On mixed generalized quasi-Einstein manifolds

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Abstract. In this paper we extend the notion of generalized quasi-Einstein manifold and name it *mixed generalized quasi-Einstein manifold*[MG(QE)$_n$]. We prove the existence of such manifolds. We also introduce the notion of *generalized quasi umbilical hypersurface of a Riemannian manifold* and show that such a manifold is a mixed generalized quasi Einstein manifold. Finally, we obtain the relation between the manifolds with mixed generalized quasi constant curvature and the mixed generalized quasi-Einstein quasi conformally flat manifolds.

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§ 1. Introduction

In a recent paper [4] U. C. De and G. C. Ghosh have defined the generalized the quasi-Einstein manifolds. A non-flat Reimannian manifold M is called a *generalized quasi-Einstein manifold* if its Ricci tensor S of type (0,2) is non-zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y),$$

where a, b, c are certain non-zero scalars and A, B are two non-zero 1-forms such that

$$g(X,U) = A(X), g(X,V) = B(X), \qquad g(U,V) = 0,$$

i.e., U and V are orthogonal vector fields on M.

In the present paper, we extend the notion of generalized quasi-Einstein manifold.

Definition 1. A non-flat Riemannian manifold is called a *mixed generalized quasi-Einstein manifold* if its Ricci tensor S of type (0,2) is non-zero and satisfies the condition

$$\begin{aligned} S(X,Y) = \ & ag(X,Y) + bK(X)K(Y) + cL(X)L(Y) + \\ & + d[K(X)L(Y) + L(X)K(Y)], \end{aligned} \tag{1.1}$$

where a, b, c, d are non-zero scalars,

$$g(X,U) = K(X) \text{ and } g(X,V) = L(X), \qquad g(U,V) = 0,$$

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K, L being two non-zero 1-forms, and U, V are unit vector fields corresponding to the 1-forms K and L respectively. We denote this type of manifold by $MG(QE)_n$. If $d = 0$, then the manifold reduces to a $G(QE)_n$.

In this paper we introduce as well another notion which generalizes the notion of a manifold of generalized quasi-constant curvature [4]. A Riemannian manifold is said to be a *manifold of generalized quasi-constant curvature* [4] if the curvature tensor $\acute{R}$ of type (0,4) satisfies the condition

$$\begin{aligned}\acute{R}(X,Y,Z,W) = \quad & p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & +q[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) \\ & +g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)] \\ & +s[g(X,W)D(Y)D(Z) - g(X,Z)D(Y)D(W) \\ & +g(Y,Z)D(X)D(W) - g(Y,W)D(X)D(Z)],\end{aligned}$$

where p, q, s are scalars, T and D are non-zero 1-forms. Here ρ and $\bar{\rho}$ are unit orthogonal vector fields such that

$$g(X,\rho) = T(X) and g(X,\bar{\rho}) = D(X), \qquad g(\rho,\bar{\rho}) = 0.$$

Definition 2. A Riemannian manifold is said to be a manifold of mixed generalized quasi-constant curvature if the curvature tensor $\acute{R}$ of type (0,4) satisfies the condition

$$\begin{aligned}\acute{R}(X,Y,Z,W) = \quad & p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & +q[g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) \\ & +g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)] \\ & +s[g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z) \\ & +g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W) \\ & +t[\{A(Y)B(Z) + B(Y)A(Z)\}g(X,W) \\ & -\{A(X)B(Z) + B(X)A(Z)\}g(Y,W) \\ & +\{A(X)B(W) + B(X)A(W)\}g(Y,Z) \\ & -\{A(Y)B(W) + B(Y)A(W)\}g(X,Z)],\end{aligned} \tag{1.2}$$

where p, q, s, t are scalars. A, B are non-zero 1-forms. Here ρ and $\bar{\rho}$ are orthonormal unit vector fields corresponding to A and B.

$$g(X,\rho) = A(X) \text{ and } g(X,\bar{\rho}) = B(X), \qquad g(\rho,\bar{\rho}) = 0.$$

§ 2. Preliminaries

We consider a mixed generalized quasi-Einstein manifold with associated scalars a, b, c and d and associated 1-forms K and L. From (1.1), we get

$$r = na + b + c,$$

where r denotes the scalar curvature of the manifold. Since U and V are orthogonal unit vector fields, therefore $g(U,U) = 1, g(V,V) = 1, g(U,V) = 0$. Putting $X = Y = U$ in (1.1), we obtain $S(U,U) = a+b$. Substituting $X = Y = V$ in (1.1) we further get

$S(V,V) = a + c$. It is known that in an n-dimensional ($n > 2$) Riemannian manifold the covariant quasi conformal curvature tensor is defind as [3]

$$
\begin{aligned}
\tilde{'C}(X,Y,Z,W) = \; & a'R(X,Y,Z,W) + b[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\
& + g(Y,Z)g(QX,W) - g(X,W)g(QY,W)] \\
& - \tfrac{r}{n}[\tfrac{a}{n-1} + 2b][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)],
\end{aligned} \tag{2.3}
$$

where

$$
g(\tilde{C}(X,Y)Z,W) = \tilde{'C}(X,Y,Z,W)
$$

$$
g(R(X,Y)Z,W) = {'R}(X,Y,Z,W).
$$

The conharmonic curvature tensor is denoted by $H(X,Y,Z,W)$ and in a $M^n (n > 2)$ it is defined as,

$$
\begin{aligned}
H(X,Y,Z,W) = \; & R(X,Y,Z,W) - \tfrac{1}{n-2}[g(X,W)S(Y,Z) - g(X,Z)S(Y,W) \\
& + g(Y,Z)S(X,W) - g(Y,W)S(X,Z)],
\end{aligned} \tag{2.4}
$$

where Q be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci-tensor S, and

$$
g(QX,Y) = S(X,Y), \;\; \forall X,Y \epsilon TM. \tag{2.5}
$$

§ 3. Existence theorem of a mixed generalized quasi-Einstein manifold

We can state the following:

Theorem 3.1. *If the Ricci tensor S of a Riemannian manifold satisfies the relation*

$$
\begin{aligned}
& S(X,W)S(Y,Z) - S(Y,W)S(X,Z) = \\
& \mu[S(Y,W)g(Z,X) + S(Z,X)g(Y,W)] \\
& + \beta[g(X,W)g(Y,Z) - g(Y,W)g(Z,X)],
\end{aligned} \tag{3.6}
$$

where μ, β are non-zero scalars, then the manifold is a mixed generalized quasi-Einstein manifold.

Proof. Let U be a vector field defined by

$$
g(X,U) = T(X), \;\; \forall X \epsilon TM. \tag{3.7}
$$

Putting $X = W = U$ in (3.6), we obtain

$$
\begin{aligned}
S(U,U)S(Y,Z) - S(Y,U)S(U,Z) = \; & \mu[S(Y,U)g(Z,U) + S(Z,U)g(Y,U)] \\
& + \beta[g(U,U)g(Y,Z) - g(Y,U)g(Z,U)].
\end{aligned}
$$

Now using (2.5) and (3.7) in the above equation, we get

$$\overline{\alpha}S(Y,Z) - T(QY)T(QZ) = \mu[T(QY)T(Z) + T(QZ)T(Y)]$$
$$+\rho[|U|^2 g(Y,Z) - T(Y)T(Z)],$$

where $S(U,U) = \overline{\alpha}$ and $T(QY) = g(QY,U) = S(Y,U)$. Therefore

$$S(Y,Z) = \alpha T(QY)T(QZ) + \mu\alpha[T(QY)T(Z) + T(QZ)T(Y)]$$
$$+\rho\alpha[|U|^2 g(Y,Z) - T(Y)T(Z)].$$

Taking $\alpha = \frac{1}{\overline{\alpha}}$ and $T(QY) = P(Y)$, we get

$$\begin{aligned} S(Y,Z) = \ & \alpha\rho[|U|^2 g(Y,Z)] + (-\alpha\rho)T(Y)T(Z) + \alpha P(Y)P(Z) \\ & +\alpha\mu[T(Y)P(Z) + P(Y)T(Z)]. \end{aligned} \tag{3.8}$$

Therefore in view of (3.8) we can conclude that the manifold is a mixed generalized quasi-Einstein manifold. □

§ 4. Example of a mixed generalized quasi-Einstein manifold

A manifold of mixed generalized quasi-constant curvature defined by (1.2) is a mixed generalized quasi-Einstein manifold. Putting $X = W = e_i$ in (1.2), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i, $1 \leq i \leq n$, we obtain

$$\begin{aligned} S(Y,Z) = \ & [p(n-1) + q + s + 2t]g(Y,Z) + q(n-2)A(Y)A(Z) \\ & +s(n-2)B(Y)B(Z) + t(n-2)[A(Y)B(Z) + B(Y)A(Z)]. \end{aligned}$$

Hence the manifold is a mixed generalized quasi-Einstein manifold.

§ 5. Hypersurfaces of the Euclidean space

Let M^n be a hypersurface of the Euclidean space E^{n+1} and the metric tensor $\tilde{g}$ of M^n is induced by E^{n+1}. The Gauss equation of M^n in E^{n+1} can be written as

$$\tilde{g}(\tilde{R}(X,Y)Z,W) = \tilde{g}(H(X,W),H(Y,Z)) - \tilde{g}(H(Y,W),H(X,Z)), \tag{5.9}$$

where $\tilde{R}$ is the Riemannian curvature tensor corresponding to the induced metric $\tilde{g}$, H is the second fundamental tensor of M^n (orthonormal to M^n) and X, Y, Z, W are vector fields tangent to M^n. If A_ξ is the (1,1) tensor corresponding to the normal valued second fundamental tensor H, then we have [2]

$$\tilde{g}(A_\xi(X),Y) = g(H(X,Y),\xi), \tag{5.10}$$

where ξ is the unit normal vector field and X, Y are tangent vector fields. Let H_ξ be the symmetric (0,2) tensor associated with A_ξ in the hypersurface defined by

$$\tilde{g}(A_\xi(X),Y) = H_\xi(X,Y). \tag{5.11}$$

A hypersurface of a Riemannian manifold (M^n, g) is called quasi-umbilical [2], if its second fundamental tensor has the form

$$H_\xi(X, Y) = \alpha\tilde{g}(X, Y) + \beta\omega(X)\omega(Y), \tag{5.12}$$

where ω is a 1-form.The vector field corresponding to the 1-form ω is a unit vector field, and α, β are scalars. If $\alpha = 0$ (resp.$\beta = 0$ or $\alpha = \beta = 0$) holds, then M^n is called cylindrical (respectively umbilical or geodesic).

In this section we define generalized quasi-umbilical hypersurface of a Riemannian manifold.

Definition 5.1. A hypersurface of a Riemannian manifold (M^n, g) is called generalized quasi-umbilical if its second fundamental tensor has the form

$$H_\xi(X, Y) = \alpha\tilde{g}(X, Y) + \beta\omega(X)\omega(Y) + \gamma\delta(X)\delta(Y),$$

where α, β, γ are scalars. The vector fields corresponding to 1-forms ω and δ are unit vector fields. If $\alpha = \beta = \gamma = 0, M^n$ is called geodesic. If $\alpha = \gamma = 0$ or $\alpha = \beta = 0, M^n$ is called cylindrical. Also M^n is called umbilical when $\beta = \gamma = 0$.

Now from (5.10), (5.11) and (5.12), we get

$$g(H(X, Y), \xi) = \alpha g(X, Y)g(\xi, \xi) + \beta\omega(X)\omega(Y)g(\xi, \xi). \tag{5.13}$$

Since ξ is the only unit normal vector, (5.13) reduces to

$$H(X, Y) = \alpha g(X, Y)\xi + \beta\omega(X)\omega(Y)\xi. \tag{5.14}$$

Let us suppose that the hypersurface is generalized quasi-umbilical. Then in view of (5.14) we have

$$H(X, Y) = \alpha g(X, Y)\xi + \beta\omega(X)\omega(Y)\xi + \gamma\delta(X)\delta(Y)\xi. \tag{5.15}$$

From (5.9) and (5.15) it follows that

$$\begin{aligned} \tilde{g}(\tilde{R}(X, Y)Z, W) = \;& \alpha^2\{g(X, W)g(Y, Z) - g(Y, W)g(X, Z)\} \\ & +\alpha\beta\{g(X, W)\omega(Y)\omega(Z) + g(Y, Z)\omega(X)\omega(W) \\ & -g(Y, W)\omega(X)\omega(Z) - g(X, Z)\omega(Y)\omega(W)\} \\ & +\alpha\gamma\{g(X, W)\delta(Y)\delta(Z) + g(Y, Z)\delta(X)\delta(W) \\ & -g(Y, W)\delta(X)\delta(Z) - g(X, Z)\delta(Y)\delta(W)\} \\ & +\beta\gamma\{\omega(X)\omega(W)\delta(Y)\delta(Z) + \omega(Y)\omega(Z)\delta(X)\delta(W) \\ & -\omega(Y)\omega(W)\delta(X)\delta(Z) - \omega(X)\omega(Z)\delta(Y)\delta(W)\} \end{aligned} \tag{5.16}$$

On contraction to (5.16) we get

$$\begin{aligned} \overline{S}(Y, Z) = \;& [\alpha^2(n-2) + \alpha\beta + \alpha\gamma]g(Y, Z) + [(n-2)\alpha\beta + \beta\gamma]\omega(Y)\omega(Z) \\ & +[(n-2)\gamma\alpha + \beta\gamma]\delta(Y)\delta(Z) - \beta\gamma[\omega(Y)\delta(Z) + \delta(Y)\omega(Z)], \end{aligned}$$

which shows that the manifold is a mixed generalized quasi-Einstein manifold. Thus we can state the following theorem

Theorem 5.1. *A generalized quasi-umbilical hypersurface of a Euclidean space is a mixed generalized quasi-Einstein manifold.*

§ 6. Relations between structures.

Since $\mathrm{MG(QE)}_n$ is quasi-conformally flat, from (2.3) we have

$$\begin{aligned} \acute{R}(X,Y,Z,W) = \; & \frac{r}{na'}\Big[\frac{a'}{n-1} + 2b'\Big][g(Y,Z)g(X,W) - g((X,Z)g(Y,W)] \\ & - \frac{b'}{a'}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ & + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)]. \end{aligned} \tag{6.17}$$

Using (1.1) in (6.17) we get

$$\begin{aligned} \acute{R}(X,Y,Z,W) = \; & \Big[\frac{r}{na'}\Big\{\frac{a'}{n-1} + 2b'\Big\} - \frac{2ab'}{a'}\Big][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & - \frac{bb'}{a'}[g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) \\ & + g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)] \\ & - \frac{b'c}{a'}[g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z) \\ & + g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W)] \\ & - \frac{b'd}{a'}[\{A(Y)B(Z) + B(Y)A(Z)\}g(X,W) \\ & - \{A(X)B(Z) + B(X)A(Z)\}g(Y,W) \\ & + \{A(X)B(W) + B(X)A(W)\}g(Y,Z) \\ & - \{A(Y)B(W) + B(Y)A(W)\}g(X,Z)]. \end{aligned} \tag{6.18}$$

From (6.18) we can state the following

Theorem 6.1. *A quasi conformally flat mixed generalized quasi-Einstein manifold is a manifold of mixed generalized quasi-constant curvature.*

From (2.4) and Theorem 6.1 we can also have the following

Corollary 6.1. *A conharmonically flat mixed generalized quasi-Einstein manifold is a manifold of mixed generalized quasi-constant curvature.*

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