cap (r_{\{i\}})^{-1}([t^\rho]) \quad \text{and} \quad D = r_{\{i\}}\big(Z \cap S^-_\nu\big) \cap S^-_{\rho,\{i\}}$$

are irreducible components of F and X, respectively. Proposition 3.10 implies then that $D = X$ and that the map $h : p \mapsto y_i\big(pt^{-\langle \alpha_i,\rho\rangle}\big)[t^\rho]$ from $\mathscr{K}$ to $\mathscr{M}_{\{i\}}$ induces a homeomorphism from $\mathbb{C}[t^{-1}]^+_n$ onto D.

Let $k \in \mathbb{N}$ and set $D' = S^-_{\rho,\{i\}} \cap \overline{S^+_{\lambda+k\alpha_i^\vee,\{i\}}}$. Then h induces a homeomorphism from $\mathbb{C}[t^{-1}]^+_{n+k}$ onto D'. Since $-\langle \alpha_i, \rho\rangle = \varepsilon_i(Z) + n$, it follows that the map $g : (p,x) \mapsto y_i\big(pt^{\varepsilon_i(Z)}\big)x$ from $\mathscr{K} \times \mathscr{M}_{\{i\}}$ to $\mathscr{M}_{\{i\}}$ induces a homeomorphism from $\mathbb{C}[t^{-1}]^+_k \times D$ onto D'. Now set

$$Z' = \tilde{e}_i^k(Z), \quad \dot{Z} = Z \cap S^-_\nu \cap \left(\overline{s_i}\, S^+_{s_i^{-1}\rho}\right) \quad \text{and} \quad \dot{Z}' = Z' \cap S^-_\nu \cap \left(\overline{s_i}\, S^+_{s_i^{-1}\rho}\right).$$

The proof of Theorem 4.1 gives us $\dot{Z} = \tilde{C} \cap (r_{\{i\}})^{-1}(D)$ and $\dot{Z}' = \tilde{C} \cap (r_{\{i\}})^{-1}(D')$. Consider the map $f : (p,z) \mapsto y_i\big(pt^{\varepsilon_i(Z)}\big)z$ from $\mathscr{K} \times \mathscr{G}$ to $\mathscr{G}$. Using that the action of the group $y_i(\mathscr{K})$ stabilizes $\tilde{C}$ and commutes with the parabolic retraction $r_{\{i\}}$, we conclude that f induces a homeomorphism from $\mathbb{C}[t^{-1}]^+_k \times \dot{Z}$ onto $\dot{Z}'$. The first assertion in Statement (ii) is thus shown.

Suppose now that $\lambda = \rho$, and denote by $\mathscr{N}$ the connected component of $\mathscr{M}_{\{i\}}$ that contains $[t^\rho]$. By Lemma 3.12, $r_{\{i\}}(Z) \cap \mathscr{N}$ is contained in both $\overline{S^+_{\lambda,\{i\}}}$ and $\overline{S^-_{\rho,\{i\}}}$, hence in their intersection $\big\{[t^\rho]\big\}$. This shows that $r_{\{i\}}(Z) \cap S^+_{\lambda,\{i\}} = \big\{[t^\rho]\big\} = r_{\{i\}}(Z) \cap S^-_{\rho,\{i\}}$, and thus that $Z \cap S^+_\lambda = Z \cap \left(\overline{s_i}\, S^+_{s_i^{-1}\rho,\{i\}}\right)$, again by Lemma 3.12. Therefore $\dot{Z} = Z \cap S^-_\nu \cap S^+_\lambda$. Now if $k = 0$, then

$$f\big(\mathbb{C}[t^{-1}]^*_k \times \dot{Z}\big) = \dot{Z} = Z \cap S^-_\nu \cap S^+_\lambda = Z' \cap S^-_\nu \cap S^+_{\mu_+(Z')};$$

and if $k > 0$, then by Proposition 3.10,

$$g(\mathbb{C}[t^{-1}]^*_k \times D) = h(\mathbb{C}[t^{-1}]^*_{n+k}) = S^-_{\rho,\{i\}} \cap S^+_{\lambda+k\alpha_i^\vee,\{i\}} = D' \cap S^+_{\lambda+k\alpha_i^\vee,\{i\}},$$

and thus by Lemma 3.12,

$$f(\mathbb{C}[t^{-1}]^*_k \times \dot{Z}) = \dot{Z}' \cap S^+_{\mu_+(Z')},$$

which is an open subset of $Z' \cap S^-_\nu \cap S^+_{\mu_+(Z')}$. This concludes the proof of Statement (ii).

We now turn to the proof of Statement (i). We first observe that $h(\mathscr{O}) = \big\{[t^\rho]\big\}$. Let $p \in \mathscr{O}$ and write $pt^{-n} = q + r$, with $q \in \mathbb{C}[t^{-1}]^+_n$ and $r \in \mathscr{O}$. For each $x \in D$, we can find $s \in \mathbb{C}[t^{-1}]_n^+$ such that $x = h(s)$, and then

$$y_i\big(pt^{\varepsilon_i(Z)}\big) \cdot x = y_i\big((q+r)t^{-\langle\alpha_i,\rho\rangle}\big) \cdot h(s) = h(q+r+s) = h(q+s) \cdot h(r) = h(q+s)$$

belongs to D. The action of $y_i\big(pt^{\varepsilon_i(Z)}\big)$ therefore stabilizes D. Since it stabilizes also $\tilde{C}$ and commutes with $r_{\{i\}}$, it stabilizes $\dot{Z}$. We conclude that it stabilizes $\overline{\dot{Z}} = Z$.

Using this, we see that

$$\big\{y_i(p)\, z \bigm| z \in Z \text{ and } p \in \mathscr{K}^\times \text{ such that } \operatorname{val}(p) = -k + \varepsilon_i(Z)\big\} = f\big(\mathbb{C}[t^{-1}]_k^* \times Z\big).$$

This set has the same closure as $f(\mathbb{C}[t^{-1}]_k^* \times \dot{Z})$, namely Z'. This completes the proof of Statement (i). □

We now recall the definition of the string parameter of an element in $\mathbf{B}(-\infty)$. To each sequence $\mathbf{i} = (i_1, \ldots, i_l)$ of elements of I, we associate an injective map $\Psi_{\mathbf{i}}$ from $\mathbf{B}(-\infty)$ to $\mathbb{N}^l \times \mathbf{B}(-\infty)$ by the following recursive definition:

- If $l = 0$, then $\Psi_{()} : \mathbf{B}(-\infty) \to \mathbf{B}(-\infty)$ is the identity map.
- If $l > 1$ and $b \in \mathbf{B}(-\infty)$, then $\Psi_{\mathbf{i}}(b) = \big(c_1, \Psi_{\mathbf{j}}(\tilde{f}_{i_1}^{c_1} b)\big)$, where $c_1 = \varphi_{i_1}(b)$ and $\mathbf{j} = (i_2, \ldots, i_l)$.

To the sequence $\mathbf{i}$, one also associates recursively an element $w_{\mathbf{i}} \in W$ by setting $w_{()} = 1$ and asking that $w_{\mathbf{i}}$ is the longest of the two elements $w_{\mathbf{j}}$ and $s_{i_1} w_{\mathbf{j}}$, where $\mathbf{j} = (i_2, \ldots, i_l)$ as above. Finally, one defines the subset

$$\mathbf{B}(-\infty)_{\mathbf{i}} = \big\{b \in \mathbf{B}(-\infty) \bigm| \exists (k_1, \ldots, k_l) \in \mathbb{N}^l,\ b = \tilde{e}_{i_1}^{k_1} \cdots \tilde{e}_{i_l}^{k_l} 1\big\}.$$

From Kashiwara's work on Demazure modules [17] (see also Section 12.4 in [18]), one deduces that:

- $\mathbf{B}(-\infty)_{\mathbf{i}}$ depends only on $w_{\mathbf{i}}$ and not on $\mathbf{i}$.
- If $\mathbf{i}$ is a reduced decomposition of the longest element w_0 of W, then $\mathbf{B}(-\infty)_{\mathbf{i}} = \mathbf{B}(-\infty)$.
- $\mathbf{B}(-\infty)_{\mathbf{i}}$ is the set of all $b \in \mathbf{B}(-\infty)$ such that $\Psi_{\mathbf{i}}(b)$ has the form $\big(\mathbf{c}_{\mathbf{i}}(b), 1\big)$ for a certain $\mathbf{c}_{\mathbf{i}}(b) \in \mathbb{N}^l$.

The map $\mathbf{c}_{\mathbf{i}} : \mathbf{B}(-\infty)_{\mathbf{i}} \to \mathbb{N}^l$ implicitly defined in the third item above is called the string parametrization in the direction $\mathbf{i}$. Its image is called the string cone and is denoted by $\mathcal{C}_{\mathbf{i}}$.

The next theorem affords an explicit description of the MV cycle $\Xi(t_0 \otimes b)$ from the string parameter of b. It shows in particular that MV cycles are rational varieties, a fact already known from Gaussent and Littelmann's work (see for instance Theorem 4 in [11]).

Theorem 4.6. *Let $\mathbf{i} \in I^l$ and $b \in \mathbf{B}(-\infty)_{\mathbf{i}}$. Write $\mathbf{c}_{\mathbf{i}}(b) = (c_1, \ldots, c_l)$, set*

$$e_j = -\sum_{k=j+1}^{l} c_k \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle$$

for each $j \in \{1, \ldots, l\}$, and set $Z = \Xi(t_0 \otimes b)$. Then the map

$$(p_1, \ldots, p_l) \mapsto \big[y_{i_1}(p_1 t^{e_1}) \cdots y_{i_l}(p_l t^{e_l})\big]$$

is an embedding of $\mathbb{C}[t^{-1}]_{c_1}^* \times \cdots \times \mathbb{C}[t^{-1}]_{c_l}^*$ as an open and dense subset of $Z \cap S_{\mu_+(Z)}^+ \cap S_{\mu_-(Z)}^-$.

Proof. We use induction on the length l of the sequence $\mathbf{i}$. The assertion certainly holds when $l=0$, for in this case $b=1$ and thus $\tilde{Y}_{\mathbf{i},\mathbf{c}} = \{[t^0]\}$.

Now let $\mathbf{i} \in I^l$ and $b \in \mathbf{B}(-\infty)_{\mathbf{i}}$. We write $\mathbf{i} = (i_1, \ldots, i_l)$ and $\mathbf{c}_{\mathbf{i}}(b) = (c_1, \ldots, c_l)$. We set $\mathbf{i}' = (i_2, \ldots, i_l)$ and $b' = \tilde{f}_{i_1}^{c_1} b$. We will apply the induction hypothesis to $\mathbf{i}'$ and b'.

We note that $\varphi_{i_1}(b') = 0$ and that $\mathbf{c}_{\mathbf{i}'}(b') = (c_2, \ldots, c_l)$. For $j \in \{1, \ldots, l\}$, we set $e_j = -\sum_{k=j+1}^{l} c_k \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle$. We set $Z = \Xi(t_0 \otimes b)$ and $Z' = \Xi(t_0 \otimes b')$; then $Z = \tilde{e}_{i_1}^{c_1}(Z')$, for Ξ is an isomorphism of crystals. The equality $\varphi_{i_1}(b') = 0$ implies that

$$\varepsilon_{i_1}(Z') = \varepsilon_{i_1}(t_0 \otimes b') = \varepsilon_{i_1}(b') = -\langle \alpha_{i_1}, \mathrm{wt}(b') \rangle = e_1.$$

Thanks to (4.2), the equality $\varphi_{i_1}(b') = 0$ also leads to

$$\mu_+(Z') = s_i \mu_+\big(\overline{s_i}^{-1} Z'\big).$$

Proposition 4.5 (ii) thus asserts that the map $(p, z) \mapsto y_{i_1}(pt^{e_1})z$ is a homeomorphism from $\mathbb{C}[t^{-1}]^*_{c_1} \times \big(Z' \cap S^+_{\mu_+(Z')} \cap S^-_{\mu_-(Z')}\big)$ onto an open and dense subset of $Z \cap S^+_{\mu_+(Z)} \cap S^-_{\mu_-(Z)}$. Theorem 4.6 then follows immediately by induction. □

4.3. **The subsets $\tilde{Y}_{\mathbf{i},\mathbf{c}}$.** Given a sequence $\mathbf{i} = (i_1, \ldots, i_l)$ of elements of I and a sequence $\mathbf{p} = (p_1, \ldots, p_l)$ of elements of $\mathscr{K}$, we form the element

$$y_{\mathbf{i}}(\mathbf{p}) = y_{i_1}(p_1) \cdots y_{i_l}(p_l).$$

Given the sequence $\mathbf{i}$ as above and a sequence $\mathbf{c} = (c_1, \ldots, c_l)$ of integers, we set

$$\tilde{Y}_{\mathbf{i},\mathbf{c}} = \big\{[y_{\mathbf{i}}(\mathbf{p})] \bigm| \mathbf{p} \in (\mathscr{K}^\times)^l \text{ such that } \mathrm{val}(p_j) = \tilde{c}_j\big\},$$

where $\tilde{c}_j = -c_j - \sum_{k=j+1}^{l} c_k \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle$.

Proposition 4.7. (i) *Let $\mathbf{i} \in I^l$, let $b \in \mathbf{B}(-\infty)_{\mathbf{i}}$ and set $\mathbf{c} = \mathbf{c}_{\mathbf{i}}(b)$. Then the MV cycle $\Xi(t_0 \otimes b)$ is the closure of $\tilde{Y}_{\mathbf{i},\mathbf{c}}$.*

(ii) *Let $\mathbf{i} = (i_1, \ldots, i_N)$ be a reduced decomposition of w_0 and let $\mathbf{c} = (c_1, \ldots, c_N)$ be an element in $\mathbb{Z}^N$. Let Z be the closure of $\tilde{Y}_{\mathbf{i},\mathbf{c}}$ and let λ be the coweight $c_1 \alpha_{i_1}^\vee + \cdots + c_N \alpha_{i_N}^\vee$. Then Z is an MV cycle, $\mu_-(Z) = 0$ and $\mu_+(Z) \geqslant \lambda$. Moreover, $\mu_+(Z) = \lambda$ if and only if $\mathbf{c} \in \mathcal{C}_{\mathbf{i}}$.*

Many assertions of this proposition follow easily from Proposition 4.5 and Theorem 4.6. The truly new points are the inequality $\mu_+(Z) \geqslant \lambda$ in Statement (ii) and the fact that the equality $\mu_+(Z) = \lambda$ holds only if $\mathbf{c} \in \mathcal{C}_{\mathbf{i}}$. We will ground our proof on the notion of $\mathbf{i}$-trail in Berenstein and Zelevinsky's work [6]. We first recall what it is about.

We denote the differential at 0 of the one-parameter subgroups x_{α_i} and $x_{-\alpha_i}$ by E_i and F_i, respectively; they are elements of the Lie algebra of G. Let $\mathbf{i} = (i_1, \ldots, i_N)$ be a reduced decomposition of w_0, let γ and δ be two weights in X, let V be a rational G-module, and write $V = \bigoplus_{\eta \in X} V_\eta$ for its decomposition in weight subspaces. According to Definition 2.1 in [6], an $\mathbf{i}$-trail from γ to δ in V is a sequence of weights $\pi = (\gamma = \gamma_0, \gamma_1, \ldots, \gamma_N = \delta)$ such that each difference $\gamma_{j-1} - \gamma_j$ has the form $n_j \alpha_{i_j}$ for some non-negative integer n_j, and such that $E_{i_1}^{n_1} \cdots E_{i_N}^{n_N}$ defines a non-zero map from V_δ to V_γ. To such an $\mathbf{i}$-trail π, Berenstein and Zelevinsky associate the sequence of integers $d_j(\pi) = \langle \gamma_{j-1} + \gamma_j, \alpha_{i_j}^\vee \rangle / 2$.

Assume moreover that the derived group (G, G) is simply connected. Then $\mathbb{Z}\Phi^\vee$ is a direct summand of the lattice Λ, and we can therefore find fundamental weights ω_i, that is, elements of X such that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$. For each $i \in I$, we can thus speak of the simple rational G-module with highest weight ω_i, which we denote by $V(\omega_i)$. Then by Theorem 3.10 in [6], the string cone $\mathcal{C}_{\mathbf{i}}$ is the set of all $(c_1, \ldots, c_N) \in \mathbb{Z}^N$ such that $\sum_j d_j(\pi) c_j \geqslant 0$ for any $i \in I$ and any $\mathbf{i}$-trail π from ω_i to $w_0 s_i \omega_i$ in $V(\omega_i)$.

The following lemma explains why $\mathbf{i}$-trails are relevant to our problem.

Lemma 4.8. *Let* $\mathbf{i}$, $\mathbf{c}$, Z *and* λ *be as in the statement of Proposition* 4.7 (ii)*, let* $i \in I$*, and assume that* (G,G) *is simply connected. Then* $\langle \omega_i, \lambda - \mu_+(Z) \rangle$ *is the minimum of the numbers* $\sum_j d_j(\pi) c_j$ *for all weights* $\delta \in X$ *and all* $\mathbf{i}$*-trails* π *from* ω_i *to* δ *in* $V(\omega_i)$*.*

Proof. Let us consider an $\mathbf{i}$-trail $\pi = (\gamma_0, \gamma_1, \ldots, \gamma_N)$ in $V(\omega_i)$ which starts from $\gamma_0 = \omega_i$. Introducing the integers n_j such that $\gamma_{j-1} - \gamma_j = n_j \alpha_{i_j}$, we obtain $\gamma_j = \omega_i - \sum_{k=1}^{j} n_k \alpha_{i_k}$ for each $j \in \{1, \ldots, N\}$ and so

$$d_j(\pi) = \langle \omega_i, \alpha_{i_j}^\vee \rangle - \sum_{k=1}^{j-1} n_k \langle \alpha_{i_k}, \alpha_{i_j}^\vee \rangle - n_j.$$

We then compute

$$\sum_{j=1}^{N} d_j(\pi) c_j - \langle \omega_i, \lambda \rangle = \sum_{j=1}^{N} \left(-n_j - \sum_{k=1}^{j-1} \langle \alpha_{i_k}, \alpha_{i_j}^\vee \rangle n_k \right) c_j = n_1 \tilde{c}_1 + \cdots + n_N \tilde{c}_N,$$

where we set as usual $\tilde{c}_j = -c_j - \sum_{k=j+1}^{N} c_k \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle$ for each $j \in \{1, \ldots, N\}$.

We adopt the notational conventions set up before Proposition 3.5. In particular, we embed $V(\omega_i)$ inside $V(\omega_i) \otimes_{\mathbb{C}} \mathscr{K}$ and we view the latter as a representation of the group $G(\mathscr{K})$. We also consider a non-degenerate contravariant bilinear form $(?,?)$ on $V(\omega_i)$; it is compatible with the decomposition of $V(\omega_i)$ as the sum of its weight subspaces and it satisfies $(v, E_i v') = (F_i v, v')$ for any $i \in I$ and any vectors v and v' in $V(\omega_i)$. We extend the contravariant bilinear form to $V(\omega_i) \otimes_{\mathbb{C}} \mathscr{K}$ by multilinearity.

By Proposition 3.3, $\langle \omega_i, \mu_+(Z) \rangle$ is the maximum of $\langle \omega_i, \nu \rangle$ for those $\nu \in \Lambda$ such that S_ν^+ meets $\tilde{Y}_{\mathbf{c},\mathbf{i}}$. Using Proposition 3.5 (ii), we deduce that

$$\begin{aligned} \langle \omega_i, \mu_+(Z) \rangle &= \max \Big\{ -\operatorname{val}\big(g^{-1} \cdot v_{\omega_i}\big) \;\Big|\; g \in G(\mathscr{K}) \text{ such that } [g] \in \tilde{Y}_{\mathbf{c},\mathbf{i}} \Big\} \\ &= \max \left\{ -\operatorname{val}\big(\big(v, y_{\mathbf{i}}(\mathbf{p})^{-1} \cdot v_{\omega_i}\big)\big) \;\middle|\; \begin{array}{c} v \in V(\omega_i),\ \mathbf{p} \in (\mathscr{K}^\times)^N \\ \text{such that } \operatorname{val}(p_j) = \tilde{c}_j \end{array} \right\}, \end{aligned}$$

where we wrote $\mathbf{p} = (p_1, \ldots, p_N)$ as usual. Moreover, we may ask that the vector v in the last maximum is a weight vector.

Let us denote by M the minimum of the numbers $\sum_j d_j(\pi) c_j$ for all $\mathbf{i}$-trails π in $V(\omega_i)$ which start from ω_i. We expand the product

$$\begin{aligned} y_{\mathbf{i}}(\mathbf{p})^{-1} &= \exp(-p_N F_{i_N}) \cdots \exp(-p_1 F_{i_1}) \\ &= \sum_{n_1, \ldots, n_N \geqslant 0} \frac{(-1)^{n_1 + \cdots + n_N}\ p_1^{n_1} \cdots p_N^{n_N}}{n_1! \cdots n_N!}\ F_{i_N}^{n_N} \cdots F_{i_1}^{n_1} \end{aligned}$$

and we substitute in $\big(v, y_{\mathbf{i}}(\mathbf{p})^{-1} \cdot v_{\omega_i}\big)$: we get a sum of terms of the form

$$\frac{(-1)^{n_1+\cdots+n_N}\ p_1^{n_1}\cdots p_N^{n_N}}{n_1!\cdots n_N!}\ \Big(v, F_{i_N}^{n_N}\cdots F_{i_1}^{n_1}\cdot v_{\omega_i}\Big).$$

If such a term is not zero, then the sequence

$$\pi = (\omega_i,\ \omega_i - n_1\alpha_{i_1},\ \omega_i - n_1\alpha_{i_1} - n_2\alpha_{i_2},\ \ldots,\ \omega_i - n_1\alpha_{i_1} - \cdots - n_N\alpha_{i_N})$$

is an $\mathbf{i}$-trail and the term has valuation

$$n_1\tilde{c}_1 + \cdots + n_N\tilde{c}_N = \sum_{j=1}^{N} d_j(\pi)c_j - \langle\omega_i, \lambda\rangle \geqslant M - \langle\omega_i, \lambda\rangle.$$

Therefore the valuation of $(v, y_{\mathbf{i}}(\mathbf{p})^{-1} \cdot v_{\omega_i})$ is greater or equal to $M - \langle\omega_i, \lambda\rangle$ for any $v \in V(\omega_i)$; we conclude that $\langle\omega_i, \mu_+(Z)\rangle \leqslant \langle\omega_i, \lambda\rangle - M$.

Conversely, let π be an $\mathbf{i}$-trail in $V(\omega_i)$ which starts from ω_i and which is such that $\sum_j d_j(\pi)c_j = M$. With this $\mathbf{i}$-trail come the numbers $n_1, \ldots, n_N$ as before. By definition of an $\mathbf{i}$-trail, there is then a weight vector $v \in V(\omega_i)$ such that

$$\Big(v, F_{i_N}^{n_N}\cdots F_{i_1}^{n_1}\cdot v_{\omega_i}\Big) \neq 0.$$

Given $(a_1, \ldots, a_N) \in (\mathbb{C}^\times)^N$, we set $\mathbf{p} = (a_1 t^{\tilde{c}_1}, \ldots, a_N t^{\tilde{c}_N})$ and look at the coefficient f of $t^{M-\langle\omega_i,\lambda\rangle}$ in $\big(v, y_{\mathbf{i}}(\mathbf{p})^{-1} \cdot v_{\omega_i}\big)$. The computation above shows that f is a polynomial in $(a_1, \ldots, a_N)$; it is not zero since the coefficient of $a_1^{n_1}\cdots a_N^{n_N}$ in f is

$$\frac{(-1)^{n_1+\cdots+n_N}}{n_1!\cdots n_N!}\Big(v, F_{i_N}^{n_N}\cdots F_{i_1}^{n_1}\cdot v_{\omega_i}\Big) \neq 0.$$

Therefore there exists $\mathbf{p} \in (\mathscr{K}^\times)^N$ with $\operatorname{val}(p_j) = \tilde{c}_j$ such that $\big(v, y_{\mathbf{i}}(\mathbf{p})^{-1}\cdot v_{\omega_i}\big)$ has valuation $\leqslant M - \langle\omega_i, \lambda\rangle$. It follows that $\langle\omega_i, \mu_+(Z)\rangle \geqslant \langle\omega_i, \lambda\rangle - M$, which completes the proof. □

Proof of Proposition 4.7. Statement (i) is established in the same fashion as Theorem 4.6, using Proposition 4.5 (i) instead of Proposition 4.5 (ii).

Now let $\mathbf{i}$, $\mathbf{c}$, Z and λ be as in Statement (ii). Applying repeatedly Proposition 4.5 (i), one shows easily that Z is an MV cycle. Furthermore by its very definition, $\tilde{Y}_{\mathbf{i},\mathbf{c}}$ is contained in S_0^-; this entails that $\mu_-(Z) = 0$.

If $\mathbf{c}$ is the string in direction $\mathbf{i}$ of an element $b \in \mathbf{B}(-\infty)$, then $Z = \Xi(t_0 \otimes b)$, and thus

$$\mu_+(Z) = \operatorname{wt}(Z) = \operatorname{wt}(t_0 \otimes b) = \operatorname{wt}(b) = \operatorname{wt}\big(\tilde{e}_{i_1}^{c_1}\cdots\tilde{e}_{i_N}^{c_N}1\big) = \lambda.$$

The equality $\mu_+(Z) = \lambda$ holds therefore for each $\mathbf{c} \in \mathcal{C}_{\mathbf{i}}$.

It remains to show that $\mu_+(Z) \geqslant \lambda$ with equality only if $\mathbf{c} \in \mathcal{C}_{\mathbf{i}}$. To establish that, we may assume, without loss of generality, that (G, G) is simply connected, because an isogeny between two connected reductive groups induces a homeomorphism between the neutral connected components of their respective affine Grassmannians. We may then make use of the fundamental weights ω_i and of the G-modules $V(\omega_i)$.

We first observe that $\mu_+(Z) - \mu_-(Z)$, $\mu_-(Z)$ and λ all belong to the coroot lattice $\mathbb{Z}\Phi^\vee$; therefore $\mu_+(Z) - \lambda$ belongs to $\mathbb{Z}\Phi^\vee$. Now let $i \in I$. The sequence

$$\pi = (\omega_i,\ s_{i_1}\omega_i,\ s_{i_2}s_{i_1}\omega_i,\ \ldots,\ w_0\omega_i)$$

is an $\mathbf{i}$-trail in $V(\omega_i)$ for which $d_j(\pi) = 0$ for each j. By Lemma 4.8, we deduce

$$\langle \omega_i, \lambda - \mu_+(Z) \rangle \leqslant \sum_j d_j(\pi) c_j = 0.$$

This is enough to guarantee that $\mu_+(Z) \geqslant \lambda$.

Suppose now that $\mu_+(Z) = \lambda$. Lemma 4.8 then implies that $\sum_j d_j(\pi) c_j \geqslant 0$ for all $i \in I$, all weights $\delta \in X$, and all $\mathbf{i}$-trails π from ω_i to δ in $V(\omega_i)$. In particular, this holds for all $i \in I$ and all $\mathbf{i}$-trails π from ω_i to $w_0 s_i \omega_i$ in $V(\omega_i)$. By Theorem 3.10 in [6], this implies $\mathbf{c} \in \mathcal{C}_{\mathbf{i}}$. □

4.4. **Lusztig's algebraic-geometric parametrization of B.** As we have seen in Section 4.2, the choice of a reduced decomposition $\mathbf{i}$ of w_0 determines a bijection $\mathbf{c}_{\mathbf{i}} : \mathbf{B}(-\infty) \to \mathcal{C}_{\mathbf{i}}$, called the "string parametrization". The decomposition $\mathbf{i}$ also determines a bijection $b_{\mathbf{i}} : \mathbb{N}^N \to \mathbf{B}(-\infty)$, called the "Lusztig parametrization", which reflects Lusztig's original construction [23] of the canonical basis on a combinatorial level. We refer the reader to [24], [29] and Section 3.1 in [6] for additional information on the map $b_{\mathbf{i}}$ and its construction.

The Lusztig parametrizations $b_{\mathbf{i}}$ are convenient because they permit a study of $\mathbf{B}(-\infty)$ by way of numerical data in a fixed domain $\mathbb{N}^N$, but they are not intrinsic, for they depend on the choice of $\mathbf{i}$. To avoid this drawback, Lusztig introduces in [25] a parametrization of $\mathbf{B}(-\infty)$ in terms of closed subvarieties in arc spaces on U^-. We will first recall briefly his construction and then we will explain a relationship with MV cycles. For simplicity, Lusztig restricts himself to the case where G is simply laced, but he explains in the introduction of [25] that his results hold in the general case as well.

Lusztig starts by recalling a general construction. To a complex algebraic variety X and a non-negative integer s, one can associate the space X_s of all jets of curves drawn on X, of order s. In formulas, one looks at the algebra $\mathbb{C}_s = \mathbb{C}[[t]]/(t^{s+1})$ and defines X_s as the set of morphisms from $\operatorname{Spec} \mathbb{C}_s$ to X. If X is smooth of dimension n, then X_s is smooth of dimension $(s+1)n$. There exist morphisms of truncation

$$\cdots \to X_{s+1} \to X_s \to \cdots \to X_1 \to X_0 = X;$$

the projective limit of this inverse system of maps is the space $X(\mathscr{O})$. Finally, the assignment $X \rightsquigarrow X_s$ is functorial, hence X_s is a group as soon as X is one.

Now let $\mathbf{i}$ be a reduced decomposition of w_0. The morphism

$$y_{\mathbf{i}} : (a_1, \ldots, a_N) \mapsto y_{i_1}(a_1) \cdots y_{i_N}(a_N)$$

from $(\mathbb{C})^N$ to U^- gives, by functoriality, a morphism $(y_{\mathbf{i}})_s : (\mathbb{C}_s)^N \to (U^-)_s$. Given an element $\mathbf{d} = (d_1, \ldots, d_N)$ in $\mathbb{N}^N$, we may look at the image of the subset

$$(t^{d_1} \mathbb{C}_s) \times \cdots \times (t^{d_N} \mathbb{C}_s) \subseteq (\mathbb{C}_s)^N$$

by $(y_{\mathbf{i}})_s$. This is a constructible, irreducible subset of $(U^-)_s$. If s is big enough, then the closure of this subset depends only on $b = b_{\mathbf{i}}(\mathbf{d})$ and not on $\mathbf{i}$ and $\mathbf{d}$ individually. (This is Lemma 5.2 of [25]; the precise condition is that s must be strictly larger than $\operatorname{ht}(\operatorname{wt} b)$.) One may therefore denote this closure by $V_{b,s}$; it is a closed irreducible subset of $(U^-)_s$ of codimension $\operatorname{ht}(\operatorname{wt} b)$. Proposition 7.5 in [25] asserts that, moreover, the assignment $b \mapsto V_{b,s}$ is injective for s big enough: there is a constant M depending only on the root system Φ such that

$$\Bigl(V_{b,s} = V_{b',s} \quad \text{and} \quad s > M \operatorname{ht}(\operatorname{wt} b) \Bigr) \quad \Longrightarrow \quad b = b'$$

for any b, $b' \in \mathbf{B}(-\infty)$. Thus $b \mapsto V_{b,s}$ may be seen as a parametrization of $\mathbf{B}(-\infty)$ by closed irreducible subvarieties of $(U^-)_s$.

Our next result shows that Lusztig's construction is related to MV cycles and to Braverman, Finkelberg and Gaitsgory's theorem. We fix a dominant coweight $\lambda \in \Lambda_{++}$. By Proposition 3.2, the map $x \mapsto x \cdot [t^{w_0\lambda}]$ from $G(\mathscr{O})$ to $\mathscr{G}$ factorizes through the reduction map $G(\mathscr{O}) \to G_s$ when s is big enough, defining thus a map

$$\Upsilon_s : \ G_s \to \mathscr{G}, \ x \mapsto x \cdot [t^{w_0\lambda}].$$

On the other hand, we may consider the two embeddings of crystals $\kappa_\lambda : \mathbf{B}(\lambda) \hookrightarrow \mathbf{B}(\infty) \otimes \mathbf{T}_\lambda$ and $\iota_{w_0\lambda} : \mathbf{B}(\lambda) \hookrightarrow \mathbf{T}_{w_0\lambda} \otimes \mathbf{B}(-\infty)$, as in Section 2.2. Finally, the isomorphism $\mathbf{B}(\infty)^\vee \cong \mathbf{B}(-\infty)$ yields a bijection $b \mapsto b^\vee$ from $\mathbf{B}(\infty)$ onto $\mathbf{B}(-\infty)$.

Proposition 4.9. *We adopt the notation above and assume that s is big enough so that the map Υ_s exists and that the closed subsets $V_{b^\vee,s}$ are defined for each $b \otimes t_\lambda$ in the image of κ_λ. Then the diagram*

$$\begin{tikzcd}
\mathbf{B}(\lambda) \arrow[r, hook, "\kappa_\lambda"] \arrow[d, hook, "\iota_{w_0\lambda}"'] & \operatorname{im}(\kappa_\lambda) \arrow[d, "b\otimes t_\lambda \mapsto \overline{\Upsilon_s(V_{b^\vee,s})}"] \\
\mathbf{T}_{w_0\lambda} \otimes \mathbf{B}(-\infty) \arrow[r, "\Xi"'] & \mathscr{Z}
\end{tikzcd}$$

commutes.

Proof. This is a consequence of Proposition 4.7 (i), combined with a result of Morier-Genoud [28]. We first look at the commutative diagram that defines the embedding $\iota_{w_0\lambda}$, namely

$$\begin{tikzcd}
 & \mathbf{B}(\lambda) \arrow[r, "\simeq"] \arrow[dl, hook, "\kappa_\lambda"'] \arrow[dr, hook, "\iota_{w_0\lambda}"] & \mathbf{B}(-w_0\lambda)^\vee \arrow[d, hook] & \mathbf{B}(-w_0\lambda) \arrow[l, dashed] \arrow[d, hook, "\kappa_{-w_0\lambda}"] \\
\mathbf{B}(\infty) \otimes \mathbf{T}_\lambda & & \mathbf{T}_{w_0\lambda} \otimes \mathbf{B}(-\infty) & \mathbf{B}(\infty) \otimes \mathbf{T}_{-w_0\lambda}. \arrow[l, dashed]
\end{tikzcd}$$

The two arrows in broken line on this diagram are the maps $b \mapsto b^\vee$; they are not morphisms of crystals. The map from $\mathbf{B}(-w_0\lambda)$ to $\mathbf{B}(\lambda)$ obtained by composing the two arrows on the top line intertwines the raising operators $\tilde{e}_i$ with their lowering counterparts $\tilde{f}_i$ and sends the highest weight element of $\mathbf{B}(-w_0\lambda)$ to the lowest weight element of $\mathbf{B}(\lambda)$; it therefore coincides with the map denoted by $\Phi_{-w_0\lambda}$ in [28].

Now let $b \in \mathbf{B}(\lambda)$. We write $\kappa_\lambda(b) = b' \otimes t_\lambda$ and $\kappa_{-w_0\lambda}(\Phi^{-1}_{-w_0\lambda}(b)) = b'' \otimes t_{-w_0\lambda}$; thus $\iota_{w_0\lambda}(b) = t_{w_0\lambda} \otimes (b'')^\vee$. We choose a reduced decomposition $\mathbf{i}$ of w_0 and we set $\mathbf{c} = (c_1, \ldots, c_N) = \mathbf{c_i}((b'')^\vee)$ and $(d_1, \ldots, d_N) = b_{\mathbf{i}}^{-1}((b')^\vee)$. We additionally set $\tilde{c}_j = -c_j - \sum_{k=j+1}^{l} c_k \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle$ for each $j \in \{1, \ldots, N\}$. Corollary 3.5 in [28] then asserts that $d_j = \langle \alpha_{i_j}, -w_0\lambda \rangle + \tilde{c}_j$ for all j. Now comparing the definition of Lusztig's subset $V_{(b')^\vee,s}$ with the definition of $\tilde{Y}_{\mathbf{i},\mathbf{c}}$ and using Proposition 4.7 (i), we compute

$$\overline{V_{(b')^\vee,s} \cdot [t^{w_0\lambda}]} = \overline{t^{w_0\lambda} \cdot \tilde{Y}_{\mathbf{i},\mathbf{c}}} = t^{w_0\lambda} \cdot \Xi\big(t_0 \otimes (b'')^\vee\big) = \Xi\big(t_{w_0\lambda} \otimes (b'')^\vee\big) = (\Xi \circ \iota_{w_0\lambda})(b).$$

$\square$

4.5. **Link with Kamnitzer's construction.** Let $b \in \mathbf{B}(-\infty)$ and let $\mathbf{i}$ be a reduced decomposition of w_0. Theorem 4.6 explains how to construct an open and dense subset in the MV cycle $\Xi(t_0 \otimes b)$ when one knows the string parameter $\mathbf{c_i}(b)$. In his work on MV polytopes, Kamnitzer [13] presents a similar result, which provides a dense subset of $\Xi(t_0 \otimes b)$ from the datum of the Lusztig parameter $b_{\mathbf{i}}^{-1}(b)$. These two results are identical: indeed, Kamnitzer's result and Proposition 4.7 (i) can be quickly derived one from the other. This section, which does not contain any formalized statement, aims at explaining how.

Our main tool here is Berenstein, Fomin and Zelevinsky's work. In a series of papers (among which [4, 5, 6]), these three authors devise an elegant method that yields all transition maps between the different parametrizations of $\mathbf{B}(-\infty)$ we have met, namely the maps

$$b_{\mathbf{j}}^{-1} \circ b_{\mathbf{i}} : \mathbb{N}^N \to \mathbb{N}^N, \quad \mathbf{c_j} \circ b_{\mathbf{i}} : \mathbb{N}^N \to \mathcal{C}_{\mathbf{j}}, \quad b_{\mathbf{j}}^{-1} \circ \mathbf{c_i}^{-1} : \mathcal{C}_{\mathbf{i}} \to \mathbb{N}^N, \quad \mathbf{c_j} \circ \mathbf{c_i}^{-1} : \mathcal{C}_{\mathbf{i}} \to \mathcal{C}_{\mathbf{j}},$$

where $\mathbf{i}$ and $\mathbf{j}$ are two reduced decomposition of w_0. In recalling their results hereafter, we will slightly modify their notation; our modifications simplify the presentation, perhaps at the price of the loss of positivity results.

We first alter the string parameter $\mathbf{c_i}$ by defining a map $\tilde{\mathbf{c}}_{\mathbf{i}}$ from $\mathbf{B}(-\infty)$ to $\mathbb{Z}^N$ as follows: an element $b \in \mathbf{B}(-\infty)$ with string parameter $\mathbf{c_i}(b) = (c_1, \ldots, c_N)$ in direction $\mathbf{i}$ is sent to the N-tuple $(\tilde{c}_1, \ldots, \tilde{c}_N)$, where $\tilde{c}_j = -c_j - \sum_{k=j+1}^N c_k \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle$. We denote the image of this map $\tilde{\mathbf{c}}_{\mathbf{i}}$ by $\tilde{\mathcal{C}}_{\mathbf{i}}$.

Let $\mathbf{i} = (i_1, \ldots, i_l)$ be a sequence of elements of I and let $\mathbf{a} = (a_1, \ldots, a_l)$ be a sequence of elements of $\mathbb{C}^\times$. Assuming that the product $s_{i_1} \cdots s_{i_l}$ is a reduced decomposition of an element $w \in W$, Theorem 1.2 in [5] implies the existence of an element $z_{\mathbf{i}}(\mathbf{a})$ in U^- whose image in $B^+ \backslash G$ is the same as $y_{\mathbf{i}}(\mathbf{a})\, \overline{w}^{-1}$; this theorem also implies that if $\mathbf{i}$ is a reduced decomposition of w_0, then the map $z_{\mathbf{i}}$ is a birational morphism from $(\mathbb{C}^\times)^N$ to U^-. Now under the same assumption, the map $y_{\mathbf{i}}$ is a birational morphism from $\mathbb{C}^N$ to U^-. If $\mathbf{i}$ and $\mathbf{j}$ are both reduced decompositions of w_0, we therefore get the birational maps

$$z_{\mathbf{j}}^{-1} \circ z_{\mathbf{i}}, \quad y_{\mathbf{j}}^{-1} \circ z_{\mathbf{i}}, \quad z_{\mathbf{j}}^{-1} \circ y_{\mathbf{i}} \quad \text{and} \quad y_{\mathbf{j}}^{-1} \circ y_{\mathbf{i}} \tag{4.4}$$

from $\mathbb{C}^N$ to itself. After extension of the base field, we may view them as birational maps from $\mathscr{K}^N$ to itself.

Now we need to define the process of tropicalization. Here we depart from Berenstein, Fomin and Zelevinsky's purely algebraic method based on total positivity and semifields and adopt a more pedestrian approach.

Let k and l be two positive integers and let $\mathbf{f} : \mathscr{K}^k \to \mathscr{K}^l$ be a rational map, represented as a sequence $(f_1, \ldots, f_l)$ of rational functions in k indeterminates. These indeterminates are collectively denoted as a sequence $\mathbf{p} = (p_1, \ldots, p_k)$. We suppose that no component f_j vanishes identically. Now choose $j \in \{1, \ldots, l\}$ and $\mathbf{m} = (m_1, \ldots, m_k) \in \mathbb{Z}^k$. There exists a non-empty (Zariski) open subset $\Omega \subseteq (\mathbb{C}^\times)^k$ such that the valuation of $f_j(a_1 t^{m_1}, \ldots, a_k t^{m_k})$ is a constant $\hat{f}_j$, independent on the point $\mathbf{a} = (a_1, \ldots, a_k)$ in Ω. (It is here implicitly understood that if $\mathbf{a} \in \Omega$, then neither the numerator nor the denominator of the rational function f_j vanishes after substitution.) The term of lowest degree in $f_j(a_1 t^{m_1}, \ldots, a_k t^{m_k})$ may then be written $\bar{f}_j(\mathbf{a}) t^{\hat{f}_j}$, where $\bar{f}_j$ is a rational function with complex coefficients in the indeterminates $a_1, \ldots, a_k$. Of course, $\hat{f}_j$ and $\bar{f}_j$ depend on the choice of $\mathbf{m} \in \mathbb{Z}^k$, but the open subset Ω may be chosen to meet the demand simultaneously for all $\mathbf{m}$. Indeed, as we make the substitution $p_i = a_i t^{m_i}$, each monomial in the indeterminates $p_1, \ldots, p_k$ in the numerator or in the denominator of f_j becomes a non-zero element of $\mathscr{K}$. To find the term $\bar{f}_j(\mathbf{a})t^{\hat{f}_j}$ of lowest degree in $f_j(a_1 t^{m_1}, \ldots, a_k t^{m_k})$, we collect the monomials in the numerator of f_j that get minimal valuation, and likewise in the denominator. The rôle of the condition $\mathbf{a} \in \Omega$ is to ensure that no accidental cancellation occurs when we make the sum of these monomials, in the numerator as well as in the denominator. Since there are only finitely many monomials, there are only finitely many possibilities for accidental cancellations, hence finitely many conditions on $\mathbf{a}$ to be prescribed by Ω. Moreover, monomials in the numerator or in the denominator of f_j are selected or discarded according to their valuation, and we can divide $\mathbb{R}^k$ into finitely many regions, say $\mathbb{R}^k = D^{(1)} \sqcup \cdots \sqcup D^{(t)}$, so that the set of selected monomials depends only on the domain $D^{(r)}$ to which $\mathbf{m}$ belongs. Since the valuation of each monomial depends affinely on $\mathbf{m}$, the regions $D^{(1)}, \ldots, D^{(t)}$ are indeed intersections of affine hyperplanes and open affine half-spaces, hence are locally closed, convex and polyhedral. For the same reason, $\hat{f}_j$ depends affinely on $\mathbf{m}$ in each region $D^{(r)}$; for its part, $\bar{f}_j$ remains constant when $\mathbf{m}$ varies inside a region $D^{(r)}$. Finally, we note that the choice of the domain $\Omega \subseteq (\mathbb{C}^\times)^k$, the decomposition $\mathbb{R}^k = D^{(1)} \sqcup \cdots \sqcup D^{(t)}$ and the reduction $f_j \mapsto (\hat{f}_j, \bar{f}_j)$ may be carried out for all $j \in \{1, \ldots, l\}$ at the same time. In particular, each $\mathbf{m} \in \mathbb{Z}^k$ yields a tuple $\hat{\mathbf{f}} = (\hat{f}_1, \ldots, \hat{f}_l)$ of integers and a rational map $\bar{\mathbf{f}} = (\bar{f}_1, \ldots, \bar{f}_l)$ from $\mathbb{C}^k$ to $\mathbb{C}^l$. We summarize these observations in a formalized statement:

Let $\mathbf{f} : \mathscr{K}^k \to \mathscr{K}^l$ be a rational map, without identically vanishing component. Then there exists a partition $\mathbb{R}^k = D^{(1)} \sqcup \cdots \sqcup D^{(t)}$ of $\mathbb{R}^k$ into finitely many locally closed polyhedral convex subsets, there exist affine maps $\hat{\mathbf{f}}^{(1)}, \ldots, \hat{\mathbf{f}}^{(t)} : \mathbb{R}^k \to \mathbb{R}^l$, there exist rational maps $\bar{\mathbf{f}}^{(1)}, \ldots, \bar{\mathbf{f}}^{(t)} : \mathbb{C}^k \to \mathbb{C}^l$, and there exists an open subset $\Omega \subseteq (\mathbb{C}^\times)^k$ with the following property: for each $r \in \{1, \ldots, t\}$, each lattice point $\mathbf{m}$ in $D^{(r)} \cap \mathbb{Z}^k$, each point $\mathbf{a} \in \Omega$, and each sequence $\mathbf{p} \in (\mathscr{K}^\times)^k$ such that the lower degree term of p_i is $a_i t^{m_i}$, the map $\mathbf{f}$ has a well-defined value in $(\mathscr{K}^\times)^l$ at $\mathbf{p}$, the map $\bar{\mathbf{f}}^{(r)}$ has a well-defined value in $(\mathbb{C}^\times)^l$ at $\mathbf{a}$, and the term of lower degree of $f_j(\mathbf{p})$ has valuation $\hat{f}_j^{(r)}(\mathbf{m})$ and coefficient $\bar{f}_j^{(r)}(\mathbf{a})$.

We define the tropicalization of $\mathbf{f}$ as the map $\mathbf{f}^{\mathrm{trop}} : \mathbb{R}^k \to \mathbb{R}^l$ whose restriction to each $D^{(r)}$ coincides with the restriction of the corresponding $\hat{\mathbf{f}}^{(r)}$; this is a continuous piecewise affine map. If the rational map $\mathbf{f}$ we started with has complex coefficients (that is, if it comes from a rational map from $\mathbb{C}^k$ to $\mathbb{C}^l$ by extension of the base field), then the convex subsets $D^{(r)}$ are cones and the affine maps $\hat{\mathbf{f}}^{(r)}$ are linear.

With this notation and this terminology, Theorems 5.2 and 5.7 in [6] implies that the maps

$$b_{\mathbf{j}}^{-1} \circ b_{\mathbf{i}} : \mathbb{N}^N \to \mathbb{N}^N, \quad \tilde{\mathbf{c}}_{\mathbf{j}} \circ b_{\mathbf{i}} : \mathbb{N}^N \to \tilde{\mathcal{C}}_{\mathbf{j}}, \quad b_{\mathbf{j}}^{-1} \circ \tilde{\mathbf{c}}_{\mathbf{i}}^{-1} : \tilde{\mathcal{C}}_{\mathbf{i}} \to \mathbb{N}^N, \quad \tilde{\mathbf{c}}_{\mathbf{j}} \circ \tilde{\mathbf{c}}_{\mathbf{i}}^{-1} : \tilde{\mathcal{C}}_{\mathbf{i}} \to \tilde{\mathcal{C}}_{\mathbf{j}}$$

are restrictions of the tropicalizations of the maps in (4.4).

One may here observe a hidden symmetry. Using the equality $\overline{w_0}^2 = (-1)^{2\rho^\vee}$, where $2\rho^\vee$ is the sum of all positive coroots in $\Phi_+^\vee$, one checks that the birational maps $y_{\mathbf{j}}^{-1} \circ z_{\mathbf{i}}$ and $z_{\mathbf{j}}^{-1} \circ y_{\mathbf{i}}$ are equal. These maps have therefore the same tropicalization. In other words, $\tilde{\mathbf{c}}_{\mathbf{j}} \circ b_{\mathbf{i}}$ and $b_{\mathbf{j}}^{-1} \circ \tilde{\mathbf{c}}_{\mathbf{i}}^{-1}$ are given by the same piecewise affine formulas. The sentence following Theorem 3.8 in [6] seems to indicate that this fact has escaped observation up to now.

In [13], Kamnitzer introduces subsets $A^{\mathbf{i}}(n_\bullet)$ in $\mathscr{G}$, where $\mathbf{i}$ is a reduced decomposition of w_0 and $n_\bullet \in \mathbb{N}^N$. Combining Theorem 4.7 in [14] with the proof of Theorem 3.1 in [13], one can see that $\Xi(t_0 \otimes b_{\mathbf{i}}(n_\bullet))$ is the closure of $A^{\mathbf{i}}(n_\bullet)$. On the other hand, Theorem 4.5 in [13] says that

$$A^{\mathbf{i}}(n_\bullet) = \bigl\{[z_{\mathbf{i}}(\mathbf{q})] \bigm| \mathbf{q} = (q_1, \dots, q_N) \in (\mathscr{K}^\times)^N \text{ such that } \operatorname{val}(q_j) = n_j\bigr\}.$$

Now fix $b \in \mathbf{B}(-\infty)$ and a reduced decomposition $\mathbf{i}$ of w_0. Call $\tilde{\mathbf{c}} = (\tilde{c}_1, \dots, \tilde{c}_N)$ the modified string parameter $\tilde{\mathbf{c}}_{\mathbf{i}}(b)$ of b in direction $\mathbf{i}$ and call $n_\bullet = (n_1, \dots, n_N)$ the Lusztig parameter $b_{\mathbf{i}}^{-1}(b)$ of b with respect to $\mathbf{i}$. The rational maps $\mathbf{f} = z_{\mathbf{i}}^{-1} \circ y_{\mathbf{i}}$ and $\mathbf{g} = y_{\mathbf{i}}^{-1} \circ z_{\mathbf{i}}$ are mutually inverse birational maps from $\mathscr{K}^N$ to itself, and by Berenstein and Zelevinsky's theorem,

$$\mathbf{f}^{\mathrm{trop}}(\tilde{\mathbf{c}}) = n_\bullet \quad \text{and} \quad \mathbf{g}^{\mathrm{trop}}(n_\bullet) = \tilde{\mathbf{c}}.$$

The analysis that we made to define the tropicalization of $\mathbf{f}$ and $\mathbf{g}$ shows the existence of open subsets Ω and Ω' of $(\mathbb{C}^\times)^N$ and of rational maps $\bar{\mathbf{f}}$ and $\bar{\mathbf{g}}$ from $\mathbb{C}^N$ to itself such that:

- For each $\mathbf{a} \in \Omega$ and $\mathbf{b} \in \Omega'$, $\bar{\mathbf{f}}(\mathbf{a})$ and $\bar{\mathbf{g}}(\mathbf{b})$ have well-defined values in $(\mathbb{C}^\times)^N$.
- For any N-tuple $\mathbf{p}$ of Laurent series whose terms of lower degree are $a_1 t^{\tilde{c}_1}$, $\dots$, $a_N t^{\tilde{c}_N}$ with $(a_1, \dots, a_N) \in \Omega$, the evaluation $\mathbf{f}(\mathbf{p})$ is a well-defined element $\mathbf{q}$ of $(\mathscr{K}^\times)^N$; moreover, the lower degree terms of the components of $\mathbf{q}$ are $\bar{f}_1(\mathbf{a}) t^{n_1}$, $\dots$, $\bar{f}_N(\mathbf{a}) t^{n_N}$.
- For any N-tuple $\mathbf{q}$ of Laurent series whose terms of lower degree are $b_1 t^{n_1}$, $\dots$, $b_N t^{n_N}$ with $(b_1, \dots, b_N) \in \Omega'$, the evaluation $\mathbf{g}(\mathbf{q})$ is a well-defined element $\mathbf{p}$ of $(\mathscr{K}^\times)^N$; moreover, the lower degree terms of the components of $\mathbf{p}$ are $\bar{g}_1(\mathbf{b}) t^{\tilde{c}_1}$, $\dots$, $\bar{g}_N(\mathbf{b}) t^{\tilde{c}_N}$.

Because $\mathbf{f}$ and $\mathbf{g}$ are mutually inverse birational maps, so are $\bar{\mathbf{f}}$ and $\bar{\mathbf{g}}$. One can then assume that these two latter maps are mutually inverse isomorphisms between Ω and Ω', by shrinking these open subsets if necessary. Thus $\mathbf{f}$ and $\mathbf{g}$ set up a bijective correspondence between

$$\widehat{\Omega} = \left\{ \mathbf{p} \in (\mathscr{K}^\times)^N \;\middle|\; \begin{array}{c} \text{each } p_j \text{ has lower degree term} \\ a_j t^{\tilde{c}_j} \text{ with } (a_1, \dots, a_N) \in \Omega \end{array} \right\}$$

and

$$\widehat{\Omega}' = \left\{ \mathbf{q} \in (\mathscr{K}^\times)^N \;\middle|\; \begin{array}{c} \text{each } q_j \text{ has lower degree term} \\ b_j t^{n_j} \text{ with } (b_1, \dots, b_N) \in \Omega' \end{array} \right\}.$$

In other words, to each $\mathbf{p} \in \widehat{\Omega}$ corresponds a $\mathbf{q} \in \widehat{\Omega}'$ such that $y_{\mathbf{i}}(\mathbf{p}) = z_{\mathbf{i}}(\mathbf{q})$, and conversely. This shows the equality

$$\bigl\{[y_{\mathbf{i}}(\mathbf{p})] \bigm| \mathbf{p} \in \widehat{\Omega}\bigr\} = \bigl\{[z_{\mathbf{i}}(\mathbf{q})] \bigm| \mathbf{q} \in \widehat{\Omega}'\bigr\}.$$

By Kamnitzer's theorem, the right-hand side is dense in $A^{\mathbf{i}}(n_\bullet)$ hence in $\Xi(t_0 \otimes b)$. We thus get another proof of our Proposition 4.7 (i), which claims that $\Xi(t_0 \otimes b)$ is the closure of the left-hand side.

Remark 4.10. We fix a reduced decomposition $\mathbf{i}$ of w_0. Each MV cycle Z such that $\mu_-(Z) = 0$ is the closure of a set $\tilde{Y}_{\mathbf{i},\mathbf{c}}$ for a certain $\mathbf{c} \in \mathcal{C}_{\mathbf{i}}$; indeed there exists

$b \in \mathbf{B}(-\infty)$ such that $Z = \Xi(t_0 \otimes b)$, and one takes then $\mathbf{c} = \mathbf{c_i}(b)$. It follows that S_0^- is contained in the union $\bigcup_{\mathbf{c}\in\mathcal{C}_\mathbf{i}} \overline{\tilde{Y}_{\mathbf{i},\mathbf{c}}}$. On the other side, each $\tilde{Y}_{\mathbf{i},\mathbf{c}}$ is contained in S_0^-. One could then hope that S_0^- is the disjoint union of the $\tilde{Y}_{\mathbf{i},\mathbf{c}}$ for $\mathbf{c} \in \mathcal{C}_\mathbf{i}$, because the analogous property $S_0^- = \bigsqcup_{n_\bullet \in \mathbb{N}^N} A^\mathbf{i}(n_\bullet)$ for Kamnitzer's subsets holds (see Proposition 4.1 in [13]).

This is alas not the case in general, as the following counterexample shows. We take $G = \mathbf{SL}_4$ with its usual pinning and enumerate the simple roots in the usual way $(\alpha_1, \alpha_2, \alpha_3)$. We choose the reduced decomposition $\mathbf{i} = (2, 1, 3, 2, 1, 3)$ and consider

$$g = y_2(-1)\ y_1(1/t)\ y_3(1/t)\ y_2(t)\ y_1(-1/t)\ y_3(-1/t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & t-1 & 1 & 0 \\ -1/t & 1 & 0 & 1 \end{pmatrix}.$$

If one tries to factorize an element in $gG(\mathscr{O}) \cap U^-(\mathscr{K})$ as a product

$$y_2(p_1)\ y_1(p_2)\ y_3(p_3)\ y_2(p_4)\ y_1(p_5)\ y_3(p_6)$$

using Berenstein, Fomin and Zelevinsky's method [4], and if after that one adjusts $\mathbf{c} = (c_1, \dots, c_6)$ so that $(\mathrm{val}(p_1), \dots, \mathrm{val}(p_6)) = (\tilde{c}_1, \dots, \tilde{c}_6)$, then one finds

$$c_1 \leqslant 0, \quad c_2 \leqslant 0, \quad c_3 \leqslant 0, \quad c_4 \geqslant 1, \quad c_5 \geqslant 1, \quad c_6 \geqslant 1.$$

These conditions on $\mathbf{c}$ must be satisfied so that $[g]$ can belong to $\tilde{Y}_{\mathbf{i},\mathbf{c}}$. However, the equations that define the cone $\mathcal{C}_\mathbf{i}$ are

$$c_1 \geqslant 0, \quad c_2 \geqslant c_6 \geqslant 0, \quad c_3 \geqslant c_5 \geqslant 0, \quad c_2 + c_3 \geqslant c_4 \geqslant c_5 + c_6.$$

We conclude that $[g] \notin \bigcup_{\mathbf{c}\in\mathcal{C}_\mathbf{i}} \tilde{Y}_{\mathbf{i},\mathbf{c}}$.

5. BFG crystal operations on MV cycles and root operators on LS galleries

Let $\lambda \in \Lambda_{++}$ be a dominant coweight. Littelmann's path model [21] affords a concrete realization of the crystal $\mathbf{B}(\lambda)$ in terms of piecewise linear paths drawn on $\Lambda \otimes_\mathbb{Z} \mathbb{R}$; it depends on the choice of a path joining 0 to λ and contained in the dominant Weyl chamber. In [11], Gaussent and Littelmann present a variation of the path model, replacing piecewise linear paths by galleries in the Coxeter complex of the affine Weyl group W^{aff}. They define a set $\Gamma^+_{\mathrm{LS}}(\gamma_\lambda)$ of "LS galleries", which depends on the choice of a minimal gallery γ_λ joining 0 to λ and contained in the dominant Weyl chamber. Defining "root operators" e_α and f_α for each simple root α in Φ, they endow $\Gamma^+_{\mathrm{LS}}(\gamma_\lambda)$ with the structure of a crystal, which happens to be isomorphic to $\mathbf{B}(\lambda)$. Using a Bott-Samelson resolution $\pi : \hat{\Sigma}(\gamma_\lambda) \to \overline{\mathscr{G}_\lambda}$ and a Białynicki-Birula decomposition of $\hat{\Sigma}(\gamma_\lambda)$ into a disjoint union of cells $C(\delta)$, Gaussent and Littelmann associate a closed subvariety $Z(\delta) = \overline{\pi(C(\delta))}$ of $\mathscr{G}$ to each LS gallery δ and show that the map Z is a bijection from $\Gamma^+_{\mathrm{LS}}(\gamma_\lambda)$ onto $\mathscr{Z}(\lambda)$.

The main result of this section is Theorem 5.8, which says that Z is an isomorphism of crystals. In other words, the root operators on LS galleries match Braverman and Gaitsgory's crystal operations on MV cycles under the bijection Z.

Strictly speaking, our proof for this comparison result is valid only when λ is regular. The advantage of this situation is that elements in $\Gamma^+_{\mathrm{LS}}(\gamma_\lambda)$ are then galleries of alcoves. In the case where λ is singular, Gaussent and Littelmann's constructions involve a more general class of galleries (see Section 4 in [11]). Such a sophistication is however not needed: our presentation of Gaussent and Littelmann's results in Section 5.2 below makes sense even if λ is singular. Within this framework, our comparison theorem is valid for any λ, regular or singular.

A key idea of Gaussent and Littelmann is to view the affine Grassmannian as a subset of the set of vertices of the (affine) Bruhat-Tits building of $G(\mathscr{K})$. In Section 5.1, we review quickly basic facts about the latter and study the stabilizer in $U^+(\mathscr{K})$ of certain of its faces. We warn here the reader that we use our own convention pertaining the Bruhat-Tits building: indeed our Iwahori subgroup is the preimage of B^- by the specialization map at $t = 0$ from $G(\mathscr{O})$ to G, whereas Gaussent and Littelmann use the preimage of B^+. Our convention is unusual, but it makes the statement of our comparison result more natural. Section 5.2 recalls the main steps in Gaussent and Littelmann's construction, in a way that encompasses the peculiarities of the case where λ is singular. The final Section 5.3 contains the proof of our comparison theorem. To prove the equality $\tilde{e}_i Z(\delta) = Z(e_{\alpha_i}\delta)$ for each LS gallery δ and each $i \in I$, we use the criterion of Proposition 4.2. The first three conditions are easily checked, while the inclusion $Z(\delta) \subseteq Z(e_{\alpha_i}\delta)$ is established in Proposition 5.11.

5.1. **Affine roots, the Coxeter complex and the Bruhat-Tits building.** We consider the vector space $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. We define a real root of the affine root system (for short, an affine root) as a pair $(\alpha, n) \in \Phi \times \mathbb{Z}$. To an affine root (α, n), we associate:

- the reflection $s_{\alpha,n} : x \mapsto x - \big(\langle \alpha, x\rangle - n\big)\,\alpha^\vee$ of $\Lambda_{\mathbb{R}}$;
- the affine hyperplane $H_{\alpha,n} = \{x \in \Lambda_{\mathbb{R}} \mid \langle \alpha, x\rangle = n\}$ of fixed points of $s_{\alpha,n}$;
- the closed half-space $H^-_{\alpha,n} = \{x \in \Lambda_{\mathbb{R}} \mid \langle \alpha, x\rangle \leqslant n\}$;
- the one-parameter additive subgroup $x_{\alpha,n} : b \mapsto x_\alpha(bt^n)$ of $G(\mathscr{K})$; here b belongs to either $\mathbb{C}$ or $\mathscr{K}$.

We denote the set of all affine roots by Φ^{aff}. We embed Φ in Φ^{aff} by identifying a root $\alpha \in \Phi$ with the affine root $(\alpha, 0)$. We choose an element 0 that does not belong to I; we set $I^{\mathrm{aff}} = I \sqcup \{0\}$ and $\alpha_0 = (-\theta, -1)$, where θ is the highest root of Φ. The elements α_i with $i \in I^{\mathrm{aff}}$ are called simple affine roots.

The group of affine transformations of $\Lambda_{\mathbb{R}}$ generated by all reflections $s_{\alpha,n}$ is called the affine Weyl group and is denoted by W^{aff}. For each $i \in I^{\mathrm{aff}}$, we set $s_i = s_{\alpha_i}$. Then W^{aff} is a Coxeter system when equipped with the set of generators $\{s_i \mid i \in I^{\mathrm{aff}}\}$. The parabolic subgroup of W^{aff} generated by the simple reflections s_i with $i \in I$ is isomorphic to W. For each $\lambda \in \mathbb{Z}\Phi^\vee$, the translation $\tau_\lambda : x \mapsto x + \lambda$ belongs to W^{aff}. All these translations form a normal subgroup in W^{aff}, isomorphic to the coroot lattice $\mathbb{Z}\Phi^\vee$, and W^{aff} is the semidirect product $W^{\mathrm{aff}} = \mathbb{Z}\Phi^\vee \rtimes W$.

The group W^{aff} acts on the set Φ^{aff} of affine roots: one demands that $w(H^-_\beta) = H^-_{w\beta}$ for each element $w \in W^{\mathrm{aff}}$ and each affine root $\beta \in \Phi^{\mathrm{aff}}$. The action of an element $w \in W$ or a translation τ_λ on an affine root $(\alpha, n) \in \Phi \times \mathbb{Z}$ is given by $w\,(\alpha, n) = (w\alpha, n)$ or $\tau_\lambda\,(\alpha, n) = \big(\alpha, n + \langle \alpha, \lambda\rangle\big)$. One checks that $ws_\alpha w^{-1} = s_{w\alpha}$ for all $w \in W^{\mathrm{aff}}$ and $\alpha \in \Phi^{\mathrm{aff}}$. Using Equation (2.1), one checks that

$$(t^\lambda\,\overline{w}\,)\,x_\alpha(a)\,(t^\lambda\,\overline{w}\,)^{-1} = x_{\tau_\lambda w(\alpha)}(\pm a) \tag{5.1}$$

in $G(\mathscr{K})$, for all $\lambda \in \mathbb{Z}\Phi^\vee$, $w \in W$, $\alpha \in \Phi^{\mathrm{aff}}$ and $a \in \mathscr{K}$.

We denote by $\mathfrak{H}$ the arrangement formed by the hyperplanes H_β, where $\beta \in \Phi^{\mathrm{aff}}$. It divides the vector space $\Lambda_{\mathbb{R}}$ into faces. Faces with maximal dimension are called alcoves; they are the connected components of $\Lambda_{\mathbb{R}} \setminus \bigcup_{H \in \mathfrak{H}} H$. Faces of codimension 1 are called facets; faces of dimension 0 are called vertices. The closure of a face is the disjoint union of faces of smaller dimension. Endowed with the set of all faces, $\Lambda_{\mathbb{R}}$ becomes a polysimplicial complex, called the Coxeter complex $\mathscr{A}^{\mathrm{aff}}$; it is endowed with an action of W^{aff}.

The dominant open Weyl chamber is the subset

$$C_{\mathrm{dom}} = \{x \in \Lambda_{\mathbb{R}} \mid \forall i \in I,\ \langle \alpha_i, x \rangle > 0\}.$$

The fundamental alcove

$$A_{\mathrm{fund}} = \{x \in C_{\mathrm{dom}} \mid \langle \theta, x \rangle < 1\}$$

is the complement of $\bigcup_{i \in I^{\mathrm{aff}}} H^-_{\alpha_i}$. We label the faces contained in $\overline{A_{\mathrm{fund}}}$ by proper subsets of I^{aff} by setting

$$\phi_J = \left(\bigcap_{i \in J} H_{\alpha_i} \right) \setminus \left(\bigcup_{i \in I^{\mathrm{aff}} \setminus J} H^-_{\alpha_i} \right)$$

for each $J \subset I^{\mathrm{aff}}$. For instance, $\phi_\varnothing$ is the alcove A_{fund} and ϕ_I is the vertex $\{0\}$. Any face of our arrangement $\mathfrak{H}$ is conjugated under the action of W^{aff} to exactly one face contained in $\overline{A_{\mathrm{fund}}}$, because the latter is a fundamental domain for the action of W^{aff} on $\Lambda_{\mathbb{R}}$. We say that a subset $J \subset I^{\mathrm{aff}}$ is the type of a face F if F is conjugated to ϕ_J under W^{aff}.

We denote by $\hat{B}$ the (Iwahori) subgroup of $G(\mathscr{K})$ generated by the torus $T(\mathscr{O})$ and by the elements $x_\alpha(ta)$ and $x_{-\alpha}(a)$, where $\alpha \in \Phi_+$ and $a \in \mathscr{O}$. In other words, $\hat{B}$ is the preimage of the Borel subgroup B^- under the specialization map at $t = 0$ from $G(\mathscr{O})$ to G. We lift the simple reflections s_i to the group $G(\mathscr{K})$ by setting

$$\overline{s_i} = x_{\alpha_i}(1) x_{-\alpha_i}(-1) x_{\alpha_i}(1) = x_{-\alpha_i}(-1) x_{\alpha_i}(1) x_{-\alpha_i}(-1)$$

for each $i \in I^{\mathrm{aff}}$. We lift any element $w \in W^{\mathrm{aff}}$ to an element $\overline{w} \in G(\mathscr{K})$ so that $\overline{w} = \overline{s_{i_1}} \cdots \overline{s_{i_l}}$ for each reduced decomposition $s_{i_1} \cdots s_{i_l}$ of w. This notation does not conflict with our earlier notation $\overline{s_i}$ for $i \in I$ and $\overline{w}$ for $w \in W$. For each $\lambda \in \mathbb{Z}\Phi^\vee$, the lift $\overline{\tau_\lambda}$ of the translation τ_λ coincides with t^λ up to a sign (that is, up to the multiplication by an element of the form $(-1)^\mu$ with $\mu \in \mathbb{Z}\Phi^\vee$).

The affine Bruhat-Tits building $\mathscr{I}^{\mathrm{aff}}$ is a polysimplicial complex endowed with an action of $G(\mathscr{K})$. The affine Coxeter complex $\mathscr{A}^{\mathrm{aff}}$ can be embedded in $\mathscr{I}^{\mathrm{aff}}$ as the subcomplex formed by the faces fixed by T; in this identification, the action of an element $w \in W^{\mathrm{aff}}$ on $\mathscr{A}^{\mathrm{aff}}$ matches the action of $\overline{w}$ on $(\mathscr{I}^{\mathrm{aff}})^T$. Each face of $\mathscr{I}^{\mathrm{aff}}$ is conjugated under the action of $G(\mathscr{K})$ to exactly one face contained in $\overline{A_{\mathrm{fund}}}$; we say that a subset $J \subset I^{\mathrm{aff}}$ is the type of a face F if F is conjugated to ϕ_J. Finally, there is a $G(\mathscr{K})$-equivariant map of the affine Grassmannian $\mathscr{G}$ into $\mathscr{I}^{\mathrm{aff}}$, which extends the map $[t^\lambda] \mapsto \{\lambda\}$ from $\mathscr{G}^T$ into $\mathscr{A}^{\mathrm{aff}} \cong (\mathscr{I}^{\mathrm{aff}})^T$.

Given a subset $J \subseteq I^{\mathrm{aff}}$, we denote by $\hat{P}_J$ the subgroup of $G(\mathscr{K})$ generated by $\hat{B}$ and the elements $\overline{s_i}$ for $i \in J$; thus $\hat{B} = \hat{P}_\varnothing$ and $G(\mathscr{O}) = \hat{P}_I$. (The subgroup $\hat{P}_J$ is the stabilizer in $G(\mathscr{K})$ of the face ϕ_J. For each $g \in G(\mathscr{K})$, the stabilizer of the face $g\phi_J$ is thus the parahoric subgroup $g\hat{P}_J g^{-1}$. This bijection between the set of faces in the affine building and the set of parahoric subgroups in $G(\mathscr{K})$ is indeed the starting point for the definition of the building; see §2.1 in [10].) To shorten the notation, we will write $\hat{P}_i$ instead of $\hat{P}_{\{i\}}$ for each $i \in I^{\mathrm{aff}}$. Similarly, for each $i \in I^{\mathrm{aff}}$, we will write W_i to indicate the subgroup $\{1, s_i\}$ of W^{aff}.

We denote the stabilizer in $U^+(\mathscr{K})$ of a face F of the affine building by $\mathrm{Stab}_+(F)$. Our last task in this section is to determine as precisely as possible the group $\mathrm{Stab}_+(F)$ and the set $\mathrm{Stab}_+(F')/\mathrm{Stab}_+(F)$ when F and F' are faces of the Coxeter complex such that $F' \subseteq \overline{F}$. We need additional notation for that. Given a real number a, we denote the smallest integer greater than a by $\lceil a \rceil$. To a face F of the Coxeter complex, Bruhat and Tits (see (7.1.1) in [10]) associate the function $f_F : \alpha \mapsto \sup_{x \in F} \langle \alpha, x \rangle$ on the dual space of $\Lambda_{\mathbb{R}}$. If $\alpha \in \Phi$, then $\lceil f_F(\alpha) \rceil$ is the smallest integer n such that F lies in the closed half-space $H^-_{\alpha,n}$. The function f_F is convex and positively homogeneous of degree 1; in particular, $f_F(i\alpha + j\beta) \leqslant i f_F(\alpha) + j f_F(\beta)$ for all roots $\alpha, \beta \in \Phi$ and all positive integers i, j. When F and F' are two faces of the Coxeter complex such that $F' \subseteq \overline{F}$, we denote by $\Phi_+^{\mathrm{aff}}(F', F)$ the set of all affine roots $\beta \in \Phi_+ \times \mathbb{Z}$ such that $F' \subseteq H_\beta$ and $F \not\subseteq H^-_\beta$; in other words, $(\alpha, n) \in \Phi_+^{\mathrm{aff}}(F', F)$ if and only if $\alpha \in \Phi_+$, $n = f_{F'}(\alpha)$ and $n + 1 = \lceil f_F(\alpha) \rceil$. We denote by $\mathrm{Stab}_+(F', F)$ the subgroup of $U^+(\mathscr{K})$ generated by the elements of the form $x_\beta(a)$ with $\beta \in \Phi_+^{\mathrm{aff}}(F', F)$ and $a \in \mathbb{C}$.

Proposition 5.1. (i) *The stabilizer* $\mathrm{Stab}_+(F)$ *of a face* F *of the Coxeter complex is generated by the elements* $x_\alpha(p)$*, where* $\alpha \in \Phi_+$ *and* $p \in \mathscr{K}$ *satisfy* $\mathrm{val}(p) \geqslant f_F(\alpha)$*.*

(ii) *Let* F *and* F' *be two faces of the Coxeter complex such that* $F' \subseteq \overline{F}$*. Then* $\mathrm{Stab}_+(F', F)$ *is a set of representatives for the right cosets of* $\mathrm{Stab}_+(F)$ *in* $\mathrm{Stab}_+(F')$*. For any total order on the set* $\Phi_+^{\mathrm{aff}}(F', F)$*, the map*

$$(a_\beta)_{\beta \in \Phi_+^{\mathrm{aff}}(F', F)} \mapsto \prod_{\beta \in \Phi_+^{\mathrm{aff}}(F', F)} x_\beta(a_\beta)$$

is a bijection from $\mathbb{C}^{\Phi_+^{\mathrm{aff}}(F', F)}$ *onto* $\mathrm{Stab}_+(F', F)$*.*

Proof. Item (i) is proved in Bruhat and Tits's paper [10]; see in particular Sections (7.4.4) and Equation (1) in Section (7.1.8). We note here that this result implies that for any total order on Φ_+, the map

$$(p_\alpha)_{\alpha \in \Phi_+} \mapsto \prod_{\alpha \in \Phi_+} x_\alpha\Big(p_\alpha t^{\lceil f_F(\alpha) \rceil}\Big)$$

is a bijection from $\mathscr{O}^{\Phi_+}$ onto $\mathrm{Stab}_+(F)$.

We now turn to Item (ii). We first observe the following property of $\Phi_+^{\mathrm{aff}}(F', F)$: for each pair i, j of positive integers and each pair $(\alpha, m), (\beta, n)$ of affine roots in $\Phi_+^{\mathrm{aff}}(F', F)$ such that $i\alpha + j\beta \in \Phi$, the affine root $(i\alpha + j\beta, im + jn)$ belongs to $\Phi_+^{\mathrm{aff}}(F', F)$. Indeed, $F' \subseteq H_{\alpha,m} \cap H_{\beta,n}$ implies $F' \subseteq H_{i\alpha+j\beta, im+jn}$, and the inequality

$$f_F(i\alpha + j\beta) \geqslant i f_F(\alpha) - j f_F(-\beta) = i f_F(\alpha) + jn > im + jn$$

shows that $F \not\subseteq H^-_{i\alpha+j\beta, im+jn}$. Standard arguments based on Chevalley's commutator formula (2.5) then show the second assertion in Item (ii).

Now the map $(\alpha, m) \mapsto \alpha$ from Φ^{aff} to Φ restricts to a bijection from $\Phi_+^{\mathrm{aff}}(F', F)$ onto a subset Φ'_+ of Φ_+. We set $\Phi''_+ = \Phi_+ \setminus \Phi'_+$. We endow Φ_+ with a total order, chosen so that every element in Φ'_+ is smaller than every element in Φ''_+, and we transport the order induced on Φ'_+ to $\Phi_+^{\mathrm{aff}}(F', F)$. By Item (i), each element in

$\mathrm{Stab}_+(F')$ may be uniquely written as a product

$$\prod_{\alpha\in\Phi_+} x_\alpha\Big(p_\alpha t^{\lceil f_{F'}(\alpha)\rceil}\Big) \tag{5.2}$$

with $(p_\alpha)_{\alpha\in\Phi_+}$ in $\mathscr{O}^{\Phi_+}$. We write $p_\alpha = a_\alpha + tq_\alpha$ for each $\alpha \in \Phi'_+$, with $a_\alpha \in \mathbb{C}$ and $q_\alpha \in \mathscr{O}$. Thus for each $(\alpha, m) \in \Phi_+^{\mathrm{aff}}(F', F)$, we have $p_\alpha t^{\lceil f_{F'}(\alpha)\rceil} = a_\alpha t^m + q_\alpha t^{\lceil f_F(\alpha)\rceil}$. On the other hand, $\lceil f_{F'}(\alpha)\rceil = \lceil f_F(\alpha)\rceil$ for each $\alpha \in \Phi''_+$. We may therefore rewrite the product in (5.2) as

$$\left(\prod_{(\alpha,m)\in\Phi_+^{\mathrm{aff}}(F',F)} x_\alpha\Big(a_\alpha t^m\Big)\, x_\alpha\Big(q_\alpha t^{\lceil f_F(\alpha)\rceil}\Big)\right)\left(\prod_{\alpha\in\Phi''_+} x_\alpha\Big(p_\alpha t^{\lceil f_F(\alpha)\rceil}\Big)\right).$$

We rearrange the first product above using again Chevalley's commutator formula: there exists a family $(r_\alpha)_{\alpha\in\Phi'_+}$ of power series such that this product is

$$\left(\prod_{(\alpha,m)\in\Phi_+^{\mathrm{aff}}(F',F)} x_\alpha\Big(a_\alpha t^m\Big)\right)\left(\prod_{\alpha\in\Phi'_+} x_\alpha\Big(r_\alpha t^{\lceil f_F(\alpha)\rceil}\Big)\right),$$

and for fixed numbers a_α, the map $(q_\alpha) \mapsto (r_\alpha)$ is a bijection from $\mathcal{O}^{\Phi'_+}$ onto itself. We conclude that the map

$$((a_\beta), (p_\alpha)) \mapsto \left(\prod_{\beta\in\Phi_+^{\mathrm{aff}}(F',F)} x_\beta\Big(a_\beta\Big)\right)\left(\prod_{\alpha\in\Phi_+} x_\alpha\Big(p_\alpha t^{\lceil f_F(\alpha)\rceil}\Big)\right)$$

is a bijection from $\mathbb{C}^{\Phi_+^{\mathrm{aff}}(F',F)} \times \mathscr{O}^{\Phi_+}$ onto $\mathrm{Stab}_+(F')$. This means exactly that the map $(g, h) \mapsto gh$ is a bijection from $\mathrm{Stab}_+(F', F)\times \mathrm{Stab}_+(F)$ onto $\mathrm{Stab}_+(F')$. The proof of Item (ii) is now complete. □

Things are more easy to grasp when F is an alcove and F' is a facet of $\overline{F}$, because then $\Phi_+^{\mathrm{aff}}(F', F)$ has at most one element. In this particular case, certain commutators involving elements of $\mathrm{Stab}_+(F')$ and $\mathrm{Stab}_+(F)$ automatically belong to $\mathrm{Stab}_+(F)$.

Lemma 5.2. *Let F be an alcove of the Coxeter complex and let F' be a facet of $\overline{F}$. Let $(\alpha, m) \in \Phi_+ \times \mathbb{Z}$ be the affine root such that F' lies in the wall $H_{\alpha,m}$ and let $(\beta, n) \in \Phi^{\mathrm{aff}}$ be such that $F \subseteq H^-_{\beta,n}$. We assume that β is either positive or is the opposite of a simple root, and that $\beta \neq -\alpha$. Then for each $q \in \mathscr{O}$ and each $v \in \mathrm{Stab}_+(F', F)$, the commutator $x_{\beta,n}(q)\, v\, x_{\beta,n}(q)^{-1}\, v^{-1}$ belongs to* $\mathrm{Stab}_+(F)$.

Proof. There is nothing to show if $F \subseteq H^-_{\alpha,m}$ since $v = 1$ in this case. We may thus assume that $\mathrm{Stab}_+(F', F) = \big\{(\alpha, m)\big\}$; then there is an $a \in \mathbb{C}$ such that $v = x_{\alpha,m}(a)$.

Suppose first that $\beta = \alpha$. Then

$$\begin{aligned} x_{\beta,n}(q)\, v\, x_{\beta,n}(q)^{-1}\, v^{-1} &= x_{\beta,n}(q)\, x_{\alpha,m}(a)\, x_{\beta,n}(-q)\, x_{\alpha,m}(-a) \\ &= x_\alpha(qt^n + at^m - qt^n - at^m) \\ &= 1. \end{aligned}$$

Therefore the assertion holds in this case.

Suppose now that $\beta \neq \alpha$. The facet F' is contained in the closure of exactly two alcoves, F and say F^*, the latter lying in $H^-_{\alpha,m}$. Then $f_{F^*}(\alpha) = m$. We observe that no wall other than $H_{\alpha,m}$ separates F^* and F. In particular, $H_{\beta,n}$ does not separate F^* and F, because $\beta \neq \pm\alpha$. Since F lies in $H^-_{\beta,n}$, so does F^*, and thus $f_{F^*}(\beta) \leqslant n$. Therefore for any pair of positive integers i, j such that $i\alpha + j\beta$ is a root, $f_{F^*}(i\alpha + j\beta) \leqslant im + jn$. This means that F^* lies in the half-space $H^-_{i\alpha+j\beta,im+jn}$. Again, the wall $H_{i\alpha+j\beta,im+jn}$ does not separate F^* and F, and we conclude that F lies in the half-space $H^-_{i\alpha+j\beta,im+jn}$. Chevalley's commutator formula (2.5) implies that

$$\begin{aligned} x_{\beta,n}(q)\, v\, x_{\beta,n}(q)^{-1}\, v^{-1} &= x_{\beta,n}(q)\, x_{\alpha,m}(a)\, x_{\beta,n}(-q)\, x_{\alpha,m}(-a) \\ &= \prod_{i,j>0} x_{i\alpha+j\beta,im+jn}\big(C_{i,j,\alpha,\beta}\, a^i (-q)^j\big). \end{aligned}$$

Here the product is taken over all pairs of positive integers i, j such that $i\alpha + j\beta$ is a root. The assumption about β in the statement of the lemma implies that such a root $i\alpha + j\beta$ is necessarily positive. By Proposition 5.1 (i), each factor $x_{i\alpha+j\beta,im+jn}\big(C_{i,j,\alpha,\beta} a^i(-q)^j\big)$ belongs to $\mathrm{Stab}_+(F)$. Thus the commutator $x_{\beta,n}(q)\, v\, x_{\beta,n}(q)^{-1}\, v^{-1}$ belongs to $\mathrm{Stab}_+(F)$. □

Remark 5.3. The first assertion in Proposition 5.1 (ii) means that $\mathrm{Stab}_+(F')$ has the structure of a bicrossed product $\mathrm{Stab}_+(F', F) \bowtie \mathrm{Stab}_+(F)$ (see [30]) whenever F and F' are two faces in the Coxeter complex such that $F' \subseteq \overline{F}$. Suppose now that F is an alcove and that F' is a facet of $\overline{F}$. Then Proposition 5.1 (i) and Lemma 5.2 imply that each element $v \in \mathrm{Stab}_+(F', F)$ normalizes the group $\mathrm{Stab}_+(F)$. Thus $\mathrm{Stab}_+(F)$ is a normal subgroup of $\mathrm{Stab}_+(F')$ and $\mathrm{Stab}_+(F')$ is the semidirect product $\mathrm{Stab}_+(F', F) \ltimes \mathrm{Stab}_+(F)$.

5.2. **Galleries, cells and MV cycles.** We fix a dominant coweight $\lambda \in \Lambda_{++}$. As usual, we denote by P_λ the standard parabolic subgroup P_J of G, where $J = \{j \in I \mid \langle \alpha_j, \lambda\rangle = 0\}$. Besides, we denote by $\{\lambda_{\mathrm{fund}}\}$ the vertex in $\overline{A_{\mathrm{fund}}}$ with the same type as $\{\lambda\}$. Finally, there is a unique element w_λ in W^{aff} with minimal length such that $\lambda = w_\lambda(\lambda_{\mathrm{fund}})$. Thus among all alcoves in $\mathscr{A}^{\mathrm{aff}}$ having $\{\lambda\}$ as vertex, $w_\lambda(A_{\mathrm{fund}})$ is the one closest to A_{fund}.

We denote the length of w_λ by p and we choose a reduced decomposition $s_{i_1} \cdots s_{i_p}$ of it, with $(i_1, \ldots, i_p) \in (I^{\mathrm{aff}})^p$. The geometric translation of this choice is the datum of the sequence

$$\gamma_\lambda = \big(\{0\} \subset \overline{\Gamma_0} \supset \Gamma'_1 \subset \overline{\Gamma_1} \supset \cdots \supset \Gamma'_p \subset \overline{\Gamma_p} \supset \{\lambda\}\big)$$

of alcoves and facets (also known as a gallery) in $\mathscr{A}^{\mathrm{aff}}$, where

$$\Gamma_j = s_{i_1} \cdots s_{i_j}(A_{\mathrm{fund}}) \quad \text{and} \quad \Gamma'_j = s_{i_1} \cdots s_{i_{j-1}}\big(\phi_{\{i_j\}}\big).$$

By Proposition 2.19 (iv) in [31], these alcoves and facets are all contained in the dominant Weyl chamber C_{dom}. The choice of the reduced decomposition $s_{i_1} \cdots s_{i_p}$ of w_λ and the notations P_λ, λ_{fund}, γ_λ will be kept for the rest of Section 5.

We define the Bott-Samelson variety as the smooth projective variety

$$\hat{\Sigma}(\gamma_\lambda) = G(\mathscr{O}) \underset{\hat{B}}{\times} \hat{P}_{i_1} \underset{\hat{B}}{\times} \cdots \underset{\hat{B}}{\times} \hat{P}_{i_p}/\hat{B}.$$

We will denote the image in $\hat{\Sigma}(\gamma_\lambda)$ of an element $(g_0, g_1, \ldots, g_p) \in G(\mathscr{O}) \times \hat{P}_{i_1} \times \cdots \times \hat{P}_{i_p}$ by the usual notation $[g_0, g_1, \ldots, g_p]$. The group $G(\mathscr{O})$ acts on $\hat{\Sigma}(\gamma_\lambda)$ by left multiplication on the first factor. There is a $G(\mathscr{O})$-equivariant map $\pi : [g_0, g_1, \ldots, g_p] \mapsto g_0 g_1 \cdots g_p \big[t^{\lambda_{\mathrm{fund}}}\big]$ from $\hat{\Sigma}(\gamma_\lambda)$ onto $\overline{\mathscr{G}_\lambda}$.

The geometric language of buildings is of great convenience in the study of the Bott-Samelson variety. Indeed, each element $d = [g_0, g_1, \ldots, g_p]$ in $\hat{\Sigma}(\gamma_\lambda)$ may be viewed as a gallery

$$\delta = \big(\{0\} = \Delta'_0 \subset \overline{\Delta_0} \supset \Delta'_1 \subset \overline{\Delta_1} \supset \cdots \supset \Delta'_p \subset \overline{\Delta_p} \supset \Delta'_{p+1}\big) \tag{5.3}$$

in $\mathscr{I}^{\mathrm{aff}}$, where

$$\begin{aligned} \Delta_j &= g_0 \cdots g_j(A_{\mathrm{fund}}) \quad \text{for } 0 \leqslant j \leqslant p, \\ \Delta'_j &= g_0 \cdots g_{j-1}\big(\phi_{\{i_j\}}\big) \quad \text{for } 1 \leqslant j \leqslant p, \\ \text{and} \quad \Delta'_{p+1} &= g_0 \cdots g_p\{\lambda_{\mathrm{fund}}\}. \end{aligned}$$

(This gallery has the same type as γ_λ, that is, each facet Δ'_j of δ has the same type as the corresponding element Γ'_j in γ_λ. We also observe that the vertex Δ'_{p+1} of the affine building corresponds to the element $\pi(d)$ of the affine Grassmannian.) Thus for instance the point $\big[1, \overline{s_{i_1}}, \overline{s_{i_2}}, \ldots, \overline{s_{i_p}}\big]$ in $\hat{\Sigma}(\gamma_\lambda)$ is viewed as the gallery γ_λ. With this picture in mind, one proves easily the following proposition.

Proposition 5.4. *The restriction of π to $\pi^{-1}(\mathscr{G}_\lambda)$ is a fiber bundle with fiber isomorphic to P_λ/B^+.*

Proof. Let $J = \{j \in I \mid \langle \alpha_j, \lambda\rangle = 0\}$ and let P_J^- be the parabolic subgroup of G generated by $B^- \cup M_J$. The set S of alcoves whose closure contains ϕ_J is in canonical bijection with the set of all Iwahori subgroups of $G(\mathscr{K})$ contained in $\hat{P}_J$, hence with $\hat{P}_J/\hat{B} \cong P_J^-/B^-$. In particular, P_J^- acts transitively on S and S is isomorphic to P_λ/B^+.

Now let $F = \pi^{-1}([t^\lambda])$ and let H be the stabilizer of $[t^\lambda]$ in $G(\mathscr{O})$; thus $H \supseteq P_J^-$. Since π is $G(\mathscr{O})$-equivariant, H acts on F and there is a commutative diagram

$$\begin{tikzcd} G(\mathscr{O}) \times_H F \arrow[r, "\simeq"] \arrow[d] & \pi^{-1}(\mathscr{G}_\lambda) \arrow[d, "\pi"] \\ G(\mathscr{O})/H \arrow[r, "\simeq"] & \mathscr{G}_\lambda. \end{tikzcd}$$

It thus suffices to prove that F is isomorphic to S.

Each element $d \in F$ can be viewed as a gallery

$$\delta = \big(\{0\} \subset \overline{\Delta_0} \supset \Delta'_1 \subset \overline{\Delta_1} \supset \cdots \supset \Delta'_p \subset \overline{\Delta_p} \supset \{\lambda\}\big)$$

in $\mathscr{I}^{\mathrm{aff}}$ stretching from $\{0\}$ to $\{\lambda\}$. We claim that $\overline{\Delta_0}$ always contains ϕ_J. When all faces of δ belong to $\mathscr{A}^{\mathrm{aff}}$, this claim follows from the proof of Proposition 2.29 in [31] (with $\mathrm{proj}_{\{0\}}\{\lambda\} = \phi_J$); the general case is obtained by retracting δ onto $\mathscr{A}^{\mathrm{aff}}$ from the fundamental alcove, see Lemma 3.6 in [31].

We finally consider the map $f : d \mapsto \Delta_0$ from F to S. Corollary 3.4 in [31] implies that f is injective, because in any apartment, there is only one non-stammering gallery of the same type as γ_λ that starts from a given chamber Δ_0. On the other side, f is H-equivariant; it is thus surjective, for P_J^- acts transitively on the codomain. We conclude that f is an isomorphism from F onto S. $\square$

This proposition implies the following equality, which we record for later use:

$$|\Phi_+|+p=\dim\hat\Sigma(\gamma_\lambda)=\dim\mathscr{G}_\lambda+\dim(P_\lambda/B^+)=\operatorname{ht}(\lambda-w_0\lambda)+\dim(P_\lambda/B^+). \tag{5.4}$$

Our next task is to obtain a Białynicki-Birula decomposition of the Bott-Samelson variety. The torus T acts on the latter by left multiplication on the first factor. If we represent an element $d\in\hat\Sigma(\gamma_\lambda)$ by a gallery δ as in (5.3), then d is fixed by T if and only if all the faces Δ_j and Δ'_j are in the Coxeter complex $\mathscr{A}^{\mathrm{aff}}\cong(\mathscr{I}^{\mathrm{aff}})^T$. We devote a word to this situation: a gallery δ as in (5.3), of the same type as γ_λ, all of whose faces are in $\mathscr{A}^{\mathrm{aff}}$, is called a combinatorial gallery. The weight ν such that $\Delta'_{p+1}=\{\nu\}$ is called the weight of δ; it belongs to $\lambda+\mathbb{Z}\Phi^\vee$, because $\{\nu\}$ has the same type as $\{\lambda\}$.

We denote the set of all combinatorial galleries by $\Gamma(\gamma_\lambda)$. This set is in bijection with $W\times W_{i_1}\times\cdots\times W_{i_p}$; indeed the map $(\delta_0,\delta_1,\ldots,\delta_p)\mapsto\big[\,\overline{\delta_0},\overline{\delta_1},\ldots,\overline{\delta_p}\,\big]$ from $W\times W_{i_1}\times\cdots\times W_{i_p}$ to $\hat\Sigma(\gamma_\lambda)$ is injective and its image is the set of T-fixed points in the codomain. Concretely this correspondence maps $(\delta_0,\delta_1,\ldots,\delta_p)\in W\times W_{i_1}\times\cdots\times W_{i_p}$ to the combinatorial gallery whose faces are

$$\Delta_j=\delta_0\cdots\delta_j(A_{\mathrm{fund}})\quad\text{and}\quad\Delta'_j=\delta_0\cdots\delta_{j-1}\big(\phi_{\{i_j\}}\big) \tag{5.5}$$

and whose weight is

$$\nu=\delta_0\delta_1\cdots\delta_p\lambda_{\mathrm{fund}}. \tag{5.6}$$

The retraction $r_\varnothing$ from $\mathscr{G}$ onto $\mathscr{G}^T\cong\Lambda$ can be extended to a map of polysimplicial complexes from $\mathscr{I}^{\mathrm{aff}}$ onto $(\mathscr{I}^{\mathrm{aff}})^T\cong\mathscr{A}^{\mathrm{aff}}$. Following Section 7 in [11], we further extend this retraction to a map from $\hat\Sigma(\gamma_\lambda)$ onto $\hat\Sigma(\gamma_\lambda)^T\cong\Gamma(\gamma_\lambda)$ by applying it componentwise to galleries. The preimage by this map of a combinatorial gallery δ will be denoted by $C(\delta)$.

Our aim now is to describe precisely the cell $C(\delta)$ associated to a combinatorial gallery δ. Representing the latter as in (5.3), we introduce the notation

$$\operatorname{Stab}_+(\delta)=\operatorname{Stab}_+(\Delta'_0,\Delta_0)\times\operatorname{Stab}_+(\Delta'_1,\Delta_1)\times\cdots\times\operatorname{Stab}_+(\Delta'_p,\Delta_p).$$

Proposition 5.5. *Let δ be a combinatorial gallery and let $(\delta_0,\delta_1,\ldots,\delta_p)$ be the sequence in $W\times W_{i_1}\times\cdots\times W_{i_p}$ associated to δ by Equation* (5.5)*. Then the map*

$$(v_0,v_1,\ldots,v_p)\mapsto\Big[v_0\,\overline{\delta_0}\,,\ \overline{\delta_0}^{-1}\,v_1\,\overline{\delta_0\delta_1}\,,\ldots,\ \overline{\delta_0\cdots\delta_{p-1}}^{-1}\,v_p\,\overline{\delta_0\cdots\delta_p}\,\Big]$$

from $\operatorname{Stab}_+(\delta)$ *to* $\hat\Sigma(\gamma_\lambda)$ *is injective and its image is* $C(\delta)$*.*

Proof. Denote

$$\operatorname{Stab}_+(\Delta'_0)\underset{\operatorname{Stab}_+(\Delta_0)}{\times}\operatorname{Stab}_+(\Delta'_1)\underset{\operatorname{Stab}_+(\Delta_1)}{\times}\cdots\underset{\operatorname{Stab}_+(\Delta_{p-1})}{\times}\operatorname{Stab}_+(\Delta'_p)/\operatorname{Stab}_+(\Delta_p).$$

by $\widetilde{\operatorname{Stab}_+}(\delta)$. From the inclusions

$$\begin{aligned}
&\operatorname{Stab}_+(\Delta_j)\subseteq\overline{\delta_0\cdots\delta_j}\;\hat B\;\overline{\delta_0\cdots\delta_j}^{-1} &&(\text{for }0\leqslant j\leqslant p),\\
&\operatorname{Stab}_+(\Delta'_0)\subseteq G(\mathscr{O})\overline{\delta_0}^{-1},\\
&\operatorname{Stab}_+(\Delta'_j)\subseteq\overline{\delta_0\cdots\delta_{j-1}}\;\hat P_{i_j}\;\overline{\delta_0\cdots\delta_j}^{-1} &&(\text{for }1\leqslant j\leqslant p),
\end{aligned}$$

standard arguments imply that the map

$$f:[v_0,v_1,\ldots,v_p]\mapsto\Big[v_0\,\overline{\delta_0}\,,\ \overline{\delta_0}^{-1}\,v_1\,\overline{\delta_0\delta_1}\,,\ldots,\ \overline{\delta_0\cdots\delta_{p-1}}^{-1}\,v_p\,\overline{\delta_0\cdots\delta_p}\,\Big]$$

from $\widetilde{\operatorname{Stab}_+}(\delta)$ to $\hat\Sigma(\gamma_\lambda)$ is well defined.

The proof of Proposition 6 in [11] says that an element $d = [g_0, g_1, \dots, g_p]$ in the Bott-Samelson variety belongs to the cell $C(\delta)$ if and only if there exists $u_0, u_1, \dots, u_p \in U^+(\mathscr{K})$ such that

$$g_0 g_1 \cdots g_j A_{\mathrm{fund}} = u_j \Delta_j \quad \text{and} \quad u_{j-1}\Delta'_j = u_j \Delta'_j$$

for each j. Setting $v_0 = u_0$ and $v_j = u_{j-1}^{-1} u_j$ for $1 \leqslant j \leqslant p$, the conditions above can be rewritten

$$g_0 g_1 \cdots g_j \hat{B} = v_0 v_1 \cdots v_j \, \overline{\delta_0 \delta_1 \cdots \delta_j} \, \hat{B} \quad \text{and} \quad v_j \in \mathrm{Stab}_+(\Delta'_j),$$

which shows that $f([v_0, v_1, \dots, v_p]) = d$. Therefore the image of f contains the cell $C(\delta)$. The reverse inclusion can be established similarly.

The map f is injective. Indeed, suppose that two elements $v = [v_0, v_1, \dots, v_p]$ and $v' = [v'_0, v'_1, \dots, v'_p]$ in $\widetilde{\mathrm{Stab}}_+(\delta)$ have the same image. Then

$$v_0 v_1 \cdots v_j \, \overline{\delta_0 \delta_1 \cdots \delta_j} \, \hat{B} = v'_0 v'_1 \cdots v'_j \, \overline{\delta_0 \delta_1 \cdots \delta_j} \, \hat{B}$$

for each $j \in \{0, \dots, p\}$. This means geometrically that

$$v_0 v_1 \cdots v_j \, \overline{\delta_0 \delta_1 \cdots \delta_j} \, A_{\mathrm{fund}} = v'_0 v'_1 \cdots v'_j \, \overline{\delta_0 \delta_1 \cdots \delta_j} \, A_{\mathrm{fund}};$$

in other words, $v_0 v_1 \cdots v_j$ and $v'_0 v'_1 \cdots v'_j$ are equal in $U^+(\mathscr{K}) / \mathrm{Stab}_+(\Delta_j)$. Since this holds for each j, the two elements v and v' are equal in $\widetilde{\mathrm{Stab}}_+(\delta)$. We conclude that f induces a bijection from $\widetilde{\mathrm{Stab}}_+(\delta)$ onto $C(\delta)$.

It then remains to observe that the map $(v_0, v_1, \dots, v_p) \mapsto [v_0, v_1, \dots, v_p]$ from $\mathrm{Stab}_+(\delta)$ to $\widetilde{\mathrm{Stab}}_+(\delta)$ is bijective. This follows from Proposition 5.1 (ii): indeed, for each $[a_0, a_1, \dots, a_p] \in \widetilde{\mathrm{Stab}}_+(\delta)$, the element $(v_0, v_1, \dots, v_p) \in \mathrm{Stab}_+(\delta)$ such that $[v_0, v_1, \dots, v_p] = [a_0, a_1, \dots, a_p]$ is uniquely determined by the condition that for all $j \in \{0, 1, \dots, p\}$,

$$v_j \in \bigl((v_0 \cdots v_{j-1})^{-1}(a_0 \cdots a_j) \, \mathrm{Stab}_+(\Delta_j)\bigr) \cap \mathrm{Stab}_+(\Delta'_j, \Delta_j). \qquad \square$$

The definition of the map π, Equation (5.6), Proposition 5.1 (ii) and Proposition 5.5 yield the following explicit description of the image of the cell $C(\delta)$ by the map π.

Corollary 5.6. *Let δ be a combinatorial gallery of weight ν, as in* (5.3)*, and equip the set $\Phi_+^{\mathrm{aff}}(\Delta'_0, \Delta_0)$ with a total order. Then $\pi(C(\delta))$ is the image of the map*

$$(a_{j,\beta}) \mapsto \prod_{j=0}^{p} \left(\prod_{\beta \in \Phi_+^{\mathrm{aff}}(\Delta'_j, \Delta_j)} x_\beta(a_{j,\beta}) \right) [t^\nu]$$

from $\prod_{j=0}^{p} \mathbb{C}^{\Phi_+^{\mathrm{aff}}(\Delta'_j, \Delta_j)}$ to $\mathscr{G}$.

Certainly the notation used in Corollary 5.6 is more complicated than really needed. Indeed, except perhaps for $j = 0$, each set $\Phi_+^{\mathrm{aff}}(\Delta'_j, \Delta_j)$ has at most one element. Each inner product is therefore almost always empty or reduced to one factor. Keeping this fact in mind may help understand the proofs of Lemma 5.10 and Proposition 5.11 in Section 5.3.

We now endow $\Gamma(\gamma_\lambda)$ with the structure of a crystal. To do that, we introduce "root operators" e_α and f_α for each simple root α of the root system Φ. These operators act on $\Gamma(\gamma_\lambda)$ and are defined by the following recipe (see Section 6 in [11]).

Let δ be a combinatorial gallery, as in Equation (5.3). We call $m \in \mathbb{Z}$ the smallest integer such that the hyperplane $H_{\alpha,m}$ contains a face Δ'_j, where $0 \leqslant j \leqslant p+1$.

- If $m = 0$, then $e_\alpha \delta$ is not defined. Otherwise we find $k \in \{1, \dots, p+1\}$ minimal such that $\Delta'_k \subseteq H_{\alpha,m}$, we find $j \in \{0, \dots, k-1\}$ maximal such that $\Delta'_j \subseteq H_{\alpha,m+1}$, and we define the combinatorial gallery $e_\alpha \delta$ as

$$\begin{aligned}\bigl(\{0\} = \Delta'_0 \subset \overline{\Delta_0} \supset \Delta'_1 \subset \overline{\Delta_1} \supset \cdots \supset \Delta'_j \\ \subset s_{\alpha,m+1}\bigl(\overline{\Delta_j}\bigr) \supset s_{\alpha,m+1}\bigl(\Delta'_{j+1}\bigr) \subset \cdots \supset s_{\alpha,m+1}\bigl(\Delta'_{k-1}\bigr) \subset s_{\alpha,m+1}\bigl(\overline{\Delta_{k-1}}\bigr) \\ \supset \tau_{\alpha^\vee}\bigl(\Delta'_k\bigr) \subset \tau_{\alpha^\vee}\bigl(\overline{\Delta_k}\bigr) \supset \cdots \subset \tau_{\alpha^\vee}\bigl(\overline{\Delta_p}\bigr) \supset \tau_{\alpha^\vee}\bigl(\Delta'_{p+1}\bigr) = \{\nu + \alpha^\vee\}\bigr).\end{aligned}$$

  Thus we reflect all faces between Δ'_j and Δ'_k across the hyperplane $H_{\alpha,m+1}$ and we translate all faces after Δ'_k by $\alpha^\vee$. (Note here that $s_{\alpha,m+1}(\Delta'_j) = \Delta'_j$ and that $s_{\alpha,m+1}(\Delta'_k) = \tau_{\alpha^\vee}(\Delta'_k)$.)
- If $m = \langle \alpha, \nu \rangle$, then $f_\alpha \delta$ is not defined. Otherwise we find $j \in \{0, \dots, p\}$ maximal such that $\Delta'_j \subseteq H_{\alpha,m}$, we find $k \in \{j+1, \dots, p+1\}$ minimal such that $\Delta'_k \subseteq H_{\alpha,m+1}$, and we define the combinatorial gallery $f_\alpha \delta$ as

$$\begin{aligned}\bigl(\{0\} = \Delta'_0 \subset \overline{\Delta_0} \supset \Delta'_1 \subset \overline{\Delta_1} \supset \cdots \supset \Delta'_j \\ \subset s_{\alpha,m}\bigl(\overline{\Delta_j}\bigr) \supset s_{\alpha,m}\bigl(\Delta'_{j+1}\bigr) \subset \cdots \supset s_{\alpha,m}\bigl(\Delta'_{k-1}\bigr) \subset s_{\alpha,m}\bigl(\overline{\Delta_{k-1}}\bigr) \\ \supset \tau_{-\alpha^\vee}\bigl(\Delta'_k\bigr) \subset \tau_{-\alpha^\vee}\bigl(\overline{\Delta_k}\bigr) \supset \cdots \subset \tau_{-\alpha^\vee}\bigl(\overline{\Delta_p}\bigr) \supset \tau_{-\alpha^\vee}\bigl(\Delta'_{p+1}\bigr) = \{\nu - \alpha^\vee\}\bigr).\end{aligned}$$

  Thus we reflect all faces between Δ'_j and Δ'_k across the hyperplane $H_{\alpha,m}$ and we translate all faces after Δ'_k by $-\alpha^\vee$. (Note here that $s_{\alpha,m}(\Delta'_j) = \Delta'_j$ and that $s_{\alpha,m}(\Delta'_k) = \tau_{-\alpha^\vee}(\Delta'_k)$.)

With the notation above, the maximal integer n such that $(e_\alpha)^n \delta$ is defined is equal to $-m$, and the maximal integer n such that $(f_\alpha)^n \delta$ is defined is equal to $\langle \alpha, \nu \rangle - m$.

The crystal structure on $\Gamma(\gamma_\lambda)$ is then defined as follows. Given $\delta \in \Gamma(\gamma_\lambda)$, written as in (5.3), and $i \in I$, we set

$$\mathrm{wt}(\delta) = \nu, \quad \varepsilon_i(\delta) = -m \quad \text{and} \quad \varphi_i(\delta) = \langle \alpha_i, \nu \rangle - m,$$

where ν is the weight of δ and $m \in \mathbb{Z}$ is the smallest integer such that the hyperplane $H_{\alpha_i,m}$ contains a face Δ'_j, with $0 \leqslant j \leqslant p+1$. Finally, $\tilde{e}_i$ and $\tilde{f}_i$ are given by the root operators e_{α_i} and f_{α_i}.

Let δ be a combinatorial gallery, written as in (5.3). We say that δ is positively folded if

$$\forall j \in \{1, \dots, p\}, \quad \Delta_{j-1} = \Delta_j \implies \Phi_+^{\mathrm{aff}}(\Delta'_j, \Delta_j) \neq \varnothing.$$

We define the dimension of δ as

$$\dim \delta = \sum_{j=0}^{p} \bigl|\Phi_+^{\mathrm{aff}}(\Delta'_j, \Delta_j)\bigr|.$$

(These are Definitions 16 and 17 in [11].) Thus, for instance, the gallery γ_λ is positively folded of dimension

$$\dim \gamma_\lambda = \bigl|\Phi_+\bigr| + p = \mathrm{ht}(\lambda - w_0\lambda) + \dim(P_\lambda/B^+), \tag{5.7}$$

by Equation (5.4). We denote the set of positively folded combinatorial gallery by $\Gamma^+(\gamma_\lambda)$. Arguing as in the proof of Proposition 4 in [11], one shows that for each $\delta \in \Gamma^+(\gamma_\lambda)$ of weight ν,

$$\dim \gamma_\lambda - \dim \delta \geqslant \mathrm{ht}(\lambda - \nu).$$

We say that a positively folded combinatorial gallery δ is an LS gallery if this inequality is in fact an equality. The set of LS galleries is denoted by $\Gamma^+_{\mathrm{LS}}(\gamma_\lambda)$. Then Corollary 2 in [11] says that $\Gamma^+_{\mathrm{LS}}(\gamma_\lambda)$ is a subcrystal of $\Gamma(\gamma_\lambda)$ and that for any gallery $\delta \in \Gamma^+_{\mathrm{LS}}(\gamma_\lambda)$, there is a sequence $(\alpha_1, \ldots, \alpha_t)$ of simple roots such that $\delta = f_{\alpha_1} \cdots f_{\alpha_t} \gamma_\lambda$. Moreover, Lemma 7 and Definition 21 in [11] say that if δ is an LS gallery, written as in (5.3), if α is a simple root, and if $m \in \mathbb{Z}$ is the smallest integer such that the hyperplane $H_{\alpha,m}$ contains a face Δ'_j, where $0 \leqslant j \leqslant p+1$, then δ does not cross $H_{\alpha,m}$; this implies that $\Delta_{j-1} = \Delta_j$ for all $j \in \{1, \ldots, p\}$ such that $\Delta'_j \subseteq H_{\alpha,m}$.

The following proposition makes the link between LS galleries and MV cycles; it is equivalent to Corollary 5 in [11] when λ is regular.

Proposition 5.7. *The map $Z : \delta \mapsto \overline{\pi(C(\delta))}$ is a bijection from $\Gamma^+_{\mathrm{LS}}(\gamma_\lambda)$ onto $\mathscr{Z}(\lambda)$; it maps a combinatorial gallery of weight ν to a MV cycle in $\mathscr{Z}(\lambda)_\nu$.*

Proof. We fix $\nu \in \Lambda$. We denote the set of combinatorial galleries of weight ν by $\Gamma(\gamma_\lambda, \nu)$ and we set $\Gamma^+(\gamma_\lambda, \nu) = \Gamma^+(\gamma_\lambda) \cap \Gamma(\gamma_\lambda, \nu)$. By construction,

$$\pi^{-1}(S^+_\nu) = \bigsqcup_{\delta \in \Gamma(\gamma_\lambda, \nu)} C(\delta).$$

We set $\mathring{\Sigma} = \pi^{-1}\bigl(\mathscr{G}_\lambda\bigr)$ and $X = \pi^{-1}\bigl(S^+_\nu \cap \mathscr{G}_\lambda\bigr)$. Since $S^+_\nu \cap \mathscr{G}_\lambda$ is of pure dimension $\mathrm{ht}(\nu - w_0\lambda)$, Proposition 5.4 and Equation (5.7) imply that X is of pure dimension

$$\mathrm{ht}(\nu - w_0\lambda) + \dim(P_\lambda/B^+) = \dim \gamma_\lambda - \mathrm{ht}(\lambda - \nu).$$

Proposition 5.4 implies also that the map $Z \mapsto \pi^{-1}(Z)$ is a bijection from the set of irreducible components of $S^+_\nu \cap \mathscr{G}_\lambda$ onto the set of irreducible components of X.

By Lemma 11 in [11], a cell $C(\delta)$ meets $\mathring{\Sigma}$ if and only if δ is positively folded. Therefore

$$X = \pi^{-1}\bigl(S^+_\nu\bigr) \cap \mathring{\Sigma} = \bigsqcup_{\delta \in \Gamma^+(\gamma_\lambda, \nu)} \bigl(C(\delta) \cap \mathring{\Sigma}\bigr).$$

Now let $\delta \in \Gamma^+(\gamma_\lambda, \nu)$. Proposition 5.5 says that the cell $C(\delta)$ is isomorphic to $\mathrm{Stab}_+(\delta)$, thus is an affine space of dimension $\dim \delta$. The intersection $C(\delta) \cap \mathring{\Sigma}$, as a non-empty open subset of $C(\delta)$, is then irreducible of dimension $\dim \delta \leqslant \dim \gamma_\lambda - \mathrm{ht}(\lambda - \nu)$. It follows that the irreducible components of X are the closures in X of the subsets $C(\delta) \cap \mathring{\Sigma}$, for δ running over the set of LS galleries of weight ν.

To conclude the proof, it remains to observe that

$$\overline{\pi\bigl(C(\delta) \cap \mathring{\Sigma}\bigr)} = \overline{\pi\bigl(C(\delta)\bigr)}$$

for each $\delta \in \Gamma^+(\gamma_\lambda, \nu)$, since $C(\delta) \cap \mathring{\Sigma}$ is dense in $C(\delta)$. □

5.3. **The comparison theorem.** The aim of this section is to show the following property of the map Z defined in Proposition 5.7.

Theorem 5.8. *The bijection $Z : \Gamma^+_{\mathrm{LS}}(\gamma_\lambda) \to \mathscr{Z}(\lambda)$ is an isomorphism of crystals.*

The existence of an isomorphism of crystals from $\mathbf{B}(\lambda)$ onto $\Gamma^+_{\mathrm{LS}}(\gamma_\lambda)$ was already known; see for instance Theorem 2 in [11] for the case λ regular. The theorem above says that the map $Z^{-1} \circ \Xi(\lambda)$ is actually such an isomorphism. For its proof, we need two propositions and a lemma.

Proposition 5.9. *Let δ be a combinatorial gallery of weight ν, written as in* (5.3), *and let $i \in I$. Call m the smallest integer such that the hyperplane $H_{\alpha_i,m}$ contains a face Δ'_j of the gallery, where $0 \leqslant j \leqslant p+1$, and set $\rho = \nu - (\langle \alpha_i, \nu\rangle - m)\,\alpha_i^\vee$. Then*

$$r_{\{i\}}(\pi(C(\delta))) = S^+_{\nu,\{i\}} \cap \overline{S^-_{\rho,\{i\}}} \quad \textit{and} \quad s_i\mu_+\Big(\overline{s_i}^{-1}\ \overline{\pi(C(\delta))}\Big) = \rho.$$

Proof. We collect in a set J the indices $j \in \{0,\dots,p\}$ such that $\Phi^{\mathrm{aff}}_+(\Delta'_j,\Delta_j)$ contains an affine root of the form (α_i, n), with $n \in \mathbb{Z}$. For each $j \in J$, there is a unique integer, say n_j, so that $(\alpha_i, n_j) \in \Phi^{\mathrm{aff}}_+(\Delta'_j,\Delta_j)$. (Thus $n_j = f_{\Delta'_j}(\alpha_i)$ in the notation of Section 5.1.)

All these integers n_j are larger or equal than m. We claim that

$$\{m, m+1, m+2, \dots\} \supseteq \{n_j \mid j \in J\} \supseteq \{m, m+1, \dots, \langle \alpha_i, \nu\rangle - 1\}. \tag{5.8}$$

Consider indeed an integer n in the right-hand side above. Since the gallery δ must go from the wall $H_{\alpha_i,m}$ to the point ν, it must cross the wall $H_{\alpha_i,n}$. More exactly, there is an index $j \in \{0,\dots,p\}$ such that $\Delta'_j \subseteq H_{\alpha_i,n}$ and $\Delta_j \not\subseteq H^-_{\alpha_i,n}$; this implies that $(\alpha_i, n) \in \Phi^{\mathrm{aff}}_+(\Delta'_j,\Delta_j)$, and thus that $j \in J$ and $n = n_j$.

We apply now the parabolic retraction $r_{\{i\}}$ to the expression given in Corollary 5.6. Equation (3.5) allows us to remove all factors in the product that belong to the unipotent radical of $P_{\{i\}}(\mathscr{K})$. We deduce that $r_{\{i\}}(\pi(C(\delta)))$ is the image of the map

$$(a_j) \mapsto \prod_{j\in J} x_{\alpha_i,n_j}(a_j)[t^\nu]$$

from $\mathbb{C}^J$ to $\mathscr{M}_{\{i\}}$. Using (5.8) and the fact that $[t^\nu]$ is fixed by all subgroups $x_{\alpha_i,n}(\mathbb{C})$ with $n \geqslant \langle \alpha_i, \nu\rangle$, we then get

$$r_{\{i\}}(\pi(C(\delta))) = \big\{x_{\alpha_i}\big(pt^{\langle\alpha,\nu\rangle}\big)[t^\nu] \;\big|\; p \in \mathbb{C}[t^{-1}]^+_{\langle\alpha,\nu\rangle - m}\big\}.$$

From there, the proposition follows easily using Proposition 3.10 (with $+$ and $-$ exchanged) and Lemma 3.12. □

For a combinatorial gallery δ, written as in Equation (5.3), and an integer $k \in \{0,\dots,p+1\}$, we set

$$\mathrm{Stab}_+(\delta)_{\geqslant k} = \mathrm{Stab}_+(\Delta'_k,\Delta_k) \times \mathrm{Stab}_+(\Delta'_{k+1},\Delta_{k+1}) \times \cdots \times \mathrm{Stab}_+(\Delta'_p,\Delta_p),$$

$$\pi(C(\delta))_{\geqslant k} = \big\{v_k v_{k+1}\cdots v_p[t^\nu] \;\big|\; (v_k, v_{k+1},\dots,v_p) \in \mathrm{Stab}_+(\delta)_{\geqslant k}\big\}.$$

Lemma 5.10. *Let δ be a combinatorial gallery, as in Equation* (5.3), *and let $k \in \{0,\dots,p+1\}$.*

(i) *Let $u \in \mathrm{Stab}_+(\Delta'_k)$. Then the left action of u on $\mathscr{G}$ leaves $\pi(C(\delta))_{\geqslant k}$ stable. More precisely, for each $(v_k,\dots,v_p) \in \mathrm{Stab}_+(\delta)_{\geqslant k}$, there exists $(v'_k,\dots,v'_p) \in \mathrm{Stab}_+(\delta)_{\geqslant k}$ such that $v'_k\cdots v'_p[t^\nu] = uv_k\cdots v_p[t^\nu]$ and*

$$\forall j \in \{k+1,\dots,p\}, \quad \Delta_{j-1} = \Delta_j \implies v_j = v'_j;$$

moreover, if $k > 0$ and $u \in \mathrm{Stab}_+(\Delta_k)$, then one can manage so that $v_k = v'_k$.

(ii) *Assume that $k > 0$, let $p \in \mathscr{O}^\times$ and let $\mu \in \Lambda$. Then the left action of p^μ on $\mathscr{G}$ leaves $\pi(C(\delta))_{\geqslant k}$ stable. Suppose, moreover, that $p \in 1 + t\mathscr{O}$ and* *let* $(v_k, \ldots, v_p) \in \mathrm{Stab}_+(\delta)_{\geqslant k}$. *Then there exists* $(v'_k, \ldots, v'_p) \in \mathrm{Stab}_+(\delta)_{\geqslant k}$ *such that* $v'_k \cdots v'_p[t^\nu] = p^\mu v_k \cdots v_p[t^\nu]$ *and*

$$\forall j \in \{k, \ldots, p\}, \quad \Delta_{j-1} = \Delta_j \implies v_j = v'_j.$$

(iii) *Assume that* $k > 0$ *and that* δ *is an LS gallery. Let* $(v_k, \ldots, v_p) \in \mathrm{Stab}_+(\delta)_{\geqslant k}$, *let* α *be a simple root of the root system* Φ, *and let* $c \in \mathbb{C}^\times$. *Call* m *the smallest integer such that the hyperplane* $H_{\alpha,m}$ *contains a face* Δ'_j, *where* $0 \leqslant j \leqslant p+1$, *form the list* $(k_1, k_2, \ldots, k_r)$ *in increasing order of all indices* $l \in \{k, \ldots, p\}$ *such that* $\Phi_+^{\mathrm{aff}}(\Delta'_l, \Delta_l) = \{(\alpha, m)\}$, *and find the complex numbers* $c_1, c_2, \ldots, c_r$ *such that* $v_{k_s} = x_{\alpha,m}(c_s)$. *Assume that* $c + c_1 + c_2 + \cdots + c_s \neq 0$ *for each* $s \in \{1, \ldots, r\}$. *Then* $x_{-\alpha,-m}(1/c)v_k \cdots v_p[t^\nu]$ *belongs to* $\pi(C(\delta))_{\geqslant k}$.

Proof. The proof of these three assertions proceeds by decreasing induction on k. For $k = p+1$, all of them hold: indeed, the element u in Assertion (i), the element p^μ in Assertion (ii) and the element $x_{-\alpha,-m}(1/c)$ in Assertion (iii) fix the point $[t^\nu]$.

Now assume that $k \leqslant p$ and that the result holds for $k+1$. If $\Phi_+^{\mathrm{aff}}(\Delta'_k, \Delta_k)$ is empty, then $\mathrm{Stab}_+(\Delta'_k, \Delta_k) = \{1\}$. Assertions (i), (ii) and (iii) then follow immediately from the inductive assumption, after one has observed that the element u in Assertion (i) belongs by assumption to $\mathrm{Stab}_+(\Delta'_k)$ and that $\mathrm{Stab}_+(\Delta'_k) = \mathrm{Stab}_+(\Delta_k) \subseteq \mathrm{Stab}_+(\Delta'_{k+1})$. In the rest of the proof, we assume that $\Phi_+^{\mathrm{aff}}(\Delta'_k, \Delta_k)$ is not empty. Let $(v_k, \ldots, v_p) \in \mathrm{Stab}_+(\delta)_{\geqslant k}$. Except in the case $k = 0$ (dealt with only in Assertion (i)), $\Phi_+^{\mathrm{aff}}(\Delta'_k, \Delta_k)$ has a unique element, say (ζ, n) with $\zeta \in \Phi_+$, and there exists $b \in \mathbb{C}$ such that $v_k = x_{\zeta,n}(b)$.

Consider first Assertion (i). The element uv_k belongs to $\mathrm{Stab}_+(\Delta'_k)$. By Proposition 5.1 (ii), there exists $v'_k \in \mathrm{Stab}_+(\Delta'_k, \Delta_k)$ and $u' \in \mathrm{Stab}_+(\Delta_k)$ such that $uv_k = v'_k u'$. The inductive assumption applied to u' and $(v_{k+1}, \ldots, v_p) \in \mathrm{Stab}_+(\delta)_{\geqslant k+1}$ asserts the existence of $(v'_{k+1}, \ldots, v'_p) \in \mathrm{Stab}_+(\delta)_{\geqslant k+1}$ such that $u' v_{k+1} \cdots v_p[t^\nu] = v'_{k+1} \cdots v'_p[t^\nu]$, with the further property that $v_j = v'_j$ for all $j > k$ verifying $\Delta_{j-1} = \Delta_j$. Certainly then $uv_k v_{k+1} \cdots v_p[t^\nu] = v'_k v'_{k+1} \cdots v'_p[t^\nu]$. Now assume that $k > 0$ and that $u \in \mathrm{Stab}_+(\Delta_k)$. By Proposition 5.1 (i), we may write u as a product of elements of the form $x_{\beta,n}(q)$ with $q \in \mathscr{O}$ and $(\beta, n) \in \Phi_+ \times \mathbb{Z}$ such that $\Delta_k \subseteq H^-_{\beta,n}$. Lemma 5.2 now implies that $uv_k \in v_k \mathrm{Stab}_+(\Delta_k)$, which establishes $v'_k = v_k$. This shows that Assertion (i) holds at k.

Consider now Assertion (ii). Let $a \in \mathbb{C}^\times$ be the constant term coefficient of p and set $q = \bigl(p^{\langle\zeta,\mu\rangle} - a^{\langle\zeta,\mu\rangle}\bigr)/t$. Then

$$p^\mu v_k = x_{\zeta,n}\bigl(bp^{\langle\zeta,\mu\rangle}\bigr)p^\mu = x_{\zeta,n}(b')u'p^\mu = v'_k u' p^\mu,$$

where $b' = ba^{\langle\zeta,\mu\rangle}$, $u' = x_{\zeta,n+1}(bq)$ and $v'_k = x_{\zeta,n}(b')$. Observing that $u' \in \mathrm{Stab}_+(\Delta_k)$ and using the inductive assumption and Assertion (i), we find a tuple $(v'_{k+1}, \ldots, v'_p) \in \mathrm{Stab}_+(\delta)_{\geqslant k+1}$ such that $u'p^\mu v_{k+1} \cdots v_p[t^\nu] = v'_{k+1} \cdots v'_p[t^\nu]$; in the case $a = 1$, we may even demand that $v_j = v'_j$ for all $j > k$ verifying $\Delta_{j-1} = \Delta_j$. Then $p^\mu v_k v_{k+1} \cdots v_p[t^\nu] = v'_k v'_{k+1} \cdots v'_p[t^\nu]$, which shows that Assertion (ii) holds at k.

It remains to prove Assertion (iii). We distinguish several cases.

Suppose first that $\zeta \neq \alpha$. By Lemma 5.2, the element

$$u = x_{-\alpha,-m}(-1/c)\ (v_k)^{-1}\ x_{-\alpha,-m}(1/c)\ v_k$$

belongs to $\mathrm{Stab}_+(\Delta_k)$. Using Assertion (i), we find $(v'_{k+1},\ldots,v'_p)\in\mathrm{Stab}_+(\delta)_{\geqslant k+1}$ such that $uv_{k+1}\cdots v_p[t^\nu]=v'_{k+1}\cdots v'_p[t^\nu]$. Moreover, since δ is an LS Gallery, we know that $\Delta_{k_s-1}=\Delta_{k_s}$ for each $s\in\{1,\ldots,r\}$, and we may thus demand that $v'_{k_s}=v_{k_s}=x_{\alpha,m}(c_s)$. Applying the inductive assumption, we find a tuple $(v''_{k+1},\ldots,v''_p)\in\mathrm{Stab}_+(\delta)_{\geqslant k+1}$ such that $x_{-\alpha,-m}(1/c)\,v'_{k+1}\cdots v'_p[t^\nu]=v''_{k+1}\cdots v''_p[t^\nu]$. Then

$$x_{-\alpha,-m}(1/c)\,v_kv_{k+1}\cdots v_p[t^\nu]=v_kv''_{k+1}\cdots v''_p[t^\nu],$$

which establishes that Assertion (iii) holds at k in this first case.

The second case is when $\zeta=\alpha$ but $n\neq m$. Then $n>m$, by the minimality of m. Let p be the square root in $1+t\mathscr{O}$ of $1+t^{n-m}b/c$. Equation (2.3) implies that

$$\begin{aligned}x_{-\alpha,-m}(1/c)v_k&=x_{-\alpha}(1/ct^m)x_\alpha(bt^n)\\&=p^{-\alpha^\vee}x_\alpha(bt^n)x_{-\alpha}(1/ct^m)p^{-\alpha^\vee}\\&=p^{-\alpha^\vee}v_kx_{-\alpha,-m}(1/c)p^{-\alpha^\vee}.\end{aligned}$$

Assertion (ii) allows us to find a tuple $(v'_{k+1},\ldots,v'_p)\in\mathrm{Stab}_+(\delta)_{\geqslant k+1}$ such that $p^{-\alpha^\vee}v_{k+1}\cdots v_p[t^\nu]=v'_{k+1}\cdots v'_p[t^\nu]$, with the further property that $v'_{k_s}=v_{k_s}=x_{\alpha,m}(c_s)$ for each $s\in\{1,\ldots,r\}$. We then apply the inductive assumption and find again $(v''_{k+1},\ldots,v''_p)\in\mathrm{Stab}_+(\delta)_{\geqslant k+1}$ such that $x_{-\alpha,-m}(1/c)\,v'_{k+1}\cdots v'_p[t^\nu]=v''_{k+1}\cdots v''_p[t^\nu]$. Then

$$x_{-\alpha,-m}(1/c)\,v_kv_{k+1}\cdots v_p[t^\nu]=p^{-\alpha^\vee}v_kv''_{k+1}\cdots v''_p[t^\nu],$$

and a final application of Assertion (ii) concludes the proof of Assertion (iii) at k in this second case.

The last case is $(\zeta,n)=(\alpha,m)$. In this case, $k_1=k$ and $b=c_{k_1}$. The assumptions of the lemma imply that $b+c\neq0$. Equation (2.3) then says that

$$x_{-\alpha,-m}(1/c)v_k=x_{\alpha,m}(bc/(b+c))(1+b/c)^{-\alpha^\vee}x_{-\alpha,-m}(1/(b+c)).$$

Applying the inductive assumption, we find $(v'_{k+1},\ldots,v'_p)\in\mathrm{Stab}_+(\delta)_{\geqslant k+1}$ such that

$$x_{-\alpha,-m}(1/(b+c))\,v_{k+1}\cdots v_p[t^\nu]=v'_{k+1}\cdots v'_p[t^\nu].$$

Now using Assertion (ii), we see that

$$x_{-\alpha,-m}(1/c)\,v_kv_{k+1}\cdots v_p[t^\nu]=x_{\alpha,m}(bc/(b+c))\,(1+b/c)^{-\alpha^\vee}v'_{k+1}\cdots v'_p[t^\nu]$$

belongs to $\pi(C(\delta))_{\geqslant k}$. This concludes the proof of Assertion (iii) at k. □

At the end of their paper [11], Gaussent and Littelmann describe several cases where the crystal structure on $\Gamma^+_{\mathrm{LS}}(\gamma_\lambda)$ controls inclusions between MV cycles. The next proposition presents a general result.

Proposition 5.11. *Let δ be an LS gallery and let α be a simple root of the system Φ. If the gallery $e_\alpha\delta$ is defined, then $Z(\delta)\subseteq Z(e_\alpha\delta)$.*

Proof. We represent δ as in (5.3). We assume that $e_\alpha\delta$ is defined and we let $m\in\mathbb{Z}$ and $0\leqslant j<k\leqslant p+1$ be as in the definition of $e_\alpha\delta$. We call $(k=k_0,k_1,\ldots,k_r)$ the list in increasing order of all indices $l\in\{1,\ldots,p\}$ such that $\Phi_+^{\mathrm{aff}}(\Delta'_l,\Delta_l)=\{(\alpha,m)\}$. Finally, we equip $\Phi_+^{\mathrm{aff}}(\Delta'_0,\Delta_0)$ with a total order.

Let $(a_{l,\beta})\in\prod_{l=0}^p\mathbb{C}^{\Phi_+^{\mathrm{aff}}(\Delta'_l,\Delta_l)}$ be a family of complex numbers such that

$$a_{k_0,(\alpha,m)}+a_{k_1,(\alpha,m)}+\cdots+a_{k_s,(\alpha,m)}\neq0$$

for each $s \in \{0, 1, \ldots, r\}$ and set

$$v_l = \prod_{\beta \in \Phi_+^{\mathrm{aff}}(\Delta'_l, \Delta_l)} x_\beta(a_{l,\beta}) \quad \text{for each } l \in \{0, 1, \ldots, p\},$$

$$A = \prod_{l=0}^{j-1} v_l \qquad \text{and} \qquad B = \prod_{l=j}^{p} v_l.$$

By Corollary 5.6, the element $AB[t^\nu]$ describes a dense subset of $Z(\delta)$ when the parameters $a_{l,\beta}$ vary. To establish the proposition, it therefore suffices to show that $AB[t^\nu]$ belongs to $Z(e_\alpha \delta)$. What we will now show is more precise:

For any non-zero complex number h, the element $Ax_{-\alpha,-m-1}(h)B[t^\nu]$ belongs to $\pi(C(e_\alpha \delta))$.

We first observe that $x_{\alpha,m+1}(1/h) \in \mathrm{Stab}_+(\Delta'_j)$, for $\Delta'_j \subseteq H_{\alpha,m+1}$. Using Lemma 5.10 (i), we find $(v'_j, v'_{j+1}, \ldots, v'_p) \in \mathrm{Stab}_+(\delta)_{\geqslant j}$ such that

$$x_{\alpha,m+1}(1/h)B[t^\nu] = v'_j v'_{j+1} \cdots v'_p [t^\nu].$$

We may, moreover, demand that $v'_{k_s} = v_{k_s} = x_{\alpha,m}(a_{k_s,(\alpha,m)})$ for all $s \in \{0, 1, \ldots, r\}$, for $\Delta_{k_s - 1} = \Delta_{k_s}$. We set

$$C = \prod_{l=j}^{k-1} v'_l \qquad \text{and} \qquad D = \prod_{l=k+1}^{p} v'_l,$$

and then $B[t^\nu] = x_{\alpha,m+1}(-1/h) C v'_k D[t^\nu]$. Using Lemma 5.10 (iii), we now find $(v''_{k+1}, v''_{k+2}, \ldots, v''_p) \in \mathrm{Stab}_+(\delta)_{\geqslant k+1}$ such that

$$x_{-\alpha,-m}\big(1/a_{k,(\alpha,m)}\big) D[t^\nu] = v''_{k+1} v''_{k+2} \cdots v''_p [t^\nu].$$

We finally set

$$E = x_{\alpha,m}\big(a_{k,(\alpha,m)}\big) x_{-\alpha,-m}\big(-1/a_{k,(\alpha,m)}\big) x_{\alpha,m}\big(a_{k,(\alpha,m)}\big),$$

$$F = x_{\alpha,m}\big(-a_{k,(\alpha,m)}\big) \prod_{l=k+1}^{p} v''_l,$$

$$K = x_{-\alpha,-m-1}(h) x_{\alpha,m+1}(-1/h).$$

Then $Ax_{-\alpha,-m-1}(h)B[t^\nu] = AKCEF[t^\nu]$.

We now observe that

$$\Phi_+^{\mathrm{aff}}\big(s_{\alpha,m+1}(\Delta'_l), s_{\alpha,m+1}(\Delta_l)\big) = \begin{cases} \big\{(\alpha, m+1)\big\} \sqcup s_{\alpha,m+1}\big(\Phi_+^{\mathrm{aff}}(\Delta'_j, \Delta_j)\big) & \text{if } l = j, \\ s_{\alpha,m+1}\big(\Phi_+^{\mathrm{aff}}(\Delta'_l, \Delta_l)\big) & \text{if } j < l < k, \end{cases}$$

and that

$$\Phi_+^{\mathrm{aff}}\big(\tau_{\alpha^\vee}(\Delta'_l), \tau_{\alpha^\vee}(\Delta_l)\big) = \tau_{\alpha^\vee}\big(\Phi_+^{\mathrm{aff}}(\Delta'_l, \Delta_l)\big) \qquad \text{if } l \geqslant k.$$

These equalities, the definition of $e_\alpha\delta$, Equation (5.1) and Proposition 5.1 (ii) imply that the sequence

$$\Big(v_0,\ \ldots,\ v_{j-1},\ x_{\alpha,m+1}(h)\big(t^{(m+1)\alpha^\vee}\overline{s_\alpha}\big)v'_j\big(t^{(m+1)\alpha^\vee}\overline{s_\alpha}\big)^{-1},$$
$$\big(t^{(m+1)\alpha^\vee}\overline{s_\alpha}\big)v'_{j+1}\big(t^{(m+1)\alpha^\vee}\overline{s_\alpha}\big)^{-1},\ \ldots,\ \big(t^{(m+1)\alpha^\vee}\overline{s_\alpha}\big)v'_{k-1}\big(t^{(m+1)\alpha^\vee}\overline{s_\alpha}\big)^{-1},$$
$$t^{\alpha^\vee}x_{\alpha,m}\big(-a_{k,(\alpha,m)}\big)t^{-\alpha^\vee},\ t^{\alpha^\vee}v''_{k+1}t^{-\alpha^\vee},\ \ldots,\ t^{\alpha^\vee}v''_pt^{-\alpha^\vee}\Big)$$

belongs to $\mathrm{Stab}_+(e_\alpha\delta)$. Proposition 5.5, Equation (5.6), and the definition of the map π then say that

$$A\,x_{\alpha,m+1}(h)\ \big(t^{(m+1)\alpha^\vee}\overline{s_\alpha}\big)\ C\ \big(t^{(m+1)\alpha^\vee}\overline{s_\alpha}\big)^{-1}\ t^{\alpha^\vee}\ F\,[t^\nu]$$

belongs to $\pi(C(e_\alpha\delta))$. An appropriate application of Lemma 5.10 (ii) shows that the element obtained by inserting extra factors $(-h)^{-\alpha^\vee}$ and $\big(-a_{k,(\alpha,m)}\big)^{-\alpha^\vee}$ in this expression, respectively after A and before $t^{\alpha^\vee}$, also belongs to $\pi(C(e_\alpha\delta))$. Now Equation (2.4) allows us to rewrite

$$K=(-h)^{-\alpha^\vee}\,x_{\alpha,m+1}(h)\ \big(t^{(m+1)\alpha^\vee}\overline{s_\alpha}\big)$$

and

$$E=\big(t^{(m+1)\alpha^\vee}\overline{s_\alpha}\big)^{-1}\ \big(-a_{k,(\alpha,m)}\big)^{-\alpha^\vee}\ t^{\alpha^\vee},$$

and we conclude that $AKCEF[t^\nu]=Ax_{-\alpha,-m-1}(h)B[t^\nu]$ belongs to $\pi(C(e_\alpha\delta))$, as announced. $\square$

Proof of Theorem 5.8. Obviously Z preserves the weight. Comparing Proposition 5.9 with Equation (4.2), we see that Z is compatible with the structure maps φ_i. The axioms of a crystal imply then that Z is compatible with the structure maps ε_i. Now let δ be an LS gallery of weight ν, let $i\in I$, and assume that the LS gallery $e_{\alpha_i}\delta$ is defined. Then the two MV cycles $Z(\delta)$ and $Z(e_{\alpha_i}\delta)$ satisfy the four conditions of Proposition 4.2. Indeed, the first and the third conditions follow immediately from the fact that $Z(\delta)\in\mathscr{Z}(\lambda)_\nu$ and $Z(e_{\alpha_i}\delta)\in\mathscr{Z}(\lambda)_{\nu+\alpha_i^\vee}$; the second condition comes from Proposition 5.9 and from the second assertion of Lemma 6 (iii) in [11]; the fourth condition comes from Proposition 5.11. Therefore, $Z(e_{\alpha_i}\delta)=\tilde{e}_iZ(\delta)$; in other words, Z intertwines the action of the root operators on $\Gamma^+_{\mathrm{LS}}(\gamma_\lambda)$ with the action of Braverman and Gaitsgory's crystal operators on $\mathscr{Z}(\lambda)$. This concludes the proof that Z is a morphism of crystals. Since Z is bijective and both crystals $\Gamma^+_{\mathrm{LS}}(\gamma_\lambda)$ and $\mathscr{Z}(\lambda)$ are normal, Z is an isomorphism. $\square$

References

1. J. Anderson, *A polytope calculus for semisimple groups,* Duke Math. J. **116** (2003), 567–588. MR1958098 (2004a:20047)
2. A. Beauville and Y. Laszlo, *Conformal blocks and generalized theta functions,* Comm. Math. Phys. **164** (1994), 385-419. MR1289330 (95k:14011)
3. A. Beilinson and V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves,* preprint available at `http://www.math.uchicago.edu/~mitya/langlands.html`.
4. A. Berenstein, S. Fomin and A. Zelevinsky, *Parametrizations of canonical bases and totally positive matrices,* Adv. Math. **122** (1996), 49–149. MR1405449 (98j:17008)
5. A. Berenstein and A. Zelevinsky, *Total positivity in Schubert varieties,* Comment. Math. Helv. **72** (1997), 128–166. MR1456321 (99g:14064)
6. A. Berenstein and A. Zelevinsky, *Tensor product multiplicities, canonical bases and totally positive varieties,* Invent. Math. **143** (2001), 77–128. MR1802793 (2002c:17005)

7. A. Białynicki-Birula, *Some theorems on actions of algebraic groups,* Ann. of Math. **98** (1973), 480–497. MR0366940 (51:3186)
8. A. Braverman, M. Finkelberg and D. Gaitsgory, *Uhlenbeck spaces via affine Lie algebras,* in *The unity of mathematics,* pp. 17–135, Progr. Math., vol. 244, Boston: Birkhäuser, 2006. MR2181803 (2007f:14008)
9. A. Braverman and D. Gaitsgory, *Crystals via the affine Grassmannian,* Duke Math. J. **107** (2001), 561–575. MR1828302 (2002e:20083)
10. F. Bruhat and J. Tits, *Groupes réductifs sur un corps local,* Publ. Math. Inst. Hautes Études Sci. **41** (1972), 5–251. MR0327923 (48:6265)
11. S. Gaussent and P. Littelmann, *LS galleries, the path model, and MV cycles,* Duke Math. J. **127** (2005), 35–88. MR2126496 (2006c:20092)
12. V. Ginzburg, *Perverse sheaves on a loop group and Langlands duality,* preprint arXiv:alg-geom/9511007.
13. J. Kamnitzer, *Mirković-Vilonen cycles and polytopes,* preprint arXiv:math.AG/0501365.
14. J. Kamnitzer, *The crystal structure on the set of Mirković-Vilonen polytopes,* Adv. Math. **215** (2007), 66–93. MR2354986
15. M. Kashiwara, *On crystal bases of the Q-analogue of universal enveloping algebras,* Duke Math. J. **63** (1991), 465–516. MR1115118 (93b:17045)
16. M. Kashiwara, *Global crystal bases of quantum groups,* Duke Math. J. **69** (1993), 455–485. MR1203234 (94b:17024)
17. M. Kashiwara, *The crystal base and Littelmann's refined Demazure character formula,* Duke Math. J. **71** (1993), 839–858. MR1240605 (95b:17019)
18. M. Kashiwara, *On crystal bases,* in *Representations of groups* (Banff, 1994), pp. 155–197, CMS Conf. Proc., vol. 16, Providence: American Mathematical Society, 1995. MR1357199 (97a:17016)
19. M. Kashiwara and Y. Saito, *Geometric construction of crystal bases,* Duke Math. J. **89** (1997), 9–36. MR1458969 (99e:17025)
20. S. Kumar, *Kac-Moody groups, their flag varieties and representation theory,* Progress in Mathematics, vol. 204, Boston: Birkhäuser, 2002. MR1923198 (2003k:22022)
21. P. Littelmann, *Paths and root operators in representation theory,* Ann. of Math. **142** (1995), 499–525. MR1356780 (96m:17011)
22. G. Lusztig, *Singularities, character formulas, and a q-analog of weight multiplicities,* in *Analyse et topologie sur les espaces singuliers* (II–III) (Luminy, 1981), pp. 208–229, Astérisque, vol. **101**–**102**, Paris: Société Mathématique de France, 1983. MR737932 (85m:17005)
23. G. Lusztig, *Canonical bases arising from quantized enveloping algebras,* J. Amer. Math. Soc. **3** (1990), 447–498. MR1035415 (90m:17023)
24. G. Lusztig, *Introduction to quantized enveloping algebras,* in *New developments in Lie theory and their applications* (Córdoba, 1989), pp. 49–65, Progr. Math., vol. **105**, Boston: Birkhäuser, 1992. MR1190735 (94b:17029)
25. G. Lusztig, *An algebraic-geometric parametrization of the canonical basis,* Adv. Math. **120** (1996), 173–190. MR1392278 (97g:17015)
26. I. Mirković and K. Vilonen, *Perverse sheaves on affine Grassmannians and Langlands duality,* Math. Res. Lett. **7** (2000), 13–24. MR1748284 (2001h:14020)
27. I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings,* Ann. of Math. **166** (2007), 95–143. MR2342692
28. S. Morier-Genoud, *Relèvement géométrique de la base canonique et involution de Schützenberger,* C. R. Math. Acad. Sci. Paris **337** (2003), 371–374. MR2015078 (2004i:22016)
29. Y. Saito, *PBW basis of quantized universal enveloping algebras,* Publ. Res. Inst. Math. Sci. **30** (1994), 209–232. MR1265471 (95e:17021)
30. M. Takeuchi, *Matched pairs of groups and bismash products of Hopf algebras,* Comm. Algebra **9** (1981), 841–882. MR611561 (83f:16013)
31. J. Tits, *Buildings of spherical type and finite BN-pairs,* Lecture Notes in Mathematics, vol. 386, Berlin: Springer-Verlag, 1974. MR0470099 (57:9866)

Institut de Recherche Mathématique Avancée, Université Louis Pasteur et CNRS, 7 rue René Descartes, 67084 Strasbourg Cedex, France
E-mail address: baumann@math.u-strasbg.fr

Institut Élie Cartan, Unité Mixte de Recherche 7502, Nancy-Université, CNRS, INRIA, Boulevard des Aiguillettes, B.P. 239, 54506 Vandœuvre-lès-Nancy Cedex, France
E-mail address: Stephane.Gaussent@iecn.u-nancy.fr