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Research Article

On Mixed Equilibrium Problems in Hadamard Spaces

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The main purpose of this paper is to study mixed equilibrium problems in Hadamard spaces. First, we establish the existence of solution of the mixed equilibrium problem and the unique existence of the resolvent operator for the problem. We then prove a strong convergence of the resolvent and a Δ-convergence of the proximal point algorithm to a solution of the mixed equilibrium problem under some suitable conditions. Furthermore, we study the asymptotic behavior of the sequence generated by a Halpern-type PPA. Finally, we give a numerical example in a nonlinear space setting to illustrate the applicability of our results. Our results extend and unify some related results in the literature.

1. Introduction

Let C be a nonempty set and Ψ be any real-valued function defined on C. The minimization problem (MP) is defined as

$$\text{find } x^* \in C \text{ such that } \Psi(x^*) \leq \Psi(y), \quad \forall y \in C. \tag{1}$$

In this case, x^* is called a minimizer of Ψ and $\operatorname{argmin}_{y\in C}\Psi(y)$ denotes the set of minimizers of Ψ. MPs are very useful in optimization theory and convex and nonlinear analysis. One of the most popular and effective methods for solving MPs is the proximal point algorithm (PPA) which was introduced in Hilbert space by Martinet [1] in 1970 and was further extensively studied in the same space by Rockafellar [2] in 1976. The PPA and its generalizations have also been studied extensively for solving MP (1) and related optimization problems in Banach spaces and Hadamard manifolds (see [3–7] and the references therein), as well as in Hadamard and p-uniformly convex metric spaces (see [8–13] and the references therein).

An important generalization of Problem (1) is the following equilibrium problem (EP), defined as

$$\text{find } x^* \in C \text{ such that } F(x^*, y) \geq 0, \quad \forall y \in C. \tag{2}$$

The point x^* for which (2) is satisfied is called an equilibrium point of F. The solution set of problem (2) is denoted by $EP(C, F)$. The EP is one of the most important problems in optimization theory that has received a lot of attention in recent time since it includes many other optimization and mathematical problems as special cases, namely, MPs, variational inequality problems, complementarity problems, fixed point problems, and convex feasibility problems, among others (see, for example, [5, 14–18]). Thus, EPs are of central importance in optimization theory as well as in nonlinear and convex analysis. As a result of this, numerous authors have studied EPs in Hilbert, Banach, and topological vector spaces (see [19, 20] and the references therein), as well as in Hadamard manifolds (see [3, 21]).

Very recently, Kumam and Chaipunya [5] extended these studies to Hadamard spaces. First, they established the existence of an equilibrium point of a bifunction satisfying some convexity, continuity, and coercivity assumptions, and they also established some fundamental properties of the resolvent of the bifunction. Furthermore, they proved that the PPA Δ-converges to an equilibrium point of a monotone bifunction in a Hadamard space. More precisely, they proved the following theorem.

Theorem 1. *Let C be a nonempty closed and convex subset of an Hadamard space X and $F: C \times C \longrightarrow \mathbb{R}$ be monotone and Δ-upper semicontinuous in the first variable such that $D(J_\lambda^F) \supset C$ for all $\lambda > 0$ (where $D(J_\lambda^F)$ means the domain of J_λ^F). Suppose that $\mathrm{EP}(C, F) \neq \varnothing$ and for an initial guess $x_0 \in C$, the sequence $\{x_n\} \subset C$ is generated by*

$$x_n := J_{\lambda_n}^F(x_{n-1}), \quad n \in \mathbb{N}, \tag{3}$$

where $\{\lambda_n\}$ is a sequence of positive real numbers bounded away from 0. Then, $\{x_n\}$ Δ-converges to an element of $\mathrm{EP}(C, F)$.

Other authors have also studied EPs in Hadamard spaces (see, for example, [14, 15]).

In the linear settings (for example, in Hilbert spaces), EPs have been generalized into what is called the mixed equilibrium problem (MEP), defined as

$$\text{find } x^* \in C \text{ such that } F(x^*, y) + \Psi(y) - \Psi(x^*) \geq 0, \quad \forall y \in C. \tag{4}$$

The MEP is an important class of optimization problems since it contains many other optimization problems as special cases. For instance, if $F \equiv 0$ in (3), then the MEP (4) reduces to MP (1). Also, if $\Psi \equiv 0$ in (3), then the MEP (4) reduces to the EP (2). The existence of solutions of the MEP (4) was established in Hilbert spaces by Peng and Yao [22] (see also [23]). More so, different iterative algorithms have been developed by numerous authors for approximating solutions of MEP (4) in real Hilbert spaces (see, for example, [22–24] and the references therein).

Since MEPs contain both MPs and EPs as special cases in Hilbert spaces, it is important to extend their study to Hadamard spaces, so as to unify other optimization problems (in particular, MPs and EPs) in Hadamard spaces. Moreover, Hadamard spaces are more suitable frameworks for the study of optimization problems and other related mathematical problems since many recent results in these spaces have already found applications in diverse fields than they do in Hilbert spaces. For instance, the minimizers of the energy functional (which is an example of a convex and lower semicontinuous functional in a Hadamard space), called harmonic maps, are very useful in geometry and analysis (see [9]). Also, the gradient flow theorem in Hadamard spaces was employed to investigate the asymptotic behavior of the Calabi flow in Kahler geometry (see [25]). Furthermore, the study of the PPA for optimization problems has successfully been applied in Hadamard spaces, for computing medians and means, which are very important in computational phylogenetics, diffusion tensor imaging, consensus algorithms, and modeling of airway systems in human lungs and blood vessels (see [26, 27], for details). It is also worthy to note that many nonconvex problems in the linear settings can be viewed as convex problems in Hadamard spaces (see Section 4 of this paper).

Therefore, it is our interest in this paper to extend the study of the MEP (4) to Hadamard spaces. First, we establish the existence of solution of the MEP (4) and the unique existence of the resolvent operator associated with F and Ψ. We then prove a strong convergence of the resolvent and a Δ-convergence of the PPA to a solution of MEP (4) under some suitable conditions on F and Ψ. Furthermore, we study the asymptotic behavior of the sequence generated by the Halpern-type PPA. Finally, we give a numerical example in a nonlinear space setting to illustrate the applicability of our results. Our results extend and unify the results of Kumam and Chaipunya [5] and Peng and Yao [22].

The rest of this paper is organized as follows: In Section 2, we recall the geometry of geodesic spaces and some useful definitions and lemmas. In Section 3, we establish the existence of solution for MEP (4) and the unique existence of the resolvent operator associated with F and Ψ. Some fundamental properties of the resolvent operator are also established in this section. In Section 4, we prove a strong convergence of the resolvent and a Δ-convergence of the PPA to a solution of MEP (4) under some suitable conditions on F and Ψ. In Section 5, we study the asymptotic behavior of the sequence generated by the Halpern-type PPA. In Section 6, we generate some numerical results in nonlinear setting for the PPA and the Halpern-type PPA, to show the applicability of our results.

2. Preliminaries

2.1. Geometry of Geodesic Spaces

Definition 1. Let (X, d) be a metric space, $x, y \in X$ and $I = [0, d(x, y)]$ be an interval. A curve c (or simply a geodesic path) joining x to y is an isometry $c: I \longrightarrow X$ such that $c(0) = x$, $c(d(x, y)) = y$, and $d(c(t), c(t') = |t - t'|)$ for all $t, t' \in I$. The image of a geodesic path is called a geodesic segment, which is denoted by $[x, y]$ whenever it is unique.

Definition 2 (see [28]). A metric space (X, d) is called a geodesic space if every two points of X are joined by a geodesic path, and X is said to be uniquely geodesic if every two points of X are joined by exactly one geodesic path. A subset C of X is said to be convex if C includes every geodesic segments joining two of its points. Let $x, y \in X$ and $t \in [0, 1]$, and we write $tx \oplus (1 - t)y$ for the unique point z in the geodesic segment joining from x to y such that

$$d(x, z) = (1 - t)d(x, y) \text{ and } d(z, y) = td(x, y). \tag{5}$$

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three vertices (points in X) with unparameterized geodesic segment between each pair of vertices. For any geodesic triangle, there is comparison (Alexandrov) triangle $\overline{\Delta} \subset \mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\overline{x}_i, \overline{x}_j)$ for $i, j \in \{1, 2, 3\}$. Let Δ be a geodesic triangle in X and $\overline{\Delta}$ be a comparison triangle for $\overline{\Delta}$, then Δ is said to satisfy the CAT(0) inequality if for all points $x, y \in \Delta$ and $\overline{x}, \overline{y} \in \overline{\Delta}$:

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{6}$$

Let x, y, and z be points in X and y_0 be the midpoint of the segment $[y, z]$; then, the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d(y, z). \quad (7)$$

Inequality (7) is known as the CN inequality of Bruhat and Titis [29].

Definition 3. A geodesic space X is said to be a CAT(0) space if all geodesic triangles satisfy the CAT(0) inequality. Equivalently, X is called a CAT(0) space if and only if it satisfies the CN inequality.

CAT(0) spaces are examples of uniquely geodesic spaces, and complete CAT(0) spaces are called Hadamard spaces.

Definition 4. Let C be a nonempty closed and convex subset of a CAT(0) space X. The metric projection is a mapping $P_C : X \longrightarrow C$ which assigns to each $x \in X$, the unique point $P_C x$ in C such that

$$d(x, P_C x) = \inf\{d(x, y) : y \in C\}. \quad (8)$$

Definition 5 (see [30]). Let X be a CAT(0) space. Denote the pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then, a mapping $\langle .,. \rangle : (X \times X) \times (X \times X) \longrightarrow \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\right), \quad \forall a, b, c, d \in X, \quad (9)$$

is called a quasilinearization mapping.

It is easy to check that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$, and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$ for all a, b, c, d, $e \in X$. A geodesic space X is said to satisfy the Cauchy–Swartz inequality if $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d) \, \forall a, b, c, d \in X$. It has been established in [30] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy–Schwartz inequality. Examples of CAT(0) spaces include Euclidean spaces $\mathbb{R}^n$, Hilbert spaces, simply connected Riemannian manifolds of nonpositive sectional curvature [31], $\mathbb{R}$-trees, and Hilbert ball [32], among others.

We end this section with the following important lemmas which characterize CAT(0) spaces.

Lemma 1. *Let X be a CAT(0) space, $x, y, z \in X$, and $t, s \in [0, 1]$. Then,*

(i) $d(tx \oplus (1-t)y, z) \leq td(x, z) + (1-t)d(y, z)$ (see [28])

(ii) $d^2(tx \oplus (1-t)y, z) \leq td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y)$ (see [28])

2.2. Notion of Δ-Convergence

Definition 6. Let $\{x_n\}$ be a bounded sequence in a geodesic metric space X. Then, the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined by

$$A(\{x_n\}) = \overline{v} \in X : \limsup_{n \longrightarrow \infty} d(\overline{v}, x_n) = \inf_{v \in X} \limsup_{n \longrightarrow \infty} d(v, x_n). \quad (10)$$

A sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $\overline{v} \in X$ if $A(\{x_{n_k}\}) = \{\overline{v}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n \longrightarrow \infty} x_n = \overline{v}$ (see [33]). The concept of Δ-convergence in metric spaces was first introduced and studied by Lim [34]. Kirk and Panyanak [35] later introduced and studied this concept in CAT(0) spaces and proved that it is very similar to the weak convergence in Banach space setting.

We now end this section with the following important lemmas which are concerned with Δ-convergence.

Lemma 2 (see [28, 36]). *Let X be an Hadamard space. Then,*

(i) Every bounded sequence in X has a Δ-convergent subsequence

(ii) Every bounded sequence in X has a unique asymptotic center

Lemma 3 ([37], Opial's Lemma). *Let X be an Hadamard space and $\{x_n\}$ be a sequence in X. If there exists a nonempty subset F in which*

(i) $\lim_{n \longrightarrow \infty} d(x_n, z)$ exists for every $z \in F$

(ii) if $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ which is Δ-convergent to x, then $x \in F$

Then, there is a $p \in F$ such that $\{x_n\}$ is Δ-convergent to p.

Lemma 4 ([14], Proposition 4.3). *Suppose that $\{x_n\}$ is Δ-convergent to q and there exists $y \in X$ such that $\limsup d(x_n, y) \leq d(q, y)$, then $\{x_n\}$ converges strongly to q.*

3. Existence and Uniqueness of Solution

In this section, we establish the existence of solution for MEP (4). We also establish the unique existence of the resolvent operator associated with the bifunction F and the convex functional Ψ. In addition, we study some fundamental properties of this resolvent operator. We begin with the following known results.

Definition 7. Let X be a CAT(0) space. A function $\Psi : D(\Psi) \subseteq X \longrightarrow \mathbb{R}$ (where $D(\Psi)$ means the domain of Ψ) is said to be convex, if

$$\Psi(tx \oplus (1-t)y) \leq t\Psi(x) + (1-t)\Psi(y), \quad \forall x, y \in X, \quad t \in (0, 1). \quad (11)$$

Ψ is lower semicontinuous (or upper semicontinuous) at a point $x \in D(\Psi)$, if

$$\Psi(x) \leq \liminf_{n \longrightarrow \infty} \Psi(x_n) \left(\text{or } \Psi(x) \geq \limsup_{n \longrightarrow \infty} \Psi(x_n) \right), \quad (12)$$

for each sequence $\{x_n\}$ in $D(\Psi)$ such that $\lim_{n \longrightarrow \infty} x_n = x$. We say that Ψ is lower semicontinuous (or upper semicontinuous) on $D(\Psi)$, if it is lower semicontinuous (or upper semicontinuous) at any point in $D(\Psi)$.

Lemma 5 (See [9]). *Let X be a Hadamard space and $\Psi : C \longrightarrow \mathbb{R}$ be a convex and lower semicontinuous function. Then, Ψ is Δ-lower semicontinuous.*

For a nonempty subset C of X, we denote by $\operatorname{conv}(C)$, the convex hull of C. That is, the smallest convex subset of X containing C. Recall that the convex hull of a finite set is the set of all convex combinations of its points.

Theorem 2 (the KKM principle) (see [5], Theorem 3.3; see also [14], Lemma 1.8). *Let C be a nonempty, closed, and convex subset of an Hadamard space X and $G : C \longrightarrow 2^C$ be a set-valued mapping with closed values. Suppose that for any finite subset $\{x_1, x_2, \ldots, x_n\}$ of C,*

$$\operatorname{conv}(\{x_1, x_2, \ldots, x_m\}) \subset \% \cup_{i=1}^{m} G(x_i). \quad (13)$$

Then, the family $\{G(x)\}_{x \in C}$ has the finite intersection property. Moreover, if $G(x_0)$ is compact for some $x_0 \in C$, then $\cap_{x \in C} G(x) \neq \emptyset$.

3.1. Existence of Solution for Mixed Equilibrium Problem

Theorem 3. *Let C be a nonempty closed and convex subset of an Hadamard space X. Let $\Psi : C \longrightarrow \mathbb{R}$ be a real-valued function and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction such that the following assumptions hold:*

(A1) $F(x, x) = 0$, $\forall x \in C$

(A2) For every $x \in C$, the set $\{y \in C : F(x, y) + \Psi(y) - \Psi(x) < 0\}$ is convex

(A3) There exists a compact subset $D \subset C$ containing a point $y_0 \in D$ such that $F(x, y_0) + \Psi(y_0) - \Psi(x) < 0$ whenever $x \in C/D$

Then, the MEP *(4) has a solution.*

Proof. For each $y \in C$, define the set-valued mapping $G : C \longrightarrow 2^C$ by

$$G(y) := \{x \in C : F(x, y) + \Psi(y) - \Psi(x) \geq 0\}. \quad (14)$$

By (A1), we obtain that, for each $y \in C$, $G(y) \neq \emptyset$ since $y \in G(y)$. Also, we obtain from (A2) that $G(y)$ is a closed subset of C for all $y \in C$.

We claim that G satisfies the inclusion (13). Suppose for contradiction that this is not true, then there exist a finite subset $\{y_1, y_2, \ldots, y_m\}$ of C and $\alpha_i \geq 0$, $\forall i = 1, 2, \ldots, m$ with $\sum_{i=1}^{m} \alpha_i = 1$ such that $y^* = \sum_{i=1}^{m} \alpha_i y_1 \notin G(y_i)$ for each $i = 1, 2, \ldots, m$. That is, there exists $y^* \in \operatorname{conv}(\{y_1, y_2, \ldots, y_m\})$ such that $y^* \notin G(y_i)$, for each $1, 2, \ldots, m$. By (14), we obtain for each $i = 1, 2, \ldots, m$ that

$$F(y^*, y_i) + \Psi(y_i) - \Psi(y^*) < 0. \quad (15)$$

Thus, for each $i = 1, 2, \ldots, m$, $y_i \in \{y \in C : F(y^*, y) + \Psi(y) - \Psi(y^*) < 0\}$, which is convex by (A2). Since $\operatorname{conv}(\{y_1, y_2, \ldots, y_m\})$ is the smallest convex set containing $y_1, y_2, \ldots, y_m$, we have that $\operatorname{conv}(\{y_1, y_2, \ldots, y_m\}) \subset \{y \in C : F(y^*, y) + \Psi(y) - \Psi(y^*) < 0\}$, which implies that $y^* \in \{y \in C : F(y^*, y) + \Psi(y) - \Psi(y^*) < 0\}$. That is, $0 = F(y^*, y^*) + \Psi(y^*) - \Psi(y^*) < 0$, which is a contradiction. Therefore, G satisfies the inclusion (13).

Now, observe that (A3) implies that there exists a compact subset D of C containing $y_0 \in D$ such that for any $x \in C/D$, we have

$$F(x, y_0) + \Psi(y_0) - \Psi(x) < 0, \quad (16)$$

which further implies that

$$G(y_0) = \{x \in C : F(x, y_0) + \Psi(y_0) - \Psi(x) \geq 0\} \subset D. \quad (17)$$

Thus, $G(y_0)$ is compact. It then follows from Theorem 2 that $\cap_{y \in C} G(y) \neq \emptyset$. This implies that there exists $x^* \in C$ such that

$$F(x^*, y) + \Psi(y) - \Psi(x^*) \geq 0, \quad \forall y \in C. \quad (18)$$

That is, MEP (4) has a solution. □

3.2. Existence and Uniqueness of Resolvent Operator

Definition 8. Let X be an Hadamard space and C be a nonempty subset of X. Let $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction, $\Psi : C \longrightarrow \mathbb{R}$ be a real-valued function, $\bar{x} \in X$, and $\lambda > 0$; then, we define the perturbation $\widetilde{F}_{\bar{x}} : C \times C \longrightarrow \mathbb{R}$ of F and Ψ, by

$$\widetilde{F}_{\bar{x}}(x, y) := F(x, y) + \Psi(y) - \Psi(x) + \frac{1}{\lambda} \langle \overrightarrow{xy}, \overrightarrow{\bar{x}x} \rangle, \quad \forall x, y \in C. \quad (19)$$

In the next theorem, we shall prove the existence and uniqueness of solution of the following auxiliary problem: find $x^* \in C$ such that

$$\widetilde{F}_{\bar{x}}(x^*, y) \geq 0, \quad \forall y \in C, \quad (20)$$

where $\widetilde{F}_{\bar{x}}$ is as defined in (19). The proof for existence is similar to the proof of Theorem 3. But for completeness, we shall give the proof here.

Theorem 4. *Let C be a nonempty closed and convex subset of an Hadamard space X. Let $\Psi : C \longrightarrow \mathbb{R}$ be a convex function and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction such that the following assumptions hold:*

(A1) $F(x, x) = 0$, $\forall x \in C$

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$, $\forall x, y, \in C$

(A3) $F(x,.) : C \longrightarrow \mathbb{R}$ *is convex* $\forall x \in C$

(A4) For each $\overline{x} \in X$ *and* $\lambda > 0$*, there exists a compact subset* $D_{\overline{x}} \subset C$ *containing a point* $y_{\overline{x}} \in D_{\overline{x}}$ such that $F(x, y_{\overline{x}}) + \Psi(y_{\overline{x}}) - \Psi(x) + (1/\lambda)\langle \overrightarrow{xy_{\overline{x}}}, \overrightarrow{\overline{x}x} \rangle < 0$*whenever* $x \in C/D_{\overline{x}}$.

Then, (20) has a unique solution.

Proof. Let $\overline{x}$ be a point in X. For each $y \in C$, define the set-valued mapping $G : C \longrightarrow 2^C$ by

$$G(y) = \left\{ x \in C : F(x, y) + \Psi(y) - \Psi(x) + \frac{1}{\lambda}\langle \overrightarrow{xy}, \overrightarrow{\overline{x}x} \rangle \geq 0 \right\}. \tag{21}$$

Then, it is easy to see that $G(y)$ is a nonempty closed subset of C. As in the proof of Theorem 3, we claim that G satisfies the inclusion (13). Suppose for contradiction that this is not true, then there exists $y^* = \sum_{i=1}^m \alpha_i y_i \in \operatorname{conv}(\{y_1, y_2, \ldots, y_m\})$ such that

$$F(y^*, y_i) + \Psi(y_i) - \Psi(y^*) + \frac{1}{\lambda}\langle \overrightarrow{y^* y_i}, \overrightarrow{\overline{x}y^*} \rangle < 0, \quad i = 1, 2, \ldots, m. \tag{22}$$

By (A3) and the convexity of Ψ, we obtain that

$$\begin{aligned} 0 &= F(y^*, y^*) + \Psi(y^*) - \Psi(y^*) + \frac{1}{\lambda}\langle \overrightarrow{y^* y^*}, \overrightarrow{\overline{x}y^*} \rangle \\ &\leq \sum_{i=1}^m \alpha_i (F(y^*, y_i) + \Psi(y_i) - \Psi(y^*)) \\ &\quad + \frac{1}{\lambda}\left(\sum_{i=1}^m \alpha_i \langle \overrightarrow{y^* y_i}, \overrightarrow{\overline{x}y^*} \rangle \right) < 0, \end{aligned} \tag{23}$$

which is a contradiction. Therefore, G satisfies the inclusion (13). By (A4), we obtain that $G(y_{\overline{x}}) \subset D_{\overline{x}}$. Thus, $G(y_{\overline{x}})$ is compact and by Theorem 2, we get that $\cap_{y \in C} G(y) \neq \emptyset$. Therefore, (20) has a solution.

Next, we show that this solution is unique. Suppose that x and x^* solve (20). Then,

$$\begin{aligned} 0 &\leq \widetilde{F}_{\overline{x}}(x, x^*) = F(x, x^*) + \Psi(x^*) - \Psi(x) + \frac{1}{\lambda}\langle \overrightarrow{\overline{x}x}, \overrightarrow{xx^*} \rangle, \\ 0 &\leq \widetilde{F}_{\overline{x}}(x^*, x) = F(x^*, x) + \Psi(x) - \Psi(x^*) + \frac{1}{\lambda}\langle \overrightarrow{\overline{x}x^*}, \overrightarrow{x^* x} \rangle. \end{aligned} \tag{24}$$

Adding both inequalities and noting that F is monotone, we obtain that

$$\begin{aligned} 0 &\leq -\frac{1}{\lambda}\left(\langle \overrightarrow{\overline{x}x}, \overrightarrow{xx^*} \rangle + \langle \overrightarrow{\overline{x}x^*}, \overrightarrow{xx^*} \rangle \right) \\ &= -\frac{1}{\lambda} d(x, x^*)^2, \end{aligned} \tag{25}$$

which implies that $x = x^*$. □

Definition 9. Let X be an Hadamard space and C be a nonempty closed and convex subset of X. Let $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction and $\Psi : C \longrightarrow \mathbb{R}$ be a convex function. Assume that (20) has a unique solution for each $\lambda > 0$ and $x \in X$. This unique solution is denoted by $J_{\lambda F}^{\Psi} x$, and it is called the resolvent operator associated with F and Ψ of order $\lambda > 0$ and at $x \in X$. In other words, the resolvent operator associated with F and Ψ is the set-valued mapping $J_{\lambda F}^{\Psi} : X \longrightarrow 2^C$ defined by

$$\begin{aligned} J_{\lambda F}^{\Psi}(x) := EP(C, \widetilde{F}_x) = \Big\{ z \in C : F(z, y) + \Psi(y) - \Psi(z) \\ + \frac{1}{\lambda}\langle \overrightarrow{zy}, \overrightarrow{xz} \rangle \geq 0, \forall y \in C \Big\}, \quad \text{for all } x \text{ in } X. \end{aligned} \tag{26}$$

Under the assumptions of Theorem 4, we have the unique existence of $J_{\lambda F}^{\Psi}(x)$. Therefore, $J_{\lambda F}^{\Psi}$ is well defined.

3.3. Fundamental Properties of the Resolvent Operator.

In the following theorem, we shall study some fundamental properties of the resolvent operator. First, we recall the following definitions which will be needed for our study.

Definition 10. Let X be a CAT(0) space. A point $x \in X$ is called a fixed point of a nonlinear mapping $T : X \longrightarrow X$, if $Tx = x$. We denote the set of fixed points of T by $\operatorname{Fix}(T)$. The mapping T is said to be

(i) Firmly nonexpansive, if

$$d^2(Tx, Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle, \quad \forall x, y \in X. \tag{27}$$

(ii) Nonexpansive, if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in X. \tag{28}$$

Theorem 5. *Let C be a nonempty closed and convex subset of an Hadamard space X. Let* $\Psi : C \longrightarrow \mathbb{R}$*be a convex function and* $F : C \times C \longrightarrow \mathbb{R}$ *be a bifunction satisfying assumptions (A1)–(A4) of Theorem 4. For* $\lambda > 0$*, we have that* $J_{\lambda F}^{\Psi}$*is single valued. Moreover, if* $C \subset D(J_{\lambda F}^{\Psi})$*,then*

(i) $J_{\lambda F}^{\Psi}$*is firmly nonexpansive restricted to C*

(ii) For $F(J_{\lambda F}^{\Psi}) \neq \emptyset$*,we have*

$$\begin{aligned} d^2\left(J_{\lambda F}^{\Psi} x, x\right) \leq d^2(x, v) - d^2\left(J_{\lambda F}^{\Psi} x, v\right), \\ \forall x \in C, \forall v \in \operatorname{fix}\left(J_{\lambda F}^{\Psi}\right), \end{aligned} \tag{29}$$

(iii) For $0 < \lambda \leq \mu$, *we have* $d(J_{\mu F}^{\Psi} x, J_{\lambda F}^{\Psi} x) \leq \sqrt{1 - (\lambda/\mu)} d(x, J_{\mu F}^{\Psi} x)$,*which implies that* $d(x, J_{\lambda F}^{\Psi} x) \leq 2d(x, J_{\mu F}^{\Psi} x)$, $\forall x \in C$

(iv) $\operatorname{Fix}(J_{\lambda F}^{\Psi}) = \operatorname{MEP}(C, F, \Psi)$

Proof. For each $x \in D(J_{\lambda F}^{\Psi})$ and $\lambda > 0$, let $z_1, z_2 \in J_{\lambda F}^{\Psi} x$. Then from (26), we have

$$\begin{aligned} &F(z_1, z_2) + \Psi(z_2) - \Psi(z_1) + \frac{1}{\lambda}\langle \overrightarrow{z_1 z_2}, \overrightarrow{x z_1}\rangle \geq 0, \\ &F(z_2, z_1) + \Psi(z_1) - \Psi(z_2) + \frac{1}{\lambda}\langle \overrightarrow{z_2 z_1}, \overrightarrow{x z_2}\rangle \geq 0. \end{aligned} \tag{30}$$

Adding both inequalities and using assumption (A2), we obtain that

$$\langle \overrightarrow{z_2 z_1}, \overrightarrow{z_1 z_2}\rangle \geq 0, \tag{31}$$

which implies that $d^2(z_1, z_2) \leq 0$. This further implies that $z_1 = z_2$. Therefore, $J_{\lambda F}^{\Psi}$ is single valued.

(i) Let $x, y \in C$, then

$$\begin{aligned} &F\left(J_{\lambda F}^{\Psi} x, J_{\lambda F}^{\Psi} y\right) + \Psi\left(J_{\lambda F}^{\Psi} y\right) - \Psi\left(J_{\lambda F}^{\Psi} x\right) \\ &\quad + \frac{1}{\lambda}\langle \overrightarrow{J_{\lambda F}^{\Psi} x J_{\lambda F}^{\Psi} y}, \overrightarrow{x J_{\lambda F}^{\Psi} x}\rangle \geq 0, \end{aligned} \tag{32}$$

and

$$\begin{aligned} &F\left(J_{\lambda F}^{\Psi} y, J_{\lambda F}^{\Psi} x\right) + \Psi\left(J_{\lambda F}^{\Psi} x\right) - \Psi\left(J_{\lambda F}^{\Psi} y\right) \\ &\quad + \frac{1}{\lambda}\langle \overrightarrow{J_{\lambda F}^{\Psi} y J_{\lambda F}^{\Psi} x}, \overrightarrow{y J_{\lambda F}^{\Psi} y}\rangle \geq 0. \end{aligned} \tag{33}$$

Adding (32) and (33), and noting that F is monotone, we obtain

$$\frac{1}{\lambda}\left(\langle \overrightarrow{x J_{\lambda F}^{\Psi} x}, \overrightarrow{J_{\lambda F}^{\Psi} x J_{\lambda F}^{\Psi} y}\rangle + \langle \overrightarrow{y J_{\lambda F}^{\Psi} y}, \overrightarrow{J_{\lambda F}^{\Psi} y J_{\lambda F}^{\Psi} x}\rangle\right) \geq 0, \tag{34}$$

which implies that

$$\langle \overrightarrow{xy}, \overrightarrow{J_{\lambda F}^{\Psi} x J_{\lambda F}^{\Psi} y}\rangle \geq \langle \overrightarrow{J_{\lambda F}^{\Psi} x J_{\lambda F}^{\Psi} y}, \overrightarrow{J_{\lambda F}^{\Psi} x J_{\lambda F}^{\Psi} y}\rangle. \tag{35}$$

That is,

$$\langle \overrightarrow{xy}, \overrightarrow{J_{\lambda F}^{\Psi} x J_{\lambda F}^{\Psi} y}\rangle \geq d^2\left(J_{\lambda F}^{\Psi} x, J_{\lambda F}^{\Psi} y\right). \tag{36}$$

(ii) It follows from (36) and the definition of quasilinearization that

$$\begin{aligned} &d^2\left(x, J_{\lambda F}^{\Psi} x\right) \leq d^2(x, v) - d^2\left(v, J_{\lambda F}^{\Psi} x\right), \\ &\forall x \in C,\ v \in \operatorname{fix}\left(J_{\lambda F}^{\Psi}\right). \end{aligned} \tag{37}$$

(iii) Let $x \in C$ and $0 < \lambda \leq \mu$, then we have that

$$\begin{aligned} &F\left(J_{\lambda F}^{\Psi} x, J_{\mu F}^{\Psi} x\right) + \Psi\left(J_{\mu F}^{\Psi} x\right) - \Psi\left(J_{\lambda F}^{\Psi} x\right) \\ &\quad + \frac{1}{\lambda}\langle \overrightarrow{x J_{\lambda F}^{\Psi} x}, \overrightarrow{J_{\lambda F}^{\Psi} x J_{\mu F}^{\Psi} x}\rangle \geq 0, \end{aligned} \tag{38}$$

and

$$\begin{aligned} &F\left(J_{\mu F}^{\Psi} x, J_{\lambda F}^{\Psi} x\right) + \Psi\left(J_{\lambda F}^{\Psi} x\right) - \Psi\left(J_{\mu F}^{\Psi} x\right) \\ &\quad + \frac{1}{\mu}\langle \overrightarrow{x J_{\mu F}^{\Psi} x}, \overrightarrow{J_{\mu F}^{\Psi} x J_{\lambda F}^{\Psi} x}\rangle \geq 0. \end{aligned} \tag{39}$$

Adding (38) and (39), and using the monotonicity of F, we obtain that

$$\langle \overrightarrow{J_{\lambda F}^{\Psi} x x}, \overrightarrow{J_{\mu F}^{\Psi} x J_{\lambda F}^{\Psi} x}\rangle \geq \frac{\lambda}{\mu}\langle \overrightarrow{J_{\mu F}^{\Psi} x x}, \overrightarrow{J_{\mu F}^{\Psi} x J_{\lambda F}^{\Psi} x}\rangle. \tag{40}$$

By quasilinearization, we obtain that

$$\begin{aligned} &\left(\frac{\lambda}{\mu} + 1\right) d^2\left(J_{\mu F}^{\Psi} x, J_{\lambda F}^{\Psi} x\right) \leq \left(1 - \frac{\lambda}{\mu}\right) d^2\left(x, J_{\mu F}^{\Psi} x\right) \\ &\quad + \left(\frac{\lambda}{\mu} - 1\right) d^2\left(x, J_{\lambda F}^{\Psi} x\right). \end{aligned} \tag{41}$$

Since $(\lambda/\mu) \leq 1$, we obtain that

$$\left(\frac{\lambda}{\mu} + 1\right) d^2\left(J_{\mu F}^{\Psi} x, J_{\lambda F}^{\Psi} x\right) \leq \left(1 - \frac{\lambda}{\mu}\right) d^2\left(x, J_{\mu F}^{\Psi} x\right), \tag{42}$$

which implies that

$$d\left(J_{\mu F}^{\Psi} x, J_{\lambda F}^{\Psi} x\right) \leq \sqrt{1 - \frac{\lambda}{\mu}}\, d\left(x, J_{\mu F}^{\Psi} x\right). \tag{43}$$

Moreover, we obtain by triangle inequality and (43) that

$$d\left(x, J_{\lambda F}^{\Psi} x\right) \leq 2d\left(x, J_{\mu F}^{\Psi} x\right). \tag{44}$$

(iv) Observe that

$$\begin{aligned} x \in \operatorname{fix}\left(J_{\lambda F}^{\Psi}\right) &\Longleftrightarrow F(x, y) + \Psi(y) \\ &\quad - \Psi(x) + \frac{1}{\lambda}\langle \overrightarrow{xx}, \overrightarrow{xy}\rangle \geq 0, \\ &\quad \forall y \in C \\ &\Longleftrightarrow F(x, y) + \Psi(y) - \Psi(x) \geq 0, \quad \forall y \in C \\ &\Longleftrightarrow x \in \operatorname{MEP}(C, F, \Psi). \end{aligned} \tag{45}$$

□

Remark 1. It follows from Cauchy–Schwartz inequality that firmly nonexpansive mappings are nonexpansive, and it is well known that the set of fixed points of nonexpansive mappings is closed and convex. Thus, by (i) and (iv) of Theorem 5, we have that MEP(C, F, Ψ) is closed and convex.

4. Convergence Results

For the rest of this paper, we shall assume that C is a nonempty closed and convex subset of an Hadamard space X and that $D(J_{\lambda F}^{\Psi}) \supset C$.

4.1. Convergence of Resolvent. In the following theorem, we shall prove that $\{J_{\lambda F}^{\Psi}x\}$ converges strongly to a solution of MEP (4) as $\lambda \longrightarrow 0$.

Theorem 6. *Let $\Psi : C \longrightarrow \mathbb{R}$ be a convex and lower semicontinuous function and $F : C \times C \longrightarrow \mathbb{R}$ be Δ-upper semicontinuous in the first argument which satisfies assumptions (A1)–(A4) of Theorem 4. If* MEP$(C, F, \Psi) \neq \emptyset$, *then $\{J_{\lambda_n F}^{\Psi}x\}$ converges strongly to $q \in$ MEP(C, F, Ψ), which is the nearest point of* MEP(C, F, Ψ) *to x as $\lambda \longrightarrow 0$.*

Proof. Let $v \in$ MEP(C, F, Ψ), since $J_{\lambda F}^{\Psi}$ is nonexpansive (by Remark 1), we obtain that $\{J_{\lambda F}^{\Psi}x\}$ is bounded. Let $\{\lambda_n\}$ be a sequence that converges to 0 as $n \longrightarrow \infty$. Then, $\{J_{\lambda_n F}^{\Psi}x\}$ is bounded. Thus, by Lemma 2(i), there exists a subsequence $\{J_{\lambda_{nk} F}^{\Psi}x\}$ of $\{J_{\lambda_n F}^{\Psi}x\}$ that Δ-converges to $q \in C$.

Now, observe that, by the definition of $J_{\lambda F}^{\Psi}$, the Δ-upper semicontinuity of F, lower semicontinuous of Ψ, and Lemma 5, we obtain that

$$F(q, y) + \Psi(y) - \Psi(q) \geq 0. \tag{46}$$

Therefore, $q \in$ MEP(C, F, Ψ). Hence, we obtain from Theorem 5(ii) that

$$d^2\left(J_{\lambda_{nk} F}^{\Psi}x, x\right) \leq d^2(x, v), \quad \forall v \in \text{MEP}(C, F, \Psi). \tag{47}$$

Since $d^2(., x)$ is Δ-lower semicontinuous, we obtain that

$$d^2(q, x) \leq \liminf_{k \longrightarrow \infty} d^2\left(J_{\lambda_{nk} F}^{\Psi}x, x\right) \leq d^2(x, v), \quad \forall v \in \text{MEP}(C, F, \Psi), \tag{48}$$

which implies that

$$d(q, x) \leq d(x, v), \quad \forall v \in \text{MEP}(C, F, \Psi). \tag{49}$$

Thus, $q = P_\Gamma x$, where P_Γ is the metric projection of X onto Γ, and $\Gamma =$ MEP(C, F, Ψ). Therefore, by taking $\lambda_{nk} = \lambda$, we have that $\{J_{\lambda F}^{\Psi}x\}$ Δ-converges to $q = P_\Gamma x$ as $\lambda \longrightarrow 0$.

Now, observe also that Theorem 5(ii) implies that

$$d\left(J_{\lambda F}^{\Psi}x, x\right) \leq d(q, x). \tag{50}$$

It then follows from Lemma 4 that $\{J_{\lambda F}^{\Psi}x\}$ converges strongly to $q = P_\Gamma x$ as $\lambda \longrightarrow 0$.

By setting $\Psi \equiv 0$ in Theorem 6, we obtain the following result which is similar to ([14], Theorem 4.4). □

Corollary 1. *Let $F : C \times C \longrightarrow \mathbb{R}$ be Δ-upper semicontinuous in the first argument which satisfies assumptions (A1)–(A4) of Theorem 4. If* MEP$(C, F) \neq \emptyset$, *then $\{J_{\lambda F}x\}$ converges strongly to $q \in$ MEP(C, F), which is the nearest point of* MEP(C, F) *to x as $\lambda \longrightarrow 0$.*

4.2. Proximal Point Algorithm. In this section, we study the Δ-convergence of the sequence generated by the following PPA for approximating solutions of MEP (4): For an initial starting point x_1 in C, define the sequence $\{x_n\}$ in C by

$$x_{n+1} = J_{\lambda_n F}^{\Psi}x_n, \quad n \geq 1, \tag{51}$$

where $\{\lambda_n\}$ is a sequence in $(0, \infty)$, $F : C \times C \longrightarrow \mathbb{R}$ is a bifunction, and $\Psi : C \longrightarrow \mathbb{R}$ is a convex function.

Recall that the PPA does not converge strongly in general without additional assumptions even for the case where $F \equiv 0$. See for example, the question of interest raised by Rockafella as to whether the PPA can be improved from weak convergence (an analogue of Δ-convergence) to strong convergence in Hilbert space settings. Several counterexamples have been constructed to resolve this question in the negative (see [38, 39]). Therefore, only weak convergence of the PPA is expected without additional assumptions. For this reason, we propose the following Δ-convergence theorem for the PPA (51).

Theorem 7. *Let $\Psi : C \longrightarrow \mathbb{R}$ be a convex and lower semicontinuous function and $F : C \times C \longrightarrow \mathbb{R}$ be Δ-upper semicontinuous in the first argument which satisfies assumptions (A1)–(A4) of Theorem 4. Let $\{\lambda_n\}$ be a sequence in $(0, \infty)$ such that $0 < \lambda \leq \lambda_n$, $\forall n \geq 1$. Suppose that* MEP$(C, F, \Psi) \neq \emptyset$, *then, the sequence given by (51) Δ-converges to an element of* MEP(C, F, Ψ).

Proof. Let $v \in$ MEP(C, F, Ψ). Then, by Remark 1 and Theorem 5(iv), we obtain that

$$d(v, x_{n+1}) = d\left(v, J_{\lambda_n F}^{\Psi}x_n\right) \leq d(v, x_n), \tag{52}$$

which implies that $\lim_{n \longrightarrow \infty} d(x_n, v)$ exists for all $v \in$ MEP(C, F, Ψ). Hence $\{x_n\}$ is bounded. It then follows from Theorem 5(ii) that

$$d^2(x_{n+1}, x_n) \leq d^2(x_n, v) - d^2(x_{n+1}, v) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{53}$$

That is,

$$\lim_{n \longrightarrow \infty} d(x_{n+1}, x_n) = 0. \tag{54}$$

Since $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ that Δ-converges to a point, say $q \in C$. From (51) and (26), we obtain that

$$\begin{aligned} F(x_{nk+1}, y) + \Psi(y) - \Psi(x_{nk+1}) &\geq -\frac{1}{\lambda_{nk}}\langle \overrightarrow{x_{nk}x_{nk+1}}, \overrightarrow{x_{nk+1}y}\rangle \\ &\geq -\frac{1}{\lambda_{nk}} d(x_{nk+1}, x_{nk}) d(x_{nk+1}, y). \end{aligned} \tag{55}$$

Since $0<\lambda\leq\lambda_{nk}$, $\{x_n\}$ is bounded, F is Δ-upper semi-continuous in the first argument and Ψ is lower semi-continuous, we obtained from (54) and (55) that

$$\begin{aligned} F(q,y)+\Psi(y)-\Psi(q) &\geq \limsup_{k\longrightarrow\infty}\left(F\left(x_{nk+1},y\right)+\Psi(y)\right) \\ &\quad -\liminf_{k\longrightarrow\infty}\Psi\left(x_{nk+1}\right) \\ &\geq -\frac{M}{\lambda}\limsup_{k\longrightarrow\infty} d\left(x_{nk+1},x_{nk}\right)=0, \end{aligned} \tag{56}$$

for some $M>0$ and for all $y\in C$. This implies that $q\in \mathrm{MEP}(C,F,\Psi)$.

It then follows from Lemma 3 that $\{x_n\}\Delta$-converges to an element of $\mathrm{MEP}(C,F,\Psi)$.

By setting $\Psi\equiv 0$ in Theorem 7, we obtain the following result which coincides with ([5], Theorem 7.3). □

Corollary 2. *Let $F: C\times C\longrightarrow\mathbb{R}$ be Δ-upper semi-continuous in the first argument which satisfies assumptions (A1)–(A4) of Theorem 4 and $\{\lambda_n\}$ be a sequence in $(0,\infty)$ such that $0<\lambda\leq\lambda_n\,\forall n\geq 1$. Suppose that $\mathrm{EP}(C,F)\neq\emptyset$; then, the sequence given for $x_1\in C$ by*

$$x_{n+1}=J_{\lambda_n F}x_n,\quad n\geq 1. \tag{57}$$

Δ-converges to an element of $\mathrm{EP}(C,F)$.

By setting $F\equiv 0$ in Theorem 7, we obtain the following corollary which is similar to ([9], Theorem 1.4).

Corollary 3. *Let $\Psi: C\longrightarrow\mathbb{R}$ be a convex and lower semicontinuous function and $\{\lambda_n\}$ be a sequence in $(0,\infty)$ such that $0<\lambda\leq\lambda_n$, $\forall n\geq 1$. Suppose that $\operatorname{argmin}_{y\in C}\Psi(y)\neq\emptyset$; then, the sequence given for $x_1\in C$ by*

$$x_{n+1}=J^{\Psi}_{\lambda_n}x_n,\quad n\geq 1. \tag{58}$$

Δ-converges to an element of $\operatorname{argmin}_{y\in C}\Psi(y)$.

5. Asymptotic Behavior of Halpern's Algorithm

To obtain strong convergence result, we modify the PPA into the following Halpern-type PPA and study the asymptotic behavior of the sequence generated by it: For $x_1, u\in C$, define the sequence $\{x_n\}\subset C$ by

$$x_{n+1}=\alpha_n u\oplus(1-\alpha_n)J^{\Psi}_{\lambda_n F}x_n, \tag{59}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$, F and Ψ are as defined in (51).

We begin by establishing the following lemmas which will be very useful to our study.

Lemma 6. *Let $\Psi: C\longrightarrow\mathbb{R}$ be a convex and lower semi-continuous function and $F: C\times C\longrightarrow\mathbb{R}$ be a bifunction satisfying (A1)–(A4) of Theorem 4. If $\lambda,\mu>0$ and $x,y\in C$, then the following inequalities hold:*

$$\begin{aligned} & d^2\left(J^{\Psi}_{\lambda F}x,J^{\Psi}_{\mu F}y\right)\leq 2\lambda F\left(J^{\Psi}_{\lambda F}x,J^{\Psi}_{\mu F}y\right)+2\lambda\left(\Psi\left(J^{\Psi}_{\mu F}y\right)-\Psi\left(J^{\Psi}_{\lambda F}x\right)\right)+d^2\left(x,J^{\Psi}_{\mu F}y\right)-d^2\left(x,J^{\Psi}_{\lambda F}x\right), \\ & (\lambda+\mu)d^2\left(J^{\Psi}_{\lambda F}x,J^{\Psi}_{\mu F}y\right)+\mu d^2\left(J^{\Psi}_{\lambda F}x,x\right)+\lambda d^2\left(J^{\Psi}_{\mu F}y,y\right)\leq\lambda d^2\left(J^{\Psi}_{\lambda F}x,y\right)+\mu d^2\left(J^{\Psi}_{\lambda F}y,x\right). \end{aligned} \tag{60}$$

Proof. We first prove (60). Let λ, $\mu>0$ and x, $y\in C$. Then, by (26), we obtain that

$$F\left(J^{\Psi}_{\lambda F}x,z\right)+\Psi(z)-\Psi\left(J^{\Psi}_{\lambda F}x\right)+\frac{1}{\lambda}\langle\overrightarrow{xJ^{\Psi}_{\lambda F}x},\overrightarrow{J^{\Psi}_{\lambda F}xz}\rangle\geq 0,\quad\forall z\in C, \tag{61}$$

which implies that

$$\begin{aligned} 2\lambda\Psi\left(J^{\Psi}_{\lambda F}x\right) &\leq 2\lambda F\left(J^{\Psi}_{\lambda F}x,z\right)+2\lambda\Psi(z)+2\langle\overrightarrow{xJ^{\Psi}_{\lambda F}x},\overrightarrow{J^{\Psi}_{\lambda F}xz}\rangle \\ &= 2\lambda F\left(J^{\Psi}_{\lambda F}x,z\right)+2\lambda\Psi(z)+d^2(x,z)-d^2\left(x,J^{\Psi}_{\lambda F}\right) \\ &\quad -d^2\left(J^{\Psi}_{\lambda F}x,z\right) \\ &\leq 2\lambda F\left(J^{\Psi}_{\lambda F}x,z\right)+2\lambda\Psi(z)+d^2(x,z) \\ &\quad -d^2\left(x,J^{\Psi}_{\lambda F}x\right). \end{aligned} \tag{62}$$

Now, set $z=tJ^{\Psi}_{\mu F}y\oplus(1-t)J^{\Psi}_{\lambda F}x$ for all $t\in(0,1)$ in (5). Since Ψ is convex and F satisfies conditions (A1) and (A3) of Theorem 4, we obtain that

$$\begin{aligned} 2\lambda\Psi\left(J^{\Psi}_{\lambda F}x\right)+d^2\left(x,J^{\Psi}_{\lambda F}x\right) &\leq 2\lambda\Big(tF\left(J^{\Psi}_{\lambda F}x,J^{\Psi}_{\mu F}y\right) \\ &\quad +(1-t)F\left(J^{\Psi}_{\lambda F}x,J^{\Psi}_{\lambda F}x\right)\Big) \\ &\quad +2\lambda\left(t\Psi\left(J^{\Psi}_{\mu F}y\right)+(1-t)\Psi\left(J^{\Psi}_{\lambda F}x\right)\right) \\ &\quad +td^2\left(x,J^{\Psi}_{\mu F}y\right)+(1-t)d^2\left(x,J^{\Psi}_{\lambda F}x\right) \\ &\quad -t(1-t)d^2\left(J^{\Psi}_{\mu F}y,J^{\Psi}_{\lambda F}x\right) \\ &= 2\lambda tF\left(J^{\Psi}_{\lambda F}x,J^{\Psi}_{\mu F}y\right) \\ &\quad +2\lambda\left(t\Psi\left(J^{\Psi}_{\mu F}y\right)+(1-t)\Psi\left(J^{\Psi}_{\lambda F}x\right)\right) \\ &\quad +td^2\left(x,J^{\Psi}_{\mu F}y\right)+(1-t)d^2\left(x,J^{\Psi}_{\lambda F}x\right) \\ &\quad -t(1-t)d^2\left(J^{\Psi}_{\mu F}y,J^{\Psi}_{\lambda F}x\right), \end{aligned} \tag{63}$$

which implies that

$$2\lambda\Psi\left(J_{\lambda F}^{\Psi}x\right)+d^{2}\left(x,J_{\lambda F}^{\Psi}x\right)\leq 2\lambda F\left(J_{\lambda F}^{\Psi}x,J_{\mu F}^{\Psi}y\right)+2\lambda\Psi\left(J_{\mu F}^{\Psi}y\right)+d^{2}\left(x,J_{\mu F}^{\Psi}y\right)-(1-t)d^{2}\left(J_{\mu F}^{\Psi}y,J_{\lambda F}^{\Psi}x\right). \quad (64)$$

As $t\longrightarrow 0$ in (64), we obtain (60).

Next, we prove (60). From (60), we obtain that

$$\mu d^{2}\left(J_{\lambda F}^{\Psi}x,J_{\mu F}^{\Psi}y\right)\leq 2\lambda\mu\left[F\left(J_{\lambda F}^{\Psi}x,J_{\mu F}^{\Psi}y\right)+\Psi\left(J_{\mu F}^{\Psi}y\right)-\Psi\left(J_{\lambda F}^{\Psi}x\right)\right]+\mu d^{2}\left(x,J_{\mu F}^{\Psi}y\right)-\mu d^{2}\left(x,J_{\lambda F}^{\Psi}x\right). \quad (65)$$

Similarly, we have

$$\lambda d^{2}\left(J_{\mu F}^{\Psi}y,J_{\lambda F}^{\Psi}x\right)\leq 2\mu\lambda\left[F\left(J_{\mu F}^{\Psi}y,J_{\lambda F}^{\Psi}x\right)+\Psi\left(J_{\lambda F}^{\Psi}x\right)-\Psi\left(J_{\mu F}^{\Psi}y\right)\right]+\lambda d^{2}\left(y,J_{\lambda F}^{\Psi}x\right)-\lambda d^{2}\left(y,J_{\mu F}^{\Psi}y\right). \quad (66)$$

Adding both inequalities and noting that F is monotone, we get

$$(\lambda+\mu)d^{2}\left(J_{\lambda F}^{\Psi}x,J_{\mu F}^{\Psi}y\right)+\mu d^{2}\left(x,J_{\lambda F}^{\Psi}x\right)+\lambda d^{2}\left(y,J_{\mu F}^{\Psi}y\right)\leq\mu d^{2}\left(x,J_{\mu F}^{\Psi}y\right)+\lambda d^{2}\left(y,J_{\lambda F}^{\Psi}x\right). \quad (67)$$

□

Lemma 7. *Let $\Psi: C\longrightarrow\mathbb{R}$ be a convex and lower semicontinuous function and $F: C\times C\longrightarrow\mathbb{R}$ be a bifunction satisfying (A1)–(A4) of Theorem 4. Let $\{\lambda_n\}$ be a sequence in $(0,\infty)$ and $\bar{v}$ be an element of C. Suppose that $\lim_{n\longrightarrow\infty}\lambda_n=\infty$ and $A(\{J_{\lambda_n}^{\Psi}x_n\})=\{\bar{v}\}$ for some bounded sequence $\{x_n\}$ in X, then $\bar{v}\in \mathrm{MEP}(C,F,\Psi)$.*

Proof. From (60), we obtain that

$$(\lambda_n+1)d^{2}\left(J_{\lambda_n F}^{\Psi}x_n,J_{F}^{\Psi}\bar{v}\right)+d^{2}\left(J_{\lambda_n F}^{\Psi}x_n,x_n\right)+\lambda_n d^{2}\left(J_{F}^{\Psi}\bar{v},\bar{v}\right)\leq d^{2}\left(J_{F}^{\Psi}\bar{v},x_n\right)+\lambda_n d^{2}\left(J_{\lambda_n F}^{\Psi}x_n,\bar{v}\right), \quad (68)$$

which implies that

$$d^{2}\left(J_{\lambda_n F}^{\Psi}x_n,J_{F}^{\Psi}\bar{v}\right)\leq\frac{1}{\lambda_n}d^{2}\left(J_{F}^{\Psi}\bar{v},x_n\right)+d^{2}\left(J_{\lambda_n F}^{\Psi}x_n,\bar{v}\right)^{2}. \quad (69)$$

Since $\lim_{n\longrightarrow\infty}\lambda_n=\infty$, $\{x_n\}$ is bounded and $A(\{J_{\lambda_n}^{\Psi}x_n\})=\{\bar{v}\}$, we obtain that

$$\limsup_{n\longrightarrow\infty}d\left(J_{\lambda_n F}^{\Psi}x_n,J_{F}^{\Psi}\bar{v}\right)\leq\limsup_{n\longrightarrow\infty}d\left(J_{\lambda_n F}^{\Psi}x_n,\bar{v}\right)=\inf_{y\in X}\limsup_{n\longrightarrow\infty}d\left(J_{\lambda_n F}^{\Psi}x_n,y\right), \quad (70)$$

which by Lemma 2(ii) and Theorem 5(iv) implies that $\bar{v}\in \mathrm{fix}(J_{F}^{\Psi})=\mathrm{MEP}(C,F,\Psi)$.

Lemma 8 (Xu, [40]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1}\leq(1-\alpha_n)a_n+\alpha_n\sigma_n+\gamma_n,\quad n\geq 0, \quad (71)$$

where (i) $\{\alpha_n\}\subset[0,1]$, $\sum\alpha_n=\infty$; (ii) $\limsup\sigma_n\leq 0$; (iii) $\gamma_n\geq 0$; $(n\geq 0)$, $\sum\gamma_n<\infty$. Then, $a_n\longrightarrow 0$ as $n\longrightarrow\infty$.

Theorem 8. *Let $\Psi: C\longrightarrow\mathbb{R}$ be a convex and lower semicontinuous function and $F: C\times C\longrightarrow\mathbb{R}$ be a bifunction satisfying (A1–A4) of Theorem 4. Let $\{x_n\}$ be a sequence defined by (59), where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,\infty)$ such that $\lim_{n\longrightarrow\infty}\lambda_n=\infty$. Then, we have the following:*

(i) The sequence $\{J_{\lambda_n F}^{\Psi}x_n\}$ is bounded if and only if $\mathrm{MEP}(C,F,\Psi)\neq\emptyset$

(ii) If $\lim_{n\longrightarrow\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$ and $\Gamma\coloneqq\mathrm{MEP}(C,F,\Psi)\neq\emptyset$, then $\{x_n\}$ and $\{J_{\lambda_n F}^{\Psi}x_n\}$ *converge to $\bar{v}=P_{\Gamma}u$, where P_{Γ} is the metric projection of X onto Γ*

Proof. (i) Suppose that $\{J_{\lambda_n}^{f}x_n\}$ is bounded. Then by Lemma 2(ii), there exists $\bar{v}\in X$ such that $A(\{J_{\lambda_n}^{f}x_n\})=\{\bar{v}\}$. From (59) and Lemma 1(i), we obtain that

$$d(x_{n+1},\bar{v})\leq\alpha_n d(u,\bar{v})+(1-\alpha_n)d\left(J_{\lambda_n F}^{\Psi}x_n,\bar{v}\right), \quad (72)$$

which implies that $\{x_n\}$ is bounded. Also, since $\lim_{n\longrightarrow\infty}\lambda_n=\infty$ and $A(\{J_{\lambda_n F}^{\Psi}x_n\})=\{\bar{v}\}$, we obtain by Lemma 7 that $\mathrm{MEP}(C,F,\Psi)\neq\emptyset$.

Conversely, let $\mathrm{MEP}(C,F,\Psi)\neq\emptyset$. Then, we may assume that $\bar{v}\in\mathrm{MEP}(C,F,\Psi)\neq\emptyset$. Thus, by (59) and Lemma 1, we obtain that

$$\begin{aligned} d(x_{n+1},\bar{v})&\leq\alpha_n d(u,\bar{v})+(1-\alpha_n)d\left(J_{\lambda_n F}^{\Psi}x_n,\bar{v}\right)\\ &\leq\alpha_n d(u,\bar{v})+(1-\alpha_n)d(x_n,\bar{v})\\ &\leq\max\{d(u,\bar{v}),d(x_n,\bar{v})\}, \end{aligned} \quad (73)$$

which implies by induction that

$$d(x_n,\bar{v})\leq\max\{d(u,\bar{v}),d(x_1,\bar{v})\},\quad\forall n\geq 1. \quad (74)$$

Therefore, $\{x_n\}$ is bounded. Consequently, $\{J_{\lambda_n F}^{\Psi}x_n\}$ is also bounded.

(ii) Since $\Gamma\coloneqq\mathrm{MEP}(C,F,\Psi)\neq\emptyset$, we obtain from (74) that $\{x_n\}$ and $\{J_{\lambda_n F}^{\Psi}x_n\}$ are bounded. Furthermore, we obtain from Lemma 1(ii) that

$$\begin{aligned} d^{2}(x_{n+1},\bar{v})&\leq\alpha_n d^{2}(u,\bar{v})+(1-\alpha_n)d^{2}\left(J_{\lambda_n F}^{\Psi}x_n,\bar{v}\right)-\alpha_n(1-\alpha_n)d^{2}\left(u,J_{\lambda_n F}^{\Psi}x_n\right)\\ &\leq\alpha_n d^{2}(u,\bar{v})+(1-\alpha_n)d^{2}(x_n,\bar{v})-\alpha_n(1-\alpha_n)d^{2}\left(u,J_{\lambda_n F}^{\Psi}x_n\right)\\ &=(1-\alpha_n)d^{2}(x_n,\bar{v})+\alpha_n\delta_n,\quad\forall n\geq 1, \end{aligned} \quad (75)$$

where $\delta_n=d^{2}(u,\bar{v})+(\alpha_n-1)d^{2}(u,J_{\lambda_n F}^{\Psi}x_n)$. Now, set $v_n=J_{\lambda_n F}^{\Psi}x_n$, $\forall n\geq 1$. Then, by the boundedness of $\{v_n\}$ and Lemma 2(i), we obtain that there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$

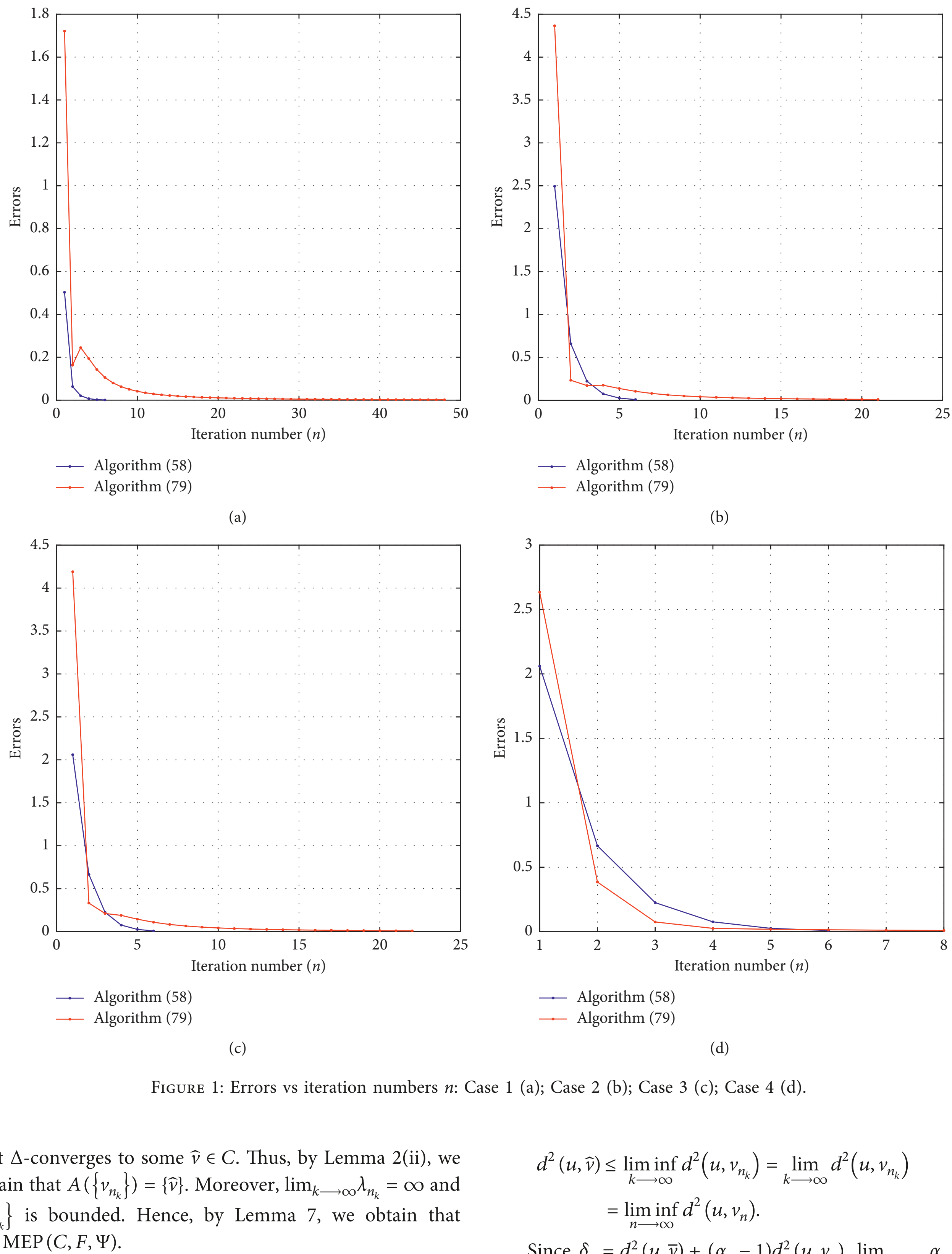

Figure 1: Errors vs iteration numbers n: Case 1 (a); Case 2 (b); Case 3 (c); Case 4 (d).

that Δ-converges to some $\widehat{v} \in C$. Thus, by Lemma 2(ii), we obtain that $A(\{v_{n_k}\}) = \{\widehat{v}\}$. Moreover, $\lim_{k\longrightarrow\infty}\lambda_{n_k} = \infty$ and $\{x_{n_k}\}$ is bounded. Hence, by Lemma 7, we obtain that $\widehat{v} \in \mathrm{MEP}(C, F, \Psi)$.

Next, we show that $\{x_n\}$ converges to $\widehat{v}$. By the Δ-lower semicontinuity of $d^2(u, .)$, we obtain that

$$
\begin{aligned}
d^2(u,\widehat{v}) &\leq \liminf_{k\longrightarrow\infty} d^2\left(u, v_{n_k}\right) = \lim_{k\longrightarrow\infty} d^2\left(u, v_{n_k}\right) \\
&= \liminf_{n\longrightarrow\infty} d^2(u, v_n).
\end{aligned}
\tag{76}
$$

Since $\delta_n = d^2(u,\overline{v}) + (\alpha_n - 1)d^2(u, v_n)$, $\lim_{n\longrightarrow\infty}\alpha_n = 0$, $\overline{v} = P_\Gamma u$, and $\widehat{v} \in \Gamma$, we obtain from the definition of P_Γ and (76) that

$$\limsup_{n\longrightarrow\infty} \delta_n \le d^2(u,\bar{v}) - \liminf_{n\longrightarrow\infty} d^2(u,v_n) \le d^2(u,\hat{v}) - \liminf_{n\longrightarrow\infty} d^2(u,v_n) \le 0. \quad (77)$$

Thus, applying Lemma 8 to (75) gives that $\{x_n\}$ converges to $\bar{v} = P_\Gamma u$. It then follows that $\{J^{\Psi}_{\lambda_n F} x_n\}$ is convergent to $\bar{v} = P_\Gamma u$.

By setting $\Psi \equiv 0$ in Theorem 8, we obtain the following new result for equilibrium problem in an Hadamard space. □

Corollary 4. *Let $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying (A1–A3) of Theorem 4 and $\{x_n\}$ be a sequence defined for u, $x_1 \in C$, by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n) J_{\lambda_n F} x_n, \quad (78)$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,\infty)$ such that $\lim_{n\longrightarrow\infty}\lambda_n = \infty$. Then, we have the following:

(i) *The sequence $\{J_{\lambda_n F} x_n\}$ is bounded if and only if* $EP(C,F) \ne \emptyset$

(ii) *If $\lim_{n\longrightarrow\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\Gamma \coloneqq EP(C,F) \ne \emptyset$*, then $\{x_n\}$ and $\{J_{\lambda_n F} x_n\}$ *converge to $\bar{v} = P_\Gamma u$, where P_Γ is the metric projection of X onto Γ*

By setting $F \equiv 0$ in Theorem 8, we obtain the following result which coincides with ([41], Theorem 5.1).

Corollary 5. *Let $\Psi : C \longrightarrow C$ be a proper convex and lower semicontinuous function and $\{x_n\}$ be a sequence defined for u, $x_1 \in C$, by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n) J^{\Psi}_{\lambda_n} x_n, \quad (79)$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,\infty)$ such that $\lim_{n\longrightarrow\infty}\lambda_n = \infty$. Then, we have the following:

(i) *The sequence $\{J^{\Psi}_{\lambda_n} x_n\}$ is bounded if and only if* $\operatorname{argmin}_{y\in C}\Psi(y) \ne \emptyset$

(ii) *If $\lim_{n\longrightarrow\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\Gamma \coloneqq \operatorname{argmin}_{y\in C}\Psi(y) \ne \emptyset$, then $\{x_n\}$ and $\{J^{\Psi}_{\lambda_n} x_n\}$ converge to $\bar{v} = P_\Gamma u$, where P_Γ is the metric projection of X onto Γ*

6. Numerical Results

In this section, we generate some numerical results in nonlinear setting for Algorithms (58) and (79).

Let $X = \mathbb{R}^2$ be endowed with a metric $d_X : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow [0,\infty)$ defined by

$$d_X(x,y) = \sqrt{(x_1-y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2}, \quad \forall x,\ y \in \mathbb{R}^2. \quad (80)$$

Then, $(\mathbb{R}^2, d_X)$ is an Hadamard space (see ([42], Example 5.2)) with the geodesic joining x to y given by

$$(1-t)x \oplus ty = \Big((1-t)x_1 + ty_1, ((1-t)x_1 + ty_1)^2 - (1-t)\big(x_1^2 - x_2\big) - t\big(y_1^2 - y_2\big)\Big). \quad (81)$$

Now, define $\Psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$\Psi(x_1,x_2) = 100\big((x_2-2) - (x_1-2)^2\big)^2 + (x_1-3)^2. \quad (82)$$

Then, it follows from ([42], Example 5.2) that Ψ is a proper convex and lower semicontinuous function in $(\mathbb{R}^2, d_X)$ but not convex in the classical sense (Figure1).

Now, take $\alpha_n = 1/(n+1)$ and $\lambda_n = n+1$ for all $n \ge 1$, then all the conditions of Corollaries 4.5 and 5.6 are satisfied. Hence, by considering the following initial vectors, we obtain the numerical results for Algorithms (58) and (79) as shown by the graphs as follows:

Case 1: $x_1 = (0.5, -0.25)^T$ and $u = (0.5, 3)^T$
Case 2: $x_1 = (-1.5, -3)^T$ and $u = (0.5, 3)^T$
Case 3: $x_1 = (0.5, 3)^T$ and $u = (-1.5, -3)^T$
Case 4: $x_1 = (0.5, 3)^T$ and $u = (0.5, -0.25)^T$

Data Availability

No data were used to support this study.

Disclosure

Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS and NRF.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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