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ON MIXED HODGE STRUCTURES OF SHIMURA VARIETIES ATTACHED TO INNER FORMS OF THE SYMPLECTIC GROUP OF DEGREE TWO

TAKAYUKI ODA AND JOACHIM SCHWERMER

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Abstract. We study arithmetic varieties V attached to certain inner forms of $\boldsymbol{Q}$-rank one of the split symplectic $\boldsymbol{Q}$-group of degree two. These naturally arise as unitary groups of a 2-dimensional non-degenerate Hermitian space over an indefinite rational quaternion division algebra. First, we analyze the canonical mixed Hodge structure on the cohomology of these quasi-projective varieties and determine the successive quotients of the corresponding weight filtration. Second, by interpreting the cohomology groups within the framework of the theory of automorphic forms, we determine the internal structure of the cohomology "at infinity" of V, that is, the part which is spanned by regular values of suitable Eisenstein series or residues of such. In conclusion, we discuss some relations between the mixed Hodge structure and the so called Eisenstein cohomology. For example, we show that the Eisenstein cohomology in degree two consists of algebraic cycles.

CONTENTS

Introduction. Let D be quaternion division algebra over $\boldsymbol{Q}$, endowed with its standard involution τ_c. Suppose that D is indefinite. Let V be a 2-dimensional vector space over D, and let f be a non-degenerate Hermitian form on V. The special unitary group

$$G := SU(f, D, \tau_c)$$

of isometries of f with respect to τ_c is an absolutely simple algebraic group defined over $\boldsymbol{Q}$. It is an inner form of $\boldsymbol{Q}$-rank 1 of the split symplectic $\boldsymbol{Q}$-group Sp_2; the latter group is of $\boldsymbol{Q}$-rank 2.

The indefinite quaternion algebra D splits over some real quadratic field extension of $\boldsymbol{Q}$. Thus, the group $G(\boldsymbol{R})$ of real points of G may be identified with the real symplectic Lie group $Sp_2(\boldsymbol{R})$. Fix a maximal compact subgroup K_0 of $G(\boldsymbol{R})$; the associated symmetric

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space $G(\boldsymbol{R})/K_0$ is denoted by X. Let Γ be a torsion free arithmetic subgroup of $G(\boldsymbol{Q})$. The group Γ acts properly discontinuously and freely on the Hermitian symmetric space X. The quotient $\Gamma\backslash X$ is a non compact Riemannian manifold of dimension 6. It admits the structure of a quasi-projective algebraic variety of dimension 3 over $\boldsymbol{C}$.

In this paper, on the one hand, we analyze the canonical mixed Hodge structure on the cohomology of these arithmetic varieties and determine the successive quotients of the weight filtration of $H^*(\Gamma\backslash X, \boldsymbol{C})$. On the other hand, these cohomology groups can be interpreted in terms of the automorphic spectrum of Γ. One has a decomposition

$$H^*(\Gamma\backslash X, \boldsymbol{C}) = H^*_{\mathrm{cusp}}(\Gamma\backslash X, \boldsymbol{C}) \oplus H^*_{\mathrm{Eis}}(\Gamma\backslash X, \boldsymbol{C})$$

into the subspace of classes represented by cuspidal automorphic forms for G with respect to Γ and the Eisenstein cohomology constructed as the cohomological space of appropriate residues or derivatives of the Eisenstein series attached to automorphic forms on the Levi components of proper parabolic $\boldsymbol{Q}$-subgroups of G. Note that there is exactly one $G(\boldsymbol{Q})$-conjugacy class P of proper parabolic $\boldsymbol{Q}$-subgroups of G in this case.

In Section 1, we review the structure theory of the split symplectic $\boldsymbol{Q}$-group Sp_2 of $\boldsymbol{Q}$-rank two and its inner forms $G = SU(f, D, \tau_c)$ that are determined by a 2-dimensional non-degenerate Hermitian space (V, f) over an indefinite quaternion division algebra D over $\boldsymbol{Q}$. We then describe the modular varieties and their cohomology groups $H^*(\Gamma\backslash X, E)$ attached to arithmetically defined subgroups of these groups G and a finite dimensional irreducible representation (τ, E) of G. In particular, we recall the description of these cohomology groups in terms of automorphic representations for the underlying group G. We then summarize some general results regarding $H^*(\Gamma\backslash X, E)$ that rely on the classification of irreducible unitary representations of the real Lie group $G(\boldsymbol{R})$ with non-vanishing cohomology.

In Section 2, the focus is on various compactifications of the quasi-projective algebraic variety $V = \Gamma\backslash X$. First, by attaching a finite number of points, to be called cusps, there is the Satake-Baily-Borel compactification V^* of V. It is a normal algebraic variety containing V as a Zariski open subset. Second, we construct the smooth toroidal compactifications $\tilde{V}$ which give a natural resolution of the singularities along the cusps so that the divisor at infinity $D = \tilde{V}\backslash V$ is the union of smooth codimension one submanifolds of $\tilde{V}$ with normal crossings.

In Section 3, we briefly recall some facts concerning Deligne's construction [8], [9] of the mixed Hodge structure on the cohomology of a smooth complex algebraic variety. In particular, we discuss the weight filtration W_* which is already defined over $\boldsymbol{Q}$ on the cohomology in question.

By analyzing the Leray spectral sequence associated to the open immersion $j : V \to \tilde{V}$, we obtain in Section 4 various results pertaining to the weight filtration of the mixed Hodge structure on $H^*(V, \boldsymbol{Q})$. In particular, we have $H^i(V, \boldsymbol{Q}) = W_i H^i(V, \boldsymbol{Q})$ for $i = 1, 2$, and that $W_3 H^3(V, \boldsymbol{Q})$ coincides with the image $H^3_!(V, \boldsymbol{Q})$ of the cohomology with compact supports.

In Section 5, we study the Leray spectral sequence associated to the open immersion $k : V \to V^*$. This provides useful information on the mixed Hodge structure on $H^0(C, R^3_{k_*}\boldsymbol{Q})$ where $C = V^*\backslash V$ is the union of a finite number of points corresponding to the cusps of V.

Section 6 contains a structural description of the individual constituents of the Eisenstein cohomology $H^*_{\mathrm{Eis}}(\Gamma\backslash X, E)$. We describe the Eisenstein series and its residues which give rise to non-trivial cohomology classes and the cuspidal automorphic forms for P to which these classes are attached. The most interesting case is the one with a trivial coefficient system $E = \boldsymbol{C}$. As pointed out in 1.6, the cohomology $H^q(\Gamma\backslash X, \boldsymbol{C})$ vanishes in degrees $q \neq 2, 3, 4, 5$. In degree 5, the cohomology

$$H^5(\Gamma\backslash X, \boldsymbol{C}) = H^5_{\mathrm{Eis}}(\Gamma\backslash X, \boldsymbol{C})$$

consists entirely of regular Eisenstein cohomology classes, and it restricts onto a subspace of codimension one under the restriction map

$$r^q : H^q(\Gamma\backslash X, \boldsymbol{C}) = H^q(\Gamma\backslash\bar{X}, \boldsymbol{C}) \to H^q(\partial(\Gamma\backslash\bar{X}), \boldsymbol{C})$$

of the cohomology of the Borel-Serre compactification onto the cohomology of its boundary. In degrees 4 and 3, the cohomology spaces $H^q_{\mathrm{Eis}}(\Gamma\backslash X, \boldsymbol{C})$, $q = 4, 3$, consist of regular Eisenstein cohomology classes as well. The restriction map r^4 is surjective whereas r^3 is not surjective. Finally, we show that the classes missing in the image of r^3 are accounted for by residual Eisenstein cohomology classes that span the subspace

$$H^2_{\mathrm{res}}(\Gamma\backslash X, \boldsymbol{C}) \subset H^2(\Gamma\backslash X, \boldsymbol{C})\,.$$

This subspace is complementary to the interior cohomology $H^2_!(\Gamma\backslash X, \boldsymbol{C})$ which is, by definition, the image of the cohomology with compact supports and coincides with $\ker r^2$. In particular, $\dim \operatorname{Im} r^2 + \dim \operatorname{Im} r^3 = \dim H^3(\partial(\Gamma\backslash\bar{X}), \boldsymbol{C})$.

In Section 7, we conclude by investigating more closely the Hodge structure on the space $H^2_{\mathrm{res}}(\Gamma\backslash X, \boldsymbol{C})$. We show that it is of $(1, 1)$-type and, thus, consists entirely out of algebraic cycles.

1. Varieties attached to $\boldsymbol{Q}$-rank one forms of Sp_2.

1.1. The symplectic group of degree two. Let H be the $\boldsymbol{Q}$-split algebraic $\boldsymbol{Q}$-group $Sp_2/\boldsymbol{Q}$, i.e. the symplectic group of degree two. The group H may even be viewed as the Chevalley group scheme over $\boldsymbol{Z}$ of all symplectic transformations on the symplectic space $\boldsymbol{Z}^4$ with its standard alternating form. For any commutative ring R with identity, one has

$$H(R) = \left\{ h = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\middle|\; \begin{array}{c} A \cdot {}^tB - B \cdot {}^tA = C \cdot {}^tD - D \cdot {}^tC = 0 \\ A \cdot {}^tD - B \cdot {}^tC = I_2 \\ A, B, C, D \in M_2(R) \end{array} \right\} \tag{1}$$

for the group of R-points of H. We let $T_0 \subset H$ be the maximal torus of diagonal matrices

$$T_0 = \{g = \mathrm{diag}(a_1, a_2, t_1, t_2) \in H \mid a_1 t_1 = a_2 t_2 = 1\}\,, \tag{2}$$

and we let $B = Q_0$ be the Borel subgroup of matrices in H with entries A, B, O, D in block form as above where A (resp. D) is an upper triangular (resp. lower-triangular) matrix. We have $Q_0 = T_0 U_0$ as a semi-direct product of T_0 and the unipotent radical U_0 of Q_0. We let $\Psi = \Psi(\mathfrak{h}_{\boldsymbol{C}}, \mathfrak{t}_{0\boldsymbol{C}})$ be the set of roots of $\mathfrak{h}_{\boldsymbol{C}}$ with respect to $\mathfrak{t}_{0\boldsymbol{C}}$. Its elements will also be viewed as roots of $H_{\boldsymbol{C}}$ with respect to T_0. Since H is split over $\boldsymbol{Q}$, we may identify Ψ with the set ${}_{\boldsymbol{R}}\Psi$ of $\boldsymbol{R}$-roots. An ordering on Ψ is fixed by requiring that the set of positive roots Ψ^+ coincides with the set $\Phi(Q_0, T_0)$ of roots of Q_0 with respect to T_0. Thinking of the entries a_i, t_i as characters of T_0, we define

$$\beta_1 = a_1 a_2^{-1}, \quad \beta_2 = a_2 \cdot t_2^{-1}, \tag{3}$$

and $\Delta_H = \{\beta_1, \beta_2\}$ is the set of simple roots with respect to the chosen ordering. The Weyl group of G with respect to T_0 is then generated by the simple reflections s_i associated to β_i, $i = 1, 2$.

The set of parabolic $\boldsymbol{Q}$-subgroups of H will be denoted by $\mathcal{P}_H$. The conjugacy classes of elements in $\mathcal{P}_H$ are parametrized by the subsets J of Δ_H. A minimal parabolic $\boldsymbol{Q}$-subgroup of H is conjugate to the standard one $Q_\emptyset = Q_0$. If Q is a maximal parabolic $\boldsymbol{Q}$-subgroup of H, then it is conjugate to the standard one

$$Q_i = Q_{\Delta_H - \{\beta_i\}} = Z(T_{0, \Delta_H - \{\beta_i\}}) \cdot U_i \supset Q_0 \tag{4}$$

given as the semi-direct product of the unipotent radical U_i by the centralizer of $T_{0, \Delta_H - \{\beta_i\}}$ where we denote $T_{0,J} = (\bigcap_{\alpha \in J} \ker \alpha)^0$ for a subset J of Δ_H. The set Δ_i of simple roots of the Levi component $L_i = Z(T_{0, \Delta_H - \{\beta_i\}})$ is $\{\beta_2\}$ if $i = 1$ and $\{\beta_1\}$ if $i = 2$. We have $L_1 \cong SL_2 \times G_m$ and $L_2 \cong GL_2$. We observe that the unipotent radical U_1 is non-abelian, and U_2 is abelian.

1.2. A $\boldsymbol{Q}$-rank one form. Let D be a quaternion division algebra with center $\boldsymbol{Q}$, i.e. D is a central simple division algebra of dimension 4 over $\boldsymbol{Q}$. One can represent D as a cyclic (crossed product) algebra over $\boldsymbol{Q}$. More precisely, there exist a separable maximal subfield k of D, necessarily of degree $[k : \boldsymbol{Q}] = 2$, and an element $v \in D$ such that

$$D = k + vk$$

as a k-vector space, $v^2 = q \in \boldsymbol{Q}^*$ and $vx = \sigma(x)v$ for all $x \in k$, where σ denotes the non-trivial $\boldsymbol{Q}$-automorphism of k. Note that D does not uniquely determine the pair (k, q).

In general, given a central simple algebra A over the field $\boldsymbol{Q}$ of rational numbers, the reduced trace and the reduced norm of an element $a \in A$ is denoted by $\mathrm{Trd}_A(a)$ and $\mathrm{Nrd}_A(a)$ respectively .

Extending the non-trivial $\boldsymbol{Q}$-automorphism σ of k, there is the k-endomorphism $d \mapsto d^*$ of D, defined by $v^* = -v$, i.e., one has $x_1 + vx_2 \mapsto \sigma(x_1) - vx_2$ for $x_1, x_2 \in k$. The $\boldsymbol{Q}$-endomorphism $*$ is an involutive automorphism of D.

The choice of the basis 1, v for D over k gives an identification $M_2(k) = \mathrm{End}_k D$, and the natural embedding $D \to \mathrm{End}_k D$ is given by $x_1 + vx_2 \mapsto \left(\begin{smallmatrix} x_1 & q\sigma(x_2) \\ x_2 & \sigma(x_1) \end{smallmatrix}\right)$ in this setting.

Suppose that the quaternion algebra D with center $\boldsymbol{Q}$ is indefinite. Let V be a vector space of dimension 2 over D, and let f be a non-degenerate hermitian sesquilinear form over

V defined with respect to the conjugation $*$. Then the special unitary group

$$SU(f) =: G \tag{1}$$

of f is a simple algebraic group defined over $\boldsymbol{Q}$. The following realization of G is useful. Let G' be the algebraic $\boldsymbol{Q}$-group whose group of rational points coincides with the group of (2×2)-matrices over D whose reduced norm is one, i.e.

$$G'(\boldsymbol{Q}) = \{M \in M_2(D) \mid \mathrm{Nrd}_{M_2(D)}(M) = 1\}. \tag{2}$$

The choice of a basis of V over D with respect to which f takes the form $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ provides an embedding $j : G \to G'$ so that the image of $G(A)$, for any commutative $\boldsymbol{Q}$-algebra A with identity, is given by

$$G(A) = \{g \in G'(A) \mid g^* J g = J\} \tag{3}$$

where

$$g^* = \begin{pmatrix} a^* c^* \\ b^* d^* \end{pmatrix} \quad \text{for} \quad g = \begin{pmatrix} ab \\ cd \end{pmatrix} \in M_2(D \otimes_{\boldsymbol{Q}} A)$$

with $* : D \otimes_{\boldsymbol{Q}} A \to D \otimes_{\boldsymbol{Q}} A$ defined by $d \otimes a \mapsto d^* \otimes a$.

Recall that a field extension L of $\boldsymbol{Q}$ is called a splitting field of a given simple algebra A over $\boldsymbol{Q}$ of dimension n^2 if $A \otimes_{\boldsymbol{Q}} L \cong M_n(L)$. A maximal subfield F of the indefinite quaternion algebra D over $\boldsymbol{Q}$ is a splitting field of D, and, thus, F is a quadratic extension of $\boldsymbol{Q}$. In turn, a quadratic splitting field of D is isomorphic to a maximal subfield of D. Note that there exists a real quadratic splitting field of D.

Given a maximal subfield F of D there is an identification $\mu : D \otimes_{\boldsymbol{Q}} F \xrightarrow{\sim} M_2(F)$ of F-algebras (use $M_2(F) = \mathrm{End}_F D$ above). Then the image of $G \times_{\boldsymbol{Q}} F$ under the composite

$$G \times_{\boldsymbol{Q}} F \xrightarrow{j_F} G' \times_{\boldsymbol{Q}} F \to GL_4 \times_{\boldsymbol{Q}} F \tag{4}$$

of j_F and the natural morphism induced by the identification μ alluded to is conjugate [via $g \mapsto C^{-1} g C$ with $C = \left(\begin{smallmatrix} I_2 & \\ & \begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} \end{smallmatrix}\right)$] to the symplectic group $Sp_2 \times_{\boldsymbol{Q}} F$ naturally embedded into $GL_4 \times_{\boldsymbol{Q}} F$. Thus there is a natural isomorphism

$$\gamma : G \times_{\boldsymbol{Q}} F \xrightarrow{\sim} Sp_2 \times_{\boldsymbol{Q}} F$$

of F-algebraic groups. A maximal F-split torus of $G \times_{\boldsymbol{Q}} F$ is $G(F)$-conjugate (under this identification) to the maximal torus $T_0 \times F$ of diagonal matrices in $Sp_2 \times_{\boldsymbol{Q}} F$.

The absolutely simple $\boldsymbol{Q}$-group G is an inner form of $Sp_2/\boldsymbol{Q}$. There is (up to conjugacy) a unique maximal $\boldsymbol{Q}$-split torus S in G, given (as a subgroup of G' as above) as

$$S(\boldsymbol{Q}) = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in G(\boldsymbol{Q}) \;\middle|\; t \in \boldsymbol{Q}^* \right\}. \tag{5}$$

Let P be the parabolic $\boldsymbol{Q}$-subgroup of G, defined by

$$P(\boldsymbol{Q}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G(\boldsymbol{Q}) \right\}. \tag{6}$$

One has $P = Z_G(S).N$ as a semi-direct product (defined over $\boldsymbol{Q}$) of $Z_G(S)$ and the unipotent radical N of P. Note that

$$N(\boldsymbol{Q}) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in G(\boldsymbol{Q}) \,\middle|\, b \in D, b = -b^* \right\} \tag{7}$$

and

$$Z_G(S)(\boldsymbol{Q}) = \left\{ \begin{pmatrix} d & 0 \\ 0 & (d^*)^{-1} \end{pmatrix} \in G(\boldsymbol{Q}) \,\middle|\, d \in D^* \right\}. \tag{8}$$

Let $\Phi(\mathfrak{g}_{\boldsymbol{Q}}, \mathfrak{s}_{\boldsymbol{Q}})$ be the set of roots of $\mathfrak{g}_{\boldsymbol{Q}}$ with respect to $\mathfrak{s}_{\boldsymbol{Q}}$. Its elements will also be viewed as roots of $G(\boldsymbol{Q})$ with respect to $S(\boldsymbol{Q})$. Fix an ordering on $\Phi(\mathfrak{g}_{\boldsymbol{Q}}, \mathfrak{s}_{\boldsymbol{Q}})$ by requiring that the set of positive roots coincides with the set $\Phi(P, S)$ of roots of P with respect to S. The group $X(Z_G(S))_{\boldsymbol{Q}}$ of $\boldsymbol{Q}$-rational characters of $Z_G(S)$ is generated by the character $\chi : Z_G(S) \to G_m$ defined by $\left(\begin{smallmatrix} d & 0 \\ 0 & (d^*)^{-1} \end{smallmatrix}\right) \mapsto \mathrm{Nrd}(d)$. Thinking of the entry s of S as a character of S we define $\alpha = s^2$. One sees $\alpha = \chi_{|S}$ and $\Delta = \{\alpha\}$ is the set of simple roots with respect to the chosen ordering. Note that the algebraic $\boldsymbol{Q}$-group G has $\boldsymbol{Q}$-rank one. A proper parabolic $\boldsymbol{Q}$-subgroup of G is conjugate to P.

Given the standard (minimal) parabolic $\boldsymbol{Q}$-subgroup P of G with Levi $\boldsymbol{Q}$-subgroup $M = Z_G(S)$ and split component A_P, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_P$. The canonical choice for $\mathfrak{h}$ is to take the Lie algebra associated to the maximal $\boldsymbol{R}$-split torus of $G \times_{\boldsymbol{Q}} \boldsymbol{R}$ corresponding to $T_0 \times_{\boldsymbol{Q}} \boldsymbol{R}$ in $Sp_2 \times_{\boldsymbol{Q}} \boldsymbol{R}$. Let $\Phi_{\boldsymbol{C}} = \Phi(\mathfrak{g}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})$ be the set of roots of $\mathfrak{g}_{\boldsymbol{C}}$ with respect to $\mathfrak{h}_{\boldsymbol{C}}$, and let $\Phi_{\boldsymbol{R}}$ be the set of $\boldsymbol{R}$-roots. Its elements will also be viewed as roots of $G_{\boldsymbol{C}}$ with respect to $Z_{G(\boldsymbol{C})}(\mathfrak{h})$ and $A_P(\boldsymbol{R})$, respectively. Choose an ordering on the real roots $\Phi_{\boldsymbol{R}}$ given by the initial choice of P.

Let α_i, $i = 1, 2$, denote the simple root corresponding to β_i (in the notation of 1.1) under the identification $\gamma : G \times_{\boldsymbol{Q}} F \xrightarrow{\sim} Sp_2 \times_{\boldsymbol{Q}} F$ of F-algebraic groups alluded to above. Then $S \times_{\boldsymbol{Q}} F = (\ker \alpha_1)^0$, and the group $P(F)$ of F-points of P is conjugate to the F-points of the maximal parabolic $\boldsymbol{Q}$-subgroup Q_2 of the symplectic group Sp_2 for any real splitting field F of D.

Let $W = W(\mathfrak{g}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})$ be the Weyl group of $\mathfrak{g}_{\boldsymbol{C}}$ with respect to $\mathfrak{h}_{\boldsymbol{C}}$, and similarly $W_P = W(\mathfrak{m}_{P,\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})$. As usual, the length $\mathfrak{l}(w)$ of w is meant with respect to the set of simple reflections $w_i = s_{\alpha_i} \in W$.

1.3. Modular varieties and their cohomology groups. The indefinite quaternion algebra D splits over some real quadratic extension of $\boldsymbol{Q}$, thus, in view of 1.2, there is an identification $D \otimes_{\boldsymbol{Q}} \boldsymbol{R} = M_2(\boldsymbol{R})$. The group $G(\boldsymbol{R}) = SU(f)(\boldsymbol{R})$ of real points of G may be identified with the symplectic group $Sp_2(\boldsymbol{R})$. Fix a maximal compact subgroup K_0 of $G(\boldsymbol{R})$; the associated symmetric space $G(\boldsymbol{R})/K_0$ is denoted by X. Let Γ be a torsion free arithmetic subgroup of $G(\boldsymbol{Q})$. The group Γ acts properly discontinuously and freely on the associated hermitian symmetric space X, and the quotient $\Gamma \backslash X$ is a non-compact complete Riemannian manifold of real dimension 6. It admits the structure of a quasi-projective algebraic variety of dimension 3 over $\boldsymbol{C}$.

It is useful to interprete these varieties in an adelic framework. Given an open compact subgroup K of $G(\boldsymbol{A}_f)$, there is the double coset space

$$S_K(\boldsymbol{C}) = G(\boldsymbol{Q})\backslash X \times G(\boldsymbol{A}_f)/K \tag{1}$$

where $g(x, x_f)k = (gx, gx_f k)$ for $g \in G(\boldsymbol{Q}),\ x \in X,\ x_f \in G(\boldsymbol{A}_f)$ and $k \in K$. Endowed with the quotient topology, this is a Hausdorff space. As an absolutely simple and simply connected algebraic group, G has the strong approximation property, that is, in particular, the algebraic group $G(\boldsymbol{Q})$ is dense in $G(\boldsymbol{A}_f)$. This gives rise to an identification

$$\Gamma\backslash X = S_K(\boldsymbol{C}) = G(\boldsymbol{Q})\backslash X \times G(\boldsymbol{A}_f)/K \tag{2}$$

where $\Gamma = G(\boldsymbol{Q}) \cap G(\boldsymbol{R})K$ is the arithmetic subgroup that is determined by the choice of K.

Associated to a given absolutely irreducible rational representation $\tau : G \times_{\boldsymbol{Q}} \bar{\boldsymbol{Q}} \to GL(E)$, where E denotes a finite dimensional $\bar{\boldsymbol{Q}}$-vector space, there is a sheaf $\tilde{E}$ on $S_K(\boldsymbol{C})$ constructed in the usual way. We are interested in the cohomology groups $H^*(S_K(\boldsymbol{C}), \tilde{E})$. Given another open compact subgroup $L \subset K$ of $G(\boldsymbol{A}_f)$ the finite covering $S_L \to S_K$ induces an inclusion $H^*(S_K(\boldsymbol{C}), \tilde{E}) \to H^*(S_L(\boldsymbol{C}), \tilde{E})$. This is a directed system of cohomology groups, and we may consider the inductive limit $\varinjlim_K H^*(S_K(\boldsymbol{C}), \tilde{E})$. Since this limit is also given as the cohomology of $S(\boldsymbol{C}) := \varprojlim_K S_K(\boldsymbol{C})$ (cf. [28]), we can write

$$H^*(S(\boldsymbol{C}), \tilde{E}) = \varinjlim_K H^*(S_K(\boldsymbol{C}), \tilde{E})\,. \tag{3}$$

The natural map $S_K(\boldsymbol{C}) \to S_{g^{-1}Kg}(\boldsymbol{C})$ given by right translation with $g \in G(\boldsymbol{A}_f)$ extends to a map between the sheaves on both sides. This induces an action of $G(\boldsymbol{A}_f)$ on the directed system of cohomology groups and gives rise to a $G(\boldsymbol{A}_f)$-module structure on $H^*(S(\boldsymbol{C}), \tilde{E})$. For a given open compact subgroup L of $G(\boldsymbol{A}_f)$, we may recover the cohomology of $S_L(\boldsymbol{C})$ by taking L invariants, i.e., if $\Gamma = G(\boldsymbol{Q}) \cap G(\boldsymbol{R})L$, then

$$H^*(\Gamma\backslash X, E) = H^*(S(\boldsymbol{C}), \tilde{E})^L\,. \tag{4}$$

1.4. The quotient $\Gamma\backslash X = S_K(\boldsymbol{C})$ may be identified with the interior of a compact manifold $\bar{S}_K(\boldsymbol{C}) = \Gamma\backslash\bar{X}$ with corners [7]; the inclusion j is a homotopy equivalence. The boundary of the Borel-Serre compactification $\Gamma\backslash\bar{X}$ is a disjoint union of a finite number of faces $e'(Q)$ which correspond bijectively to the Γ-conjugacy classes of proper parabolic $\boldsymbol{Q}$-subgroups of G. In the adelic setting, the boundary $\partial S_K(\boldsymbol{C})$ is (up to homotopy equivalence) described as

$$\partial S_K(\boldsymbol{C}) = P(\boldsymbol{Q})\backslash G(\boldsymbol{A})/K_0 \cdot K$$

where P denotes the standard minimal parabolic $\boldsymbol{Q}$-subgroup of G.

The interior cohomology $H_!^*(S_K(\boldsymbol{C}), \tilde{E})$ is, by definition, the image of the natural map $j_K^* : H_c^*(S_K(\boldsymbol{C}), \tilde{E}) \to H^*(S_K(\boldsymbol{C}), \tilde{E})$ of the cohomology with compact supports to the cohomology of $S_K(\boldsymbol{C})$. The long exact cohomology sequence of the pair $(\bar{S}_K(\boldsymbol{C}), \partial S_K(\boldsymbol{C}))$ gives rise to

$$\cdots \to H_c^*(S_K(\boldsymbol{C}), \tilde{E}) \to H^*(S_K(\boldsymbol{C}), \tilde{E}) = H^*(\bar{S}_K(\boldsymbol{C}), \tilde{E}) \to H^*(\partial S_K(\boldsymbol{C}), \tilde{E}) \to \cdots\,.$$

Thus the interior cohomology coincides with the kernel of the natural restriction map

$$r^* : H^*(S_K(\boldsymbol{C}), \tilde{E}) \to H^*(\partial S_K(\boldsymbol{C}), \tilde{E}) .$$

1.5. Automorphic cohomology. By the work of Franke [10], the cohomology of $S(\boldsymbol{C})$ and $S_K(\boldsymbol{C})$ can be interpreted respectively in terms of relative Lie algebra cohomology with respect to the automorphic spectrum of the arithmetic groups involved. There is a sum decomposition.

$$(1) \qquad H^*(S(\boldsymbol{C}),\ E) = H^*_{\text{cusp}}(S(\boldsymbol{C}),\ E) \oplus H^*_{\text{Eis}}(S(\boldsymbol{C}),\ E)$$

into the subspace of classes represented by cuspidal automorphic forms for G and the Eisenstein cohomology constructed as the cohomological space of appropriate residues or derivatives of Eisenstein series attached to cuspidal automorphic forms on the Levi component of the proper parabolic $\boldsymbol{Q}$-subgroup P. This simple description is due to the fact that the underlying $\boldsymbol{Q}$-group G has $\boldsymbol{Q}$-rank one, hence there is exactly one class of associated proper parabolic $\boldsymbol{Q}$-subgroups of G. It is the class $\{P\}$ represented by the standard minimal parabolic $\boldsymbol{Q}$-subgroup of G. Thus, the Eisenstein cohomology $H^*_{\text{Eis}}(S(\boldsymbol{C}), \tilde{E})$ is the relative Lie algebra cohomology

$$(2) \qquad H^*_{\text{Eis}}(S(\boldsymbol{C}), \tilde{E}) := H^*(\mathfrak{g}_{\boldsymbol{C}}, K_0 ; \mathcal{A}_{E,\{P\}} \otimes E)$$

in the notation of Franke-Schwermer [11] (see also [20, Section 2]).

Following [11, Theorem 2.3], the Eisenstein cohomology classes can be arranged according to the cuspidal support of the Eisenstein series involved. This internal structure of the Eisenstein cohomology will be discussed in detail in Section 6.

The following proposition summarizes some general results regarding $H^*(S(\boldsymbol{C}), \tilde{E})$. In particular, we give a vanishing result for the cuspidal cohomology $H^*_{\text{cusp}}(S(\boldsymbol{C}), \tilde{E})$. The results are valid with regard to $S(\boldsymbol{C})$ as well as $S_K(\boldsymbol{C})$.

PROPOSITION 1.6

(1) *One has* $H^q(S(\boldsymbol{C}), \tilde{E}) = 0$ *for* $q \neq 0, 2, 3, 4, 5$.

(2) *Let* $H^*_{(2)}(S(\boldsymbol{C}), \tilde{E})$ *be the subspace of square integrable cohomology classes in* $H^*(S(\boldsymbol{C}), \tilde{E})$. *One has the natural inclusions*

$$H^*_{\text{cusp}}(S(\boldsymbol{C}), \tilde{E}) \subset H^*_!(S(\boldsymbol{C}), \tilde{E}) \subset H^*_{(2)}(S(\boldsymbol{C}), \tilde{E}) .$$

(3) *The space* $H^q_{(2)}(S(\boldsymbol{C}), \tilde{E})$ *vanishes in degrees* $q \neq 2, 3, 4$; *it coincides with the cuspidal cohomology* $H^3_{\text{cusp}}(S(\boldsymbol{C}), \tilde{E})$ *in degree* 3.

PROOF. ad (1): By Corollary 11.4.3 in [7], the virtual cohomological dimension of an arithmetic subgroup of G is equal $\dim S(\boldsymbol{C}) - \operatorname{rank}_{\boldsymbol{Q}} G = 6 - 1 = 5$. Thus, one has the vanishing result for $q > 5$. The corresponding result in degree 1 is a consequence of the congruence subgroup property for G.

ad (2): These inclusions are a consequence of the results in [5, Section 5], in particular [5, Corollary 5.5].

ad (3): The cohomology $H^*_{(2)}(S(\boldsymbol{C}), \tilde{E})$, interpreted in terms of relative Lie algebra cohomology, decomposes as a finite algebraic sum

$$\bigoplus [H^*(\mathfrak{g}_{\boldsymbol{C}}, K_0; H_{\pi_\infty} \otimes H_{\pi_f}]^{m(\pi)}$$

where the sum ranges over all automorphic representations occurring in the square integrable spectrum of $G(\boldsymbol{A})$ for which the infinitesimal character of its archimedean component matches the one of the representation E^* contragredient to E. Thus, one is led to determine (up to equivalence) all irreducible unitary representations of $G(\boldsymbol{R}) = Sp_2(\boldsymbol{R})$ with non-vanishing cohomology. In this specific case, as a consequence of the general classification in [39], the resulting finite list is given in [26, Section 2.4]. This implies the vanishing result in degrees $q \neq 0, 2, 3, 4$. Only the trivial representation contributes to the cohomology in degree 0. In degree 3, the irreducible unitary representations with non-zero cohomology are discrete series representations of $G(\boldsymbol{R})$. These are tempered representations, hence, by [40], the last assertion in (3).

REMARK 1.7. In the case of the trivial coefficient system $E = \boldsymbol{C}$, there are (up to equivalence) exactly four discrete series representations (σ, H_σ) of the group $G(\boldsymbol{R})$ with non-vanishing cohomology $H^*(\mathfrak{g}_{\boldsymbol{C}}, K_0; H_\sigma \otimes E)$. Beside the holomorphic and antiholomorphic discrete series representations $D^{(3,0)}$ and $D^{(0,3)}$, there are two non-holomorphic discrete series representations, to be denoted $D^{(2,1)}$ and $D^{(1,2)}$, respectively. The upper index (i, j) denotes the unique bidegree in which the corresponding relative Lie algebra cohomology does not vanish. These non-holomorphic representations occur as subrepresentations of a principal series representation of $G(\boldsymbol{R})$. More precisely, we have the sequence

$$0 \to D^{(2,1)} \oplus D^{(1,2)} \to \mathrm{Ind}_{Q_2}^{G}(D_2 \otimes 1_N) \to J^{(1,1)} \to 0\,,$$

where the middle term denotes the induced representation determined by the discrete series representation D_2 on ${}^0L_2$ and the character $(1/3)\rho_{Q_2}$. This representation is reducible and has a unique Langlands quotient, to be denoted $J^{(1,1)}$. The latter representation is unitary as well and has non-vanishing cohomology exactly in the bidegrees $(1, 1)$ and $(2, 2)$.

There are two other irreducible unitary representations of $G(\boldsymbol{R})$ with non-vanishing cohomology which are given as Langlands quotients of an induced representation attached to data on the other maximal parabolic subgroup $Q_1(\boldsymbol{R})$ of $Sp_2(\boldsymbol{R}) = G(\boldsymbol{R})$. These two representations will be denoted by $J^{(2,0)}$ and $J^{(0,2)}$. The first one has non-vanishing cohomology in bidegrees $(2, 0)$ and $(3, 1)$, the second one in bidegrees $(0, 2)$ and $(1, 3)$.

Finally, the trivial representation has non-vanishing cohomology in bidegrees (i, i) for $i = 1, 2, 3, 4$.

2. Toroidal compactifications. In this section, we describe various compactifications of the quasi-projective algebraic variety $V = \Gamma\backslash X$ for a given arithmetic group. Attaching a finite number of points corresponding to cusps to the variety V, we have a compact normal algebraic variety V^* which contains V as a Zariski open subset. We refer to this compactification as the Satake-Baily-Borel compactification, or simply as the minimal compactification of V. The toroidal compactifications give a natural resolution of singularities along cusps, see e.g. [1], [16].

2.1. Basic Notations. Since the group G is of $\boldsymbol{Q}$-rank 1, the rational boundary components attached to X with respect to Γ are of dimension 0. Choose a rational boundary component p, and fix it once for all. Let $P = P_p$ be the stabilizer of p in G, and let N be the unipotent radical of P.

The intersection $N_\Gamma := N(\boldsymbol{Q}) \cap \Gamma$ is a free $\mathbf{Z}$-module of rank 3, which is a lattice in $N_{\boldsymbol{R}} := N(\boldsymbol{R}) \cong \boldsymbol{R}^{\oplus 3}$. We can consider a realization of X as a Siegel domain of the third kind. In our case N is abelian, hence X is realized as a subdomain in $N(\boldsymbol{C}) \cong \boldsymbol{C}^{\oplus 3}$ by $N_{\boldsymbol{R}} + \sqrt{-1}\Omega$. Hence Ω is a convex cone in N, which is isomorphic to $\{(y_1, y_2, y_3) \in \boldsymbol{R}^3 \mid y_1 > 0, y_3 > 0, y_1 y_3 - y_2^2 > 0\}$.

Let us denote by M the Levi component of P. Then Γ_M acts on N_Γ by the adjoint action. The group $P_\Gamma = P(\boldsymbol{Q}) \cap \Gamma$ acts on $N_{\boldsymbol{R}} + \sqrt{-1}\Omega$ as a group of affine transformations. Especially the subgroup N_Γ acts via translations in the real direction in $N_{\boldsymbol{R}} + \sqrt{-1}\Omega$.

Put $\boldsymbol{R}_+ = \{r \in \boldsymbol{R} \mid r > 0\}$, and consider the scalar action of $\boldsymbol{R}_+$ on $N_{\boldsymbol{R}}$. Then Ω is stable under the action of $\boldsymbol{R}_+$, and the action of Γ_M and $\boldsymbol{R}_+$ are compatible. Let $\bar{\Omega}$ be the quotient $\Omega/\boldsymbol{R}_+$. Then $\bar{\Omega}$ is isomorphic to a 2-dimensional hyperbolic space. If Γ is small enough, the induced action of Γ_M on $\bar{\Omega}$ is properly discontinuous and free.

From now on we assume this.

2.2. A triangulation of $\bar{\Omega}$. We refer to [23] for the basic terminology concerning rational partial polyhedral decompositions (an r.p.p. decomposition, for short).

Recall that there exists a rational partial polyhedral decomposition Σ of N such that

(1) $\bigcup_{\sigma \in \Sigma \backslash \{0\}} (\sigma \backslash \{0\}) = \Omega$;
(2) for any compact subset F of Ω, the cardinality $\#\{\sigma \in \Sigma \mid \sigma \cap F = \emptyset\}$ is finite;
(3) Σ is Γ_M-invariant;
(4) the action of Γ_M on $\Sigma \backslash \{\mathbf{0}\}$ is free; and
(5) the quotient $(\Sigma \backslash \{\mathbf{0}\})/\Gamma_M$ is finite.

Here $\mathbf{0}$ is the cone $\{0\}$.

Moreover, by taking a Γ_M-invariant subdivision of Σ, if necessary, we may assume that

(6) for any $\sigma, \tau \in \Sigma$, the cardinality $\#\{\gamma \in \Gamma_M \mid \gamma(\sigma) \cap \tau \neq \{\mathbf{0}\}\}$ is at most one; and
(7) every $\sigma \in \Sigma$ is a nonsingular cone, i.e., σ is spanned by a part of a $\mathbf{Z}$-basis of N_Γ.

Let us recall the triangulation of $\bar{\Omega}$ described in [16], [36]. Denote by Σ_k the set of k-dimensional cones in Σ. For each one-dimensional cone $\sigma \in \Sigma_1$, we denote by $v(\sigma)$ the primitive element of N with $\boldsymbol{R}_0 v(\sigma) = \sigma$, where $\boldsymbol{R}_0 = \{c \in \boldsymbol{R} \mid c \geq 0\}$. Since each element of Σ is nonsingular, we get a $(k-1)$-dimensional simplex $\bar{\sigma}$ in $N_{\boldsymbol{R}}$ spanned by $\{v(\tau) \mid \tau \in \Sigma_1,\ \tau \prec \sigma\}$ for each $\sigma \in \Sigma_k$ $(k = 1, 2, 3)$. If we put

$$\tilde{K} = \{\bar{\sigma} \mid \sigma \in \Sigma \backslash \{\mathbf{0}\}\},$$

then we know that the geometric realization $|\tilde{K}|$ of the simplicial complex $\tilde{K}$ is isomorphic to $\bar{\Omega}$ through the canonical projection $\Omega \to \bar{\Omega} = \Omega/\boldsymbol{R}_+$, and $\tilde{K}$ gives rise to a triangulation of $\bar{\Omega}$, equivariant under the action of Γ_M.

Let K be the quotient $\Gamma\backslash\tilde{K}$. Then, by the condition (vi) of the previous subsection, we know that K is a triangulation of the two-dimensional topological manifold $\Gamma_M\backslash\bar{\Omega}$ into a finite simplicial complex. We denote by K_0 the set of 0-simplices, i.e. the set of vertices of K.

2.3. Torus embeddings. Let T_p be the algebraic torus $N_\Gamma \otimes \boldsymbol{C}^*$. By the assumption that Σ is nonsingular, the associated T_p-embedding Z_p is nonsingular. Since Σ is Γ_M-invariant, the group Γ_M acts on Z_p. Let ord: $T_p = N_\Gamma \otimes \boldsymbol{C}^* \to N_{\boldsymbol{R}} = N_\Gamma \otimes \boldsymbol{R}$ be the homomorphism $1_{N_\Gamma} \otimes (-\log|\ |)$. Then the union $\bar{W} = \mathrm{ord}^{-1}(\Omega) \cup (Z_p\backslash T_p)$ is a Γ_M-invariant open set of Z_p in the classical topology. The action of Γ_M on $\bar{W}$ is free and properly discontinous, and the reduced analytic subspace $\tilde{D}_p = Z_p\backslash T_p \subset \tilde{W}$ is invariant under this action. We denote by W the quotient analytic manifold $\Gamma_M\backslash\tilde{W}$, and we denote $D_p = \Gamma_M\backslash\tilde{D}_p$. By construction, the pair (W, D_p) has the following properties:

(1) D_p is the union of $\#\{K_0\}$ many compact irreducible analytic subspaces $D_{p,v}$ associated to $v \in K_0$.

(2) For any subset I of K_0, the intersection $D_{p,I} = \bigcap_{v\in I} D_{p,v}$ is non-empty if and only if $I \in K$.

(3) For each $I \in K$, the analytic space $D_{p,I}$ is isomorphic to a nonsingular torus embedding of dimension $n - \#I$.

In particular, D_p is a simple normal crossing divisor on W.

2.4. Toroidal compactification. Now we piece together the quotient V and the toroidal embedding discussed in the previous section. With respect to the rational boundary component p, we have a realization of X as a Siegel domain of the third kind: $X = N_{\boldsymbol{R}} + \sqrt{-1}\Omega$. Then if we consider a subdomain

$$X_R = N_{\boldsymbol{R}} + \sqrt{-1}\Omega_R$$

with $\Omega_R = \{(y_1, y_2, y_3) \in \Omega \mid y_1y_3 - y_2^2 > R\}$ for a sufficiently large real number R, then $\gamma \in P_\Gamma$ if and only if $X_R \cap \gamma X_R$ is non-empty.

Let $\boldsymbol{C} \to \boldsymbol{C}^*$ be the exponential map $z \mapsto \exp(2\pi z)$, which induces a map $N_{\boldsymbol{C}} = N_\Gamma \otimes \boldsymbol{C} \to T_p = N_\Gamma \otimes_{\boldsymbol{Z}} \boldsymbol{C}^*$ with kernel N_Γ. Especially, $N_\Gamma\backslash X_R$ is mapped injectively to T_p as an open subset. The composition with $1_{N_\Gamma} \otimes (-\log|\ |)$ maps $N_\Gamma\backslash X_R$ to $2\pi\Omega_R = \Omega_{(2\pi)^2R}$. We put $\tilde{W}_R = \mathrm{ord}^{-1}(\Omega_{(2\pi)^2R}) \cup Z_p\backslash T_p$. Then $N_\Gamma\backslash X_R = \mathrm{ord}^{-1}(\Omega_{(2\pi)^2R})$ is an open analytic submanifold of $\tilde{W}_R$. Passing to the quotients with respect to P_Γ, we get

$$P_\Gamma\backslash X_R \to \Gamma_M\backslash\tilde{W}_R = W_R\,.$$

Patching $V = \Gamma\backslash X$ and W_R with respect to $P_\Gamma\backslash X_R$, we have a local toroidal compactification along the cusp p. Using similar constructions along other cusps, we obtain a compactification $\tilde{V}$ of $V = \Gamma\backslash X$, which is a smooth projective algebraic variety of dimension 3. Put $D = \tilde{V} - V$. Then D is a divisor with normal crossings, and $D = \bigcup_{p\in C} D_p$. Here $C = V^* - V \cong \mathcal{P}/\Gamma$ is the set of Γ-equivalence classes of cusps for Γ.

Given $D = \sum D_i$ $(i = 1, \ldots, \#C)$ we put for an index set $I = \{i_1, \cdots, i_k\} \subset \{1, \ldots, \#C\}$, $|I| = k$, $D_I = D_{i_1} \cap \cdots \cap D_{i_k}$, and we define $D^{[k]}$ as the disjoint union of the D_I where I

runs through the index sets of cardinality k. One obtains $D^{[1]} = \coprod D_i$, and D_i is a rational surface. By the above, $D^{[2]}$ is a disjoint union of projective lines $\mathbf{P}^1\boldsymbol{C}$, and $D^{[3]}$ is a finite set of points.

3. Mixed Hodge structures. We briefly recall some facts pertaining to Deligne's construction [8], [9] of a mixed Hodge structure on the cohomology of a smooth complex algebraic variety. This is mainly done to fix the notation.

3.1. The canonical mixed Hodge structure. Consider a smooth (connected) algebraic variety V over $\boldsymbol{C}$, and let $\tilde{V}$ be a smooth compactification of V such that the divisor at infinity $D = \tilde{V}\backslash V$ is a union of smooth codimension one submanifolds of $\tilde{V}$ with normal crossings. This assumption on D means that, in suitable local coordinates $z_1, \dots, z_n$ on $\tilde{V}$, the divisor is given by an equation $z_1 \cdots z_k = 0$ for some $k \leq n$. Let $j : V \to \tilde{V}$ denote the natural inclusion. Then the holomorphic differential forms on V with logarithmic poles along D are the sections of a subcomplex $\Omega^*_{\tilde{V}}(\log D)$ of $j_*\Omega^*_{\tilde{V}}$. In local terms, the sections $dz_1/z_1, \dots, dz_k/z_k, dz_{k+1}, \dots, dz_n$ generate $\Omega^1_{\tilde{V}}(\log D)$ as a free $O_{\tilde{V}}$-module, and one has $\Omega^p_{\tilde{V}}(\log D) = \bigwedge^p_{O_{\tilde{V}}} \Omega^1_{\tilde{V}}(\log D)$. This leads to an interpretation of the cohomology groups of V in terms of sheaf cohomology of $\tilde{V}$. The inclusion of complexes $\Omega^*_{\tilde{V}}(\log D) \to j_*\Omega^*_{\tilde{V}}$ is a quasiisomorphism. Thus one has

$$H^*(V, \boldsymbol{C}) = \boldsymbol{H}^*(\tilde{V}, R_{j_*}\boldsymbol{C}_V) = \boldsymbol{H}^*(\tilde{V}, j_*\Omega^*_V) = \boldsymbol{H}^*(\tilde{V}, \Omega^*_{\tilde{V}}(\log D)) \tag{1}$$

where $\boldsymbol{H}^*$ denotes the hyper-cohomology groups of sheaf complexes on $\tilde{V}$.

For any complex C in an abelian category, one has a filtration $\sigma_{\geq}$ of C defined in the following way: The subcomplex $\sigma_{\geq p}C$ of C is given by $(\sigma_{\geq p}C)^i = C^i$ if $i \geq p$ and $(\sigma_{\geq p})^i = 0$ if $i < p$. Applying this construction to the log-complex $\Omega^*_{\tilde{V}}(\log D)$, one writes

$$F^p\Omega^*_{\tilde{V}}(\log D) := \sigma_{\geq p}\Omega^*_{\tilde{V}}(\log D)\,.$$

The inclusion $\sigma_{\geq p}\Omega^*_{\tilde{V}}(\log D) \to \Omega^*_{\tilde{V}}(\log D)$ induces a map

$$\alpha_p : \boldsymbol{H}^k(\tilde{V}, F^p\Omega^*_{\tilde{V}}(\log D)) \to \boldsymbol{H}^k(\tilde{V}, \Omega^*_{\tilde{V}}(\log D)) = H^k(V, \boldsymbol{C})\,. \tag{2}$$

By $F^pH^k(V, \boldsymbol{C}) :=$ (image of α_p), a decreasing filtration on $H^k(V, \boldsymbol{C})$ is defined for any k. This filtration is called the Hodge filtration.

On the other hand, there is the weight filtration W on $H^*(V, \boldsymbol{C})$. It is obtained by defining a filtration W on $\Omega^*_{\tilde{V}}(\log D)$:

$$W_m\Omega^p_{\tilde{V}}(\log D) = \begin{cases} 0 & \text{for } m < 0 \\ \Omega^p_{\tilde{V}}(\log D) & \text{for } m > p \\ \Omega^{p-m}_{\tilde{V}} \wedge \Omega^m_{\tilde{V}}(\log D) & \text{for } 0 \leq m \leq p\,. \end{cases}$$

If the map induced by the natural inclusion of complexes is denoted by

$$\beta_m : \boldsymbol{H}^k(\tilde{V}, W_{m-k}\Omega^*_{\tilde{V}}(\log D)) \to \boldsymbol{H}^k(\tilde{V}, \Omega^*_{\tilde{V}}(\log D)) = H^k(V, \boldsymbol{C}) \tag{3}$$

the weight filtration is defined by

$$W_m H^k(V, \boldsymbol{C}) = \text{image of } \beta_m\,.$$

This increasing filtration W of $H^k(V, \boldsymbol{C})$ for any k is already defined over $\boldsymbol{Q}$. The two filtrations F and W define a mixed Hodge structure on $H^k(V, \boldsymbol{Z})$, $k \in \boldsymbol{N}$, and this is the canonical functorial mixed Hodge structure on the cohomology groups of V as constructed by Deligne [9]. It is independent of the choice of the smooth compactification $\tilde{V}$ of V.

3.2. The Poincaré residue map. Let $D_1, \dots, D_n$ be the irreducible components of the divisor at infinity $D = \tilde{V} \backslash V$. A component D_i is smooth and projective by assumption on $\tilde{V}$. Given $D = \sum D_i (i = 1, \dots, n)$ we put $D_I = D_{i_1} \cap \dots \cap D_{i_m}$ for an index set $I = \{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$, $|I| = m$, and we define $D^{[m]}$ as the disjoint union of the D_I's where I runs through the index sets of cardinality m. It is a complex manifold of dimension $\dim(\tilde{V}) - m$. One defines $D^{[0]} := \tilde{V}$ for $m = 0$. If $a_m : D^{[m]} \to \tilde{V}$ denotes the natural map, there is the residue map

$$\mathrm{Res}_m : W_m \Omega^*_{\tilde{V}}(\log D) \to (a_m)_* \Omega^*_{D^{[m]}}\,; \tag{1}$$

it is defined in local coordinates on $\tilde{V}$ by

$$\mathrm{Res}_m \left[\frac{dz_{i_1}}{z_{i_1}} \wedge \dots \wedge \frac{dz_{i_m}}{z_{i_m}} \wedge w \right] = [\text{restriction of } w \text{ to } z_{i_1} = \dots = z_{i_m} = 0] \tag{2}$$

where w is holomorphic and the order of the components of D given by z_{i_j}, $j = 1, \dots, m$, is increasing. This Poincaré residue map Res_k commutes with $d, \partial, \bar{\partial}$; it is surjective and trivial on $W_{k-1}\Omega^*_{\tilde{V}}(\log D)$. It induces an isomorphism

$$\mathrm{Gr}^W_m \Omega^*_{\tilde{V}}(\log D) \to (a_m)_* \Omega^*_{D^{[m]}}[-m]\,. \tag{3}$$

As noted above, the weight filtration W of $H^k(V, \boldsymbol{C})$ is already defined over $\boldsymbol{Q}$, i.e. W is a finite increasing filtration on $H^k(V, \boldsymbol{Q})$ for any k. One has

$$W_m H^k(V, \boldsymbol{Q}) = 0 \quad \text{for } m < k \tag{4}$$

$$W_k H^k(V, \boldsymbol{Q}) = \text{Image of } H^k(\tilde{V}, \boldsymbol{Q}) \to H^k(V, \boldsymbol{Q})\,. \tag{5}$$

The Poincaré residue map induces a map in cohomology to be denoted

$$\mathrm{Res}[i] : W_{k+i} H^k(V, \boldsymbol{C}) \to H^{k-i}(D^{[i]}, \boldsymbol{C})\,. \tag{6}$$

The kernel of $\mathrm{Res}[i]$ is given as $W_{k+i-1}H^k(V, \boldsymbol{C})$.

4. Some simple facts concerning the weight filtration. Let $j : V \to \tilde{V}$ be the open immersion of $V = \Gamma \backslash X$ into a smooth toroidal compactification. The E_2-terms of the Leray spectral sequence associated to j are given as

$$E_2^{p,q} = H^p(\tilde{V}, R^q j_* \boldsymbol{Q}) \Rightarrow H^{p+q}(V, \boldsymbol{Q})$$

abutting to the cohomology of V. This spectral sequence degenerates at E_3 ([8, 3.2.13]). Since this spectral sequence is up to some renumbering nothing else than the spectral sequence defining the weight filtration on $H^*(V, \boldsymbol{Q})$ (cf. [8, 3.2.4]), our study leads to some facts concerning the weight filtration of the mixed Hodge structure on $H^*(V, \boldsymbol{Q})$.

We freely use some facts concerning Deligne's construction of a mixed Hodge structure on the cohomology of a smooth complex algebraic variety [8], [9] as, for example, recalled in Section 3.

4.1. The complex S_*. One sees that $j_*\boldsymbol{Q} = \boldsymbol{Q}$ holds; thus $R^*j_*\boldsymbol{Q}$ is identified with the constant sheaf $\boldsymbol{Q}$ on $D^{[k]}$ if $1 \le k \le 3$ and $R^k j_*\boldsymbol{Q} = \{0\}$ if $k \ge 4$. Now we consider the following complex S_* defined by

$$S_k = E_2^{2k,3-k} = H^{2k}(\tilde{V}, R^{3-k}j_*\boldsymbol{Q}) = H^{2k}(D^{[3-k]}, \boldsymbol{Q}(-3-k)) \quad (k = 0, 1, 2, 3)\,,$$

and the coboundary map $S_k \to S_{k+1}$ is given by the transgression $d_2^{2k,3-k} : E_2^{2k,3-k} \to E_2^{2k+2,3-k-1}$ of the spectral sequence.

We immediately see, that except for $S_3 = H^6(\tilde{V}, \boldsymbol{Q})$, S_k $(k = 0, 1, 2)$ are defined by data on the boundary components of $\tilde{V}$. Then for each cusp p, we denote by $D_p^{[i]}$ the subset of $D^{[i]}$ consisting of irreducible components contained in $\pi^{-1}(p)$. Here $\pi : \tilde{V} \to V^*$ is the resolution map. Put $S(p)_k = H^{2k}(D_p^{[3-k]}, \boldsymbol{Q}(-3-k))$. Then $\{S(p)_k\}_{(k=0,1,2)}$ defines a subcomplex of $\{S_k\}_{(k=0,1,2,3)}$, and $\{S_k\}_{(k=0,1,2)}$ is a direct sum $\{\bigoplus_{p\in V^*-V} S(p)_k\}_{(k=0,1,2)}$.

Now we want to compute the cohomology of $\{S(p)_k\}_{(k=0,1,2)}$ for each p.

PROPOSITION 4.2. *Let K be the finite simplicial complex defined in* 2.2 *for a fixed rational boundary component p. There is a canonical isomorphism of complexes*:

$$\begin{array}{ccccc} C_2(K) & \to & C_1(K) & \to & C_0(K) \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ S(p)_0 & \to & S(p)_1 & \to & S(p)_2. \end{array}$$

Especially, we have isomorphisms $H_i(K, \boldsymbol{Q}) = H_i(\Gamma_M\backslash\bar{\Omega}, \boldsymbol{Q}) = H^{2-i}(S(p)_*)$ $(i = 0, 1, 2)$.

PROOF. With each simplex I of dimension k of K $(k = 0, 1, 2)$, we can associate a stratum $D_{p,I}$ of dimension $2-k$ of D_p. Since D_p is a simple normal crossing divisor, $D_{p,I}$ naturally defines an irreducible component of $D_p^{[k+1]}$. The fundamental class of $H^{4-2k}(D_{p,I}, \boldsymbol{Q}(-k-1))$ defines an element c_I of $H^{4-2k}(D_p^{[k+1]}, \boldsymbol{Q}(-k-1))$. We extend the map $I \to c_I$ linearly to obtain a $\boldsymbol{Q}$-linear map $C_k(K) \to S(p)_{k-2}$. The compatibility with the boundary maps is clear from their construction.

Concerning certain successive quotients of the weight filtration of $H^*(V, \boldsymbol{Q})$ we get as a

COROLLARY 4.3. *We have the following isomorphisms*

$$W_6H^3(V, \boldsymbol{Q})/W_5H^3(V, \boldsymbol{Q}) = E_3^{0,3} = \bigoplus_{p\in C} H^0(S(p)_*) \cong H_2(\Gamma_M\backslash\bar{\Omega}, \boldsymbol{Q})$$

$$W_6H^4(V, \boldsymbol{Q})/W_5H^4(V, \boldsymbol{Q}) = E_3^{2,2} = \bigoplus_{p\in C} H^1(S(p)_*) \cong H_1(\Gamma_M\backslash\bar{\Omega}, \boldsymbol{Q})$$

$$W_6H^5(V, \boldsymbol{Q})/W_5H^5(V, \boldsymbol{Q}) = E_3^{4,1} = \mathrm{Ker}\left\{\bigoplus_{p\in C} H^2(S(p)_*) \to H^6(\tilde{V}, \boldsymbol{Q})\right\}$$
$$\cong \mathrm{Ker}\left\{\bigoplus_{p\in C} H_0(\Gamma_M\backslash\bar{\Omega}, \boldsymbol{Q}) \to H^6(\tilde{V}, \boldsymbol{Q})\right\}.$$

Now we want to see that the weight filtrations of H^1 and H^2 are trivial, i.e. both are pure Hodge structures of weight 1 and 2, respectively.

PROPOSITION 4.4. *Let $j : V \to \tilde{V}$ be the open immersion of $V = \Gamma\backslash X$ into a smooth toroidal compactification $\tilde{V}$. Then the restriction map $H^i(\tilde{V}, \boldsymbol{Q}) \to H^i(V, \boldsymbol{Q})$ $(i = 1, 2)$ is surjective. Especially $H^i(V, \boldsymbol{Q}) = W_iH^i(V, \boldsymbol{Q})$ $(i = 1, 2)$, i.e., the weight filtration is trivial.*

PROOF. Let V^* be the minimal compactification of V. Then $C = V^* - V$ consists of a finite number of points, hence C is of codimenion 3 in the total space V^*.

Let $IH^2(V^*, \boldsymbol{Q})$ be the intersection cohomology group of degree 2 of V^* with middle perversity which we can compute using the stratification $V^* \supset C$. If $k : V \to V^*$ denotes the open immersion, the intermediate extension $k_{!*}\boldsymbol{Q}$ to V^* of $\boldsymbol{Q}_V$ is the truncation $\tau_{\leq 3}$ of $Rk_*\boldsymbol{Q}$. This does not affect the computation of H^1 and H^2 of $Rk_*\boldsymbol{Q}$. Thus we have

$$IH^i(V^*, \boldsymbol{Q}) = H^i(V^*, Rk_*\boldsymbol{Q}) = H^i(V, \boldsymbol{Q}) \quad \text{for } i = 1, 2\,.$$

From the diagram $V \to \tilde{V} \to V^*$, we have $\boldsymbol{H}^i(V^*, \boldsymbol{Q}) \to H^i(\tilde{V}, \boldsymbol{Q}) \to H^i(V, \boldsymbol{Q})$ $(i = 1, 2)$. Since the composition of the above maps is an isomorphism, the restriction map $H^i(\tilde{V}, \boldsymbol{Q}) \to H^i(V, \boldsymbol{Q})$ is surjective.

4.5. The complex T_*. For later use we introduce the complex $T_0 \to T_1 \to T_2$ defined by $E_2^{0,2} \to E_2^{2,1} \to E_2^{4,0}$ or written as

$$H^0(D^{[2]}, \boldsymbol{Q}(-2)) \to H^2(D^{[1]}, \boldsymbol{Q}(-1)) \to H^4(\tilde{V}, \boldsymbol{Q})$$

with the obvious maps given by the transgression map of the Leray spectral sequence. Since $E_3^{0,2} = \mathrm{Ker}(T_0 \to T_1)$ is $W_4H^2(V, \boldsymbol{Q})/W_3H^2(V, \boldsymbol{Q}) = \{0\}$, the map $T_0 \to T_1$ is injective.

REMARK. The term $E_3^{2,1} = H^1(T_0 \to T_1 \to T_2) = W_4H^3(V, \boldsymbol{Q})/W_3H^3(V, \boldsymbol{Q})$ is difficult to control. This delicate problem is described by Eisenstein cohomology classes (see 7.3).

Now we identify the interior cohomology group of degree 3 with the homogeneous part of weight 3 of the third cohomology group.

PROPOSITION 4.6. *Let $\tilde{V}$ be a smooth toroidal compactification of V. Then the image $H_!^3(V, \boldsymbol{Q})$ of the cohomology with compact support under the natural map coincides with $W_3H^3(V, \boldsymbol{Q})$, i.e. one has*

$$H_!^3(V, \boldsymbol{Q}) = W_3H^3(V, \boldsymbol{Q}) \cong \textit{image of } H^3(\tilde{V}, \boldsymbol{Q}) \textit{ in } H^3(V, \boldsymbol{Q})\,.$$

PROOF. By the surjectivity of the map $H^2(\tilde{V}, \boldsymbol{Q}) \to H^2(V, \boldsymbol{Q})$, we have $E_3^{1,1} = \{0\}$ in the Leray spectral sequence. This, in turn, implies that the sequence

$$\{0\} \to H^1(D^{[1]}, \boldsymbol{Q})(-1) \xrightarrow{i_*} H^3(\tilde{V}, \boldsymbol{Q}) \to H^3(V, \boldsymbol{Q})$$

is exact where i_* denotes the Gysin homomorphism. In our case $D^{[1]}$ is a disjoint union of rational surfaces, hence $H^1(D^{[1]}, \boldsymbol{Q}) = \{0\}$. Thus $H^3(\tilde{V}, Q)$ is a subspace of $H^3(V, \boldsymbol{Q})$. By Poincaré duality, this implies the surjectivity of $H_c^3(V, \boldsymbol{Q}) \to H^3(\tilde{V}, \boldsymbol{Q})$. By a general result of Deligne, $W_3 H^3(V, \boldsymbol{Q}) =$ is equal to the image of $(H^3(\tilde{V}, \boldsymbol{Q}) \to H^3(V, \boldsymbol{Q}))$. Hence the proposition follows from the commutative diagram:

$$\begin{array}{ccccc} & & H^3(\tilde{V}, \boldsymbol{Q}) & & \\ & & \uparrow & \searrow & \\ H_c^3(V, \boldsymbol{Q}) & = & H^3(V^*, \boldsymbol{Q}) & \to & H^3(V, \boldsymbol{Q}). \end{array}$$

5. Mixed Hodge structures on the cohomology of the boundary. As a prerequisite in dealing with the cohomology $H^2(V, \boldsymbol{Q})$ in degree 2, it is useful to consider the long exact cohomology sequence

$$\to H_c^n(V, \boldsymbol{Q}) \to H^n(V, \boldsymbol{Q}) \to H^n(\partial V, \boldsymbol{Q}) \to 0$$

for the pair $(\bar{V}, \partial \bar{V})$ via the Leray spectral sequence associated to the open immersion k of $V = \Gamma \backslash X$ into the minimal compactification V^*. As before, $C = V^* - V$, which is a union of a finite number of points corresponding to the equivalence classes of cusps.

5.1. Given the open immersion $k : V \to V^*$ let $k_! \boldsymbol{Q}$ be the extension of the constant sheaf $\boldsymbol{Q}_V$ by zero to V^*. Since $\boldsymbol{Q}_{V^*}/k_! \boldsymbol{Q}$ is equal to $\boldsymbol{Q}_C$, we have that $H_c^1(V, \boldsymbol{Q}) \to H^1(V^*, \boldsymbol{Q})$ is surjective, and that $H_c^n(V, \boldsymbol{Q}) \cong H^n(V^*, \boldsymbol{Q})$ for $n \geq 2$. Now we analyze the Leray spectral sequence associated to k. The E_2-terms are given as

$$E_2^{p,q} = H^p(V^*, R^q k_* \boldsymbol{Q}) \Rightarrow H^{p+q}(V, \boldsymbol{Q})$$

abutting to the cohomology of V. Observing $k_* \boldsymbol{Q} = \boldsymbol{Q}$ and that $R^q k_* \boldsymbol{Q}$ has support only on the finite set C for $q > 0$, one sees that $E_2^{p,q} = \{0\}$ if $p > 0$ and $q > 0$. One obtains a long exact sequenc

$$0 \to E_2^{1,0} \to H^1(V, \boldsymbol{Q}) \to E_2^{0,1} \xrightarrow{d_2} \to E_2^{2,0} \to H^2(V, \boldsymbol{Q}) \to$$
$$\to E_2^{3,0} \to H^3(V, \boldsymbol{Q}) \to E_2^{0,3} \xrightarrow{d_3} \to E_2^{4,0} \to H^4(V, \boldsymbol{Q}) \to \cdots.$$

This gives the long exact sequence

$$0 \to H^1(V^*, \boldsymbol{Q}) \to H^1(V, \boldsymbol{Q}) \to H^0(C, R^1 k_* \boldsymbol{Q}) \to H_c^2(V, \boldsymbol{C}) \to$$
$$\to H^2(V, \boldsymbol{Q}) \to H^0(C, R^2 k_* \boldsymbol{Q}) \to H_c^3(V, \boldsymbol{Q}) \to H^3(V, \boldsymbol{Q}) \to \cdots.$$

There is an analoguous sequence by using local cohomology, given as

$$\to H^n(V^*, \boldsymbol{Q}) \to H^n(V, \boldsymbol{Q}) \to H_c^{n+1}(V^*, \boldsymbol{Q}) \to .$$

Thus we have a canonical isomorphism $H^0(C, R^n k_* \boldsymbol{Q}) = H_c^{n+1}(V^*, \boldsymbol{Q})$ for $n \geq 0$ and both groups are isomorphic to $H^n(\partial \bar{V}, \boldsymbol{Q})$ for $n > 0$.

5.2. Let $\tilde{V}$ be a smooth toroidal compactification of V, and let $\pi : \tilde{V} \to V^*$ be the resolution map. We may write the open immersion $k : V \to V^*$ as a composite $k = \pi \circ j$ with the open immersion $j : V \to \tilde{V}$.

We are going to determine $H^0(C, R^3 k_* \boldsymbol{Q})$. Having the spectral sequence $E_2^{p,q} = R^p \pi_* R^q j_* \boldsymbol{Q} \Rightarrow R_{k*}^{p+q} \boldsymbol{Q}$ we consider the stalk $\{R^p \pi_* R^q j_* \boldsymbol{Q}\}_p$ for $p \in C,\ p + q = 3$. Since π is a proper map one has

$$\{E_2^{p,q}\}_p := \{R^p \pi_* R^q j_* \boldsymbol{Q}\}_p = H^p(\pi^{-1}(p), R^q j_* \boldsymbol{Q}) = H^p(D_p, R^q j_* \boldsymbol{Q}) .$$

Recall that D_p is of complex dimension two. This enables us to prove the following

PROPOSITION 5.3. *The natural mixed Hodge structure on* $H^3(\partial \bar{V}, \boldsymbol{Q}) = H^0(C, R^3 k_* \boldsymbol{Q})$ *has the following weight filtration*:

(1) $W_6 H^3(\partial V, \boldsymbol{Q}) = H^3(\partial V, \boldsymbol{Q})$,
$W_5 H^3(\partial V, \boldsymbol{Q}) = W_4 H^3(\partial V, \boldsymbol{Q})$,
$W_3 H^3(\partial V, \boldsymbol{Q}) = \{0\}$.

(2) *The quotient* $W_{2i} H^3(\partial V, \boldsymbol{Q}) / W_{2i-1} H^3(\partial V, \boldsymbol{Q}),\ i = 2, 3,$ *is a direct sum of the Tate Hodge structures* $\boldsymbol{Q}(-i)$ *of weight* $2i$.

PROOF. Denote by $\sigma : D^{[3]} \to \tilde{V}$ the composite of the normalization and the closed immersion; then $R^3 j_* \boldsymbol{Q}$ is isomorphic to $\sigma_* \boldsymbol{Q}(-3)$. Thus $\{E_2^{0,3}\}_p = H^0(D_p^{[3]}, \boldsymbol{Q})(-3)$ is a direct sum of the Tate Hodge structures $\boldsymbol{Q}(-3)$ of weight 6.

For $q = 2$, $R^2 j_* \boldsymbol{Q}$ is isomorphic to $\delta_* \boldsymbol{Q}(-2)$ where $\delta : D^{[2]} \to \tilde{V}$ again denotes the natural map. Then $\{E_2^{1,2}\}_p = H^1(D_p^{[2]}, \boldsymbol{Q})(-2) = \{0\}$ because each connected component of $D^{[2]}$ is a rational surface, hence its irregularity is zero.

For $q = 1$, $R^1 j_* \boldsymbol{Q}$ is isomorphic to $\tau_* \boldsymbol{Q}(-1)$ where $\tau : D^{[1]} \to \tilde{V}$ is the composite of the normalization $D^{[1]} \to D$ and the closed immersion. Hence, one has $\{E_2^{2,1}\}_p = \bigoplus H^2(D_i, \boldsymbol{Q})(-1)$ where the sum ranges over all components $D_i \subset D_p$; but this is homogeneous of weight 4. Now, each D_i is a rational surface, birational to $\boldsymbol{P}^2\boldsymbol{C}$, thus $H^2(D_i, \boldsymbol{Q})$ is generated by algebraic cycles. This implies that $\{E_2^{2,1}\}_p$ is a direct sum of $\boldsymbol{Q}(-2)$.

For $q = 0$, one has $R^0 j_* \boldsymbol{Q} = j_* \boldsymbol{Q} = \boldsymbol{Q}$, and thus $E_2^{3,0} = R^3 \pi_* \boldsymbol{Q}$. Since $\pi_{|V}$ is an isomorphism $R^3 \pi_* \boldsymbol{Q}$ has only support over C. For each $p \in C$, one has $\{E_2^{3,0}\}_p = H^3(D_p, \boldsymbol{Q})$. For an irreducible component D_i of D_p, $H^3(D_i, \boldsymbol{Q}) = \{0\}$ holds because D_i is a rational surface. A fortiori, $H^3(D_p, \boldsymbol{Q}) = \{0\}$ holds by a Mayer-Vietoris sequence argument, because the intersection of two irreducible components is of complex dimension one. This implies $E_2^{3,0} = \{0\}$.

Therefore only the terms $E_2^{0,3}$ and $E_2^{2,1}$ contribute to $R^3 k_* \boldsymbol{Q}$. This proves our claim.

6. Eisenstein cohomology. In this section we describe the internal structure of the Eisenstein cohomology $H_{\mathrm{Eis}}^*(S(\boldsymbol{C}), E)$ in some more detail.

We have to assume familiarity with the construction of Eisenstein cohomology classes ([30], [13]). The general case of groups of $\boldsymbol{Q}$-rank 1 is dealt with (up to the actual existence of poles for the Eisenstein series to be considered) in Harder [13]. The unpublished thesis of D. Osenberg [27] treats the case we are interested in. This thesis was written under the supervision of the second-named author in 1993. However, the exposition given here relies on some subsequent general results as, for example, contained in [11]. We suppose that $E = \boldsymbol{C}$ is the trivial coefficient system. The general case of an arbitrary coefficient system E can be dealt with in the same way. If the highest weight of the representation (τ, E) is regular the final result that describes the Eisenstein cohomology is much easier to obtain. We refer to [33], [20].

6.1. There are some simple observations regarding the cohomology of the arithmetic varieties $S_K(\boldsymbol{C})$, their Borel-Serre compactification $\bar{S}_K(\boldsymbol{C})$ and its boundary $\partial S_K(\boldsymbol{C})$ one should keep in mind. Let

$$r^* : H^*(\bar{S}_K(\boldsymbol{C})) \to H^*(\partial S_K(\boldsymbol{C}))$$

be the natural restriction map. We denote the image of r^q by I^q for $q \geq 0$. There exists a pairing on $H^*(\partial S_K(\boldsymbol{C}))$ such that, if $s = \dim \partial S_K(\boldsymbol{C})$, I^{s-q} is the orthogonal complement to I^q for $q = 0, \ldots, 5$ with respect to this pairing ([33, Section 5]). In fact, it is a consequence of a general result using duality for compact manifolds with boundary.

Within the long exact cohomology sequence of the pair $(\bar{S}_K(\boldsymbol{C}), \partial S_K(\boldsymbol{C}))$ the term $H^6(\bar{S}_K(\boldsymbol{C}), \partial S_K(\boldsymbol{C}))$ equals $\boldsymbol{C}$, and $H^6(\bar{S}_K(\boldsymbol{C}))$ vanishes. Thus, one obtains

$$\dim I^5 = \dim H^5(\partial S_K(\boldsymbol{C}), \boldsymbol{C}) - 1\,.$$

In turn, $H^0(S_K(\boldsymbol{C}), \boldsymbol{C}) = \boldsymbol{C}$, and the map r^0 is injective. As noted in 1.6., the cohomology $H^1(S_K(\boldsymbol{C}), \boldsymbol{C})$ in degree one vanishes, therefore, the map r^4 is surjective. The most interesting case is the one of the cohomology in degree 2 and 3. This will be the main focus in the subsequent paragraphs.

However, with regard to the Eisenstein cohomology an analysis similiar to the one in [31], [34] leads to the following results (we refer to [30], [33] for unexplained notations and notions).

6.2. Following [11, Theorem 2.3], the Eisenstein cohomology can be decomposed according to the cuspidal support for the Eisenstein series involved. This leads to a decomposition

$$H^*_{\mathrm{Eis}}(S(\boldsymbol{C}),\ E) = \bigoplus_{\varphi \in \Phi_{E,\{P\}}} H^*(\mathfrak{g}_C, K_0; \mathcal{A}_{E,\{P\},\varphi} \otimes E)$$

where the sum ranges over the set $\Phi_{E,\{P\}}$ of classes $\varphi = (\varphi_Q)_{Q\in\{P\}}$ of associate irreducible cuspidal automorphic representations of the Levi components of elements of $\{P\}$, subject to certain compatibility conditions given in [11, 1.2]. By definition, given $\varphi \in \Phi_{E,\{P\}}$, the space $\mathcal{A}_{E,\{P\},\varphi}$ is the span of all possible residues and derivatives with respect to the parameter Λ of Eisenstein series $E(\psi, \Lambda)$ starting from cuspidal automorphic forms ψ of type φ at values in the positive Weyl chamber defined by Q for which the infinitesimal character of E^* is matched.

Recall that the quotient of P by its unipotent radical N is a reductive algebraic $\boldsymbol{Q}$-group $P/N = M$. The $\boldsymbol{Q}$-rank of the derived group of M is zero. Note that the group of points of M over some splitting field F of D is isomorphic to $GL_2(F)$. The canonical projection $\kappa : P \to M$ induces a fibration

$$P(\boldsymbol{Q})\backslash P(\boldsymbol{A})/K_0^P K^P \to M(\boldsymbol{Q})\backslash M(\boldsymbol{A})/K_0^M K^M$$

with fiber $F_K^N := N(\boldsymbol{Q})\backslash N(\boldsymbol{A})/K^N$ where $K^N := K \cap N(\boldsymbol{A}_f)$, $K^M := \kappa(K^P)$ and $K_0^M := \kappa(K_0^P)$. This fibration gives rise to a spectral sequence in cohomology which degenerates at E_2 [30, 2.7], i.e., the cohomology of $\partial S(\boldsymbol{C})$ is given as a $G(\boldsymbol{A}_f)$-module by

$$H^*(\partial S(\boldsymbol{C}), \pi_p^* E) = \operatorname{Ind}_{P(\boldsymbol{A}_f)}^{G(\boldsymbol{A}_f)}[H^*(S^M(\boldsymbol{C}), H^*(\mathfrak{n}, E))]$$

where $S^M(\boldsymbol{C})$ denotes the limit $\varprojlim_K M(\boldsymbol{Q})\backslash M(\boldsymbol{A})K_0^M K^M$. The coefficient sheaf is determined by the Lie algebra cohomology $H^*(\mathfrak{n}, E)$ endowed with the natural $M \times_Q \bar{\boldsymbol{Q}}$-module structure.

6.3. A decomposition of $H^*(\mathfrak{n}, E)$. Next we are going to describe the $M \times_Q \bar{\boldsymbol{Q}}$-module structure of $H^*(\mathfrak{n}, E)$. We have to recall a result due to Kostant ([18, 5.13]). Let W^P be the set of minimal coset representatives for the right cosets of W_P in the Weyl group W (see e.g. [33, Section 4]). In this special case, one has, using the notation of 1.2 for the simple reflections in W,

$$W^P = \{1, w_2, w_2 \cdot w_1, w_2 \cdot w_1 \cdot w_2\}. \tag{1}$$

Given the irreducible representation (τ, E_λ) of $G \times_Q \bar{\boldsymbol{Q}}$ of the highest weight $\lambda \in \mathfrak{h}^*$, the Lie algebra cohomology $H^*(\mathfrak{n}, E_\lambda)$ decomposes as an $M \times_Q \bar{\boldsymbol{Q}}$-module

$$H^*(\mathfrak{n}, E_\lambda) = \bigoplus_{w \in W^P, l(w)=q} F_{\mu_{w,\lambda}} \tag{2}$$

into irreducible $M \times_Q \bar{\boldsymbol{Q}}$-modules $F_{\mu_{w,\lambda}}$ with highest weight $\mu_{w,\lambda} = w(\lambda+\rho) - \rho$. The sum ranges over all $w \in W^P$ with $l(w) = q$. The weights $\mu_{w,\lambda}$ are all dominant and distinct as w ranges through the set W^P. By [33, 4.9], if the highest weight λ is regular, then the highest weight $\mu_{w,\lambda}$ is regular.

For later use, it is helpful to carry out the following calculations. Let $\lambda_j \in \mathfrak{h}^*$, for $j = 1, 2$ be the fundamental dominant weights; one has $\lambda_1 = \alpha_1 + (1/2)\alpha_2$, $\lambda_2 = \alpha_1 + \alpha_2$. Let the highest weight λ of (τ, E_λ) be given by $\lambda = c_1\lambda_1 + c_2\lambda_2$ with non-negative integers c_j for $j = 1, 2$. Then one can verify the following facts concerning the weights $\mu_{w,\lambda}$ and the parameter $\Lambda_w := -w(\lambda + \rho)_{|\mathfrak{a}_P}$. The latter one plays a role in constructing Eisenstein cohomology classes; it is given in the form $\Lambda_w = ?\rho_P$, $\rho_P = (2/3)\alpha_2$:

$l(w)$	$\mu_{w,\lambda}$	$\Lambda_w = ?\rho_P$
0	λ	$-\big(\frac{1}{3}c_1 + \frac{2}{3}c_2\big) - 1$
1	$(c_1 + c_2)\alpha_1 + \big(\frac{1}{2}c_1 - 1\big)\alpha_2$	$-\frac{1}{3}(c_1 + 1)$
2	$(c_2 - 1)\alpha_1 + \big(-\frac{1}{2}c_1 - 2\big)\alpha_2$	$\frac{1}{3}(c_1 + 1)$
3	$(-c_2 - 3)\alpha_1 + \big(-\frac{1}{2}c_1 - c_2 - 3\big)\alpha_2$	$\big(\frac{1}{3}c_1 + \frac{2}{3}c_2\big) + 1$

It is immediate from the first column that $F_{\mu_{w,\lambda}}$ is trivial (as $M^{\mathrm{der}} \times_{\boldsymbol{Q}} \bar{\boldsymbol{Q}}$-module) if and only if $c_1 = 0$ and $l(w) = 0$ or 3.

6.4. As a consequence of the preceding discussion, there are two parameters in the description of $H^*(\partial S(\boldsymbol{C}), E)$ which matter in the construction of Eisenstein cohomology classes. First, we have the elements $w \in W^P$ and the associated module $F_{\mu_{w,\lambda}} \subset H^*(\mathfrak{n}, E)$. Second, we have all irreducible automorphic representations (π, V_π) of $M(\boldsymbol{A})$ with non-trivial cohomology with respect to a fixed module $F_{\mu_{w,\lambda}} \subset H^*(\mathfrak{n}, E)$, i.e., H_π occurs as an $M(\boldsymbol{A})$-submodule with multiplicity $m(\pi)$ in the space $L^2(M(\boldsymbol{Q})A_P\backslash M(\boldsymbol{A}))$, and $H^*({}^0\mathfrak{m}, K_0^M, \pi_\infty \otimes F_{\mu_{w,\lambda}}) \otimes \pi_f$ does not vanish. Note that $M(\boldsymbol{Q})A_P\backslash M(\boldsymbol{A})$ is compact, hence $H^*(S^M(\boldsymbol{C}), H^*(\mathfrak{n}, E))$ coincides with the subspace of square integrable cohomology classes. A cohomology class in $H^*(S^M(\boldsymbol{C}), H^*(\mathfrak{n}, E))$ is said to be of type (π, w) if it is an element of the summand $H^*({}^0\mathfrak{m}, K_0^M; \pi_\infty \otimes F_{\mu_{w,\lambda}}) \otimes \pi_f$ in the underlying decomposition of $H^*(S^M(\boldsymbol{C}), H^*(\mathfrak{n}, E))$.

6.5. We recall the actual contribution of cohomology classes in $H^*(\mathfrak{g}_{\boldsymbol{C}}, K_0; \mathcal{A}_{E,\{P\},\varphi} \otimes E)$. We consider an Eisenstein series $E_Q(\psi, \Lambda)$ attached to a non-trivial cohomology classes of type (π, w) for $\pi \in \varphi_Q$ and $w \in W^Q$. As shown in [30, 3.4 and 4.3], the analytic behaviour of $E_Q(\psi, \Lambda)$ at the point

$$\Lambda_w = -w(\lambda + \rho)_{|\mathfrak{a}_Q}$$

is decisive in constructing a class in $H^*_{Eis}(S(\boldsymbol{C}), E)$. The element Λ_w is real and uniquely determined by (π, w). One has

THEOREM 6.6 ([30, 4.11]). *If the Eisenstein series $E_Q(\psi, \Lambda)$ attached to a class of type (π, w) for $\pi \in \varphi$ and $w \in W^Q$ is regular at the point Λ_w, then the Eisenstein series evaluated at Λ_w gives rise to a non-trivial cohomology class $[E_Q(\psi, \Lambda_w)]$ in $H^*(\mathfrak{g}_{\boldsymbol{C}}, K_0; \mathcal{A}_{E,\{P\},\varphi})$. Its degree is the degree of the class started with.*

Note that the points Λ_w are determined in 6.2. Such a class is called a regular Eisenstein cohomology class.

This result is supplemented by

THEOREM 6.7. *If the Eisenstein series $E_Q(\psi, \Lambda)$ attached to a class of type (π, w) for $\pi \in \varphi$ and $w \in W^Q$ has a pole at Λ_w then the residue* $\mathrm{Res}_{\Lambda=\Lambda_w}(E_Q(\varphi, \Lambda))$ *gives rise to a non-trivial cohomology class* $[\mathrm{Res}_{\Lambda=\Lambda_w} E_Q(\varphi, \Lambda)]$ *in $H^*(\mathfrak{g}_{\boldsymbol{C}}, K_0; \mathcal{A}_{E,\{P\},\varphi})$.*

6.8. We have to determine the actual summand $H^*({}^0\mathfrak{m}, K_0^M, \pi_\infty \otimes F_{\mu_{w,\lambda}}) \otimes \pi_f$ in the decomposition of the cohomology $H^*(S^M(\boldsymbol{C}), H^*(\mathfrak{n}, E))$, i.e., we have to describe the possible types (π, w) occuring in the sense of 6.4. Recall that the group $M(F)$ of points of M over some splitting field F of D is isomorphic to $GL_2(F)$. Note that there ia s real splitting field of D. One has ${}^0M(\boldsymbol{R}) \cong SL_2^{\pm}(\boldsymbol{R})$, resp. ${}^0M(\boldsymbol{C}) \cong SL_2(\boldsymbol{C})$ for the Lie groups in question.

First, given a non-negative integer $m \in \boldsymbol{Z}$ and $\varepsilon \in 0, 1$, let $V(m, \varepsilon)$ denote the finite dimensional space of homogeneous ploynomials $q \in \boldsymbol{C}[X, Y]$ of degree m. It can be endowed with the structure of an $({}^0M \times_{\boldsymbol{Q}} \boldsymbol{C})$-module in the way

$$\left(\begin{pmatrix} a & b \\ a & d \end{pmatrix}, q(X, Y)\right) \mapsto \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\varepsilon} q(aX + cY, bX + dY)\,.$$

Then the family $V(m, \varepsilon)$ parametrized by $m \in \boldsymbol{Z}$ with $m \geq 0$ and $\varepsilon \in \{0,\ 1\}$ exhausts (up to isomorphism) the irreducible finite dimensional $({}^0M \times_{\boldsymbol{Q}} \boldsymbol{C})$-modules.

Secondly, consider a fixed $({}^0M \times_{\boldsymbol{Q}} \boldsymbol{C})$-module $V(m,\ \varepsilon)$. Let D_n be the irreducible unitary $({}^0\mathfrak{m},\ K_0^M)$-module determined by the discrete series representation of ${}^0M(\boldsymbol{R})$ with lowest K_0^M-type $(n + 2)$. Then an irreducible unitary $({}^0\mathfrak{m},\ K_0^M)$-module H_σ with the non-zero cohomology space

$$H^*({}^0\mathfrak{m},\ K_0^M;\ H_\sigma \otimes V(m, \varepsilon))$$

is given by

$$\begin{aligned} &H_\sigma \cong D_n \quad \text{if } n > 0\,. \\ &H_\sigma \cong D_0\,, \quad V(0, 0) \quad \text{or} \quad V(0, 1) \quad \text{if } n = 0 \end{aligned}$$

up to equivalence. Note that $V(0, 0) \cong \boldsymbol{C}$ is the trivial module. Both assertions are a consequence of Frobenius reciprocity and the corresponding statements in the $SL_2(\boldsymbol{R})$-situation (see e.g. [30]). In each case, the cohomology is computable as

$$H^q({}^0\mathfrak{m},\ K_0^M;\ D_n \otimes V(n, \varepsilon)) = \begin{cases} \boldsymbol{C} & q = 1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and

$$H^q({}^0\mathfrak{m},\ K_0^M;\ V(0, \varepsilon) \otimes \boldsymbol{C}) = \begin{cases} \boldsymbol{C} & \varepsilon = 0\,,\ q = 0, 2\,, \\ \boldsymbol{C} & \varepsilon = 1\,,\ q = 2\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

6.9. Suppose that the given irreducible representation (τ, E) of $G \times_{\boldsymbol{Q}} \bar{\boldsymbol{Q}}$ is the trivial one. In this case, the summands $F_{\mu_w,\lambda}$ for $w \in W^P$ occuring in the decomposition 6.3 of $H^*(\mathfrak{n}, \boldsymbol{C})$ as an $({}^0M \times \boldsymbol{C})$-module, are made precise in the following list, based on the results in 6.3.

$$
\begin{aligned}
H^0(\mathfrak{n},\ \boldsymbol{C}) &\cong V(0,0)\,, & w &= 1\,.\\
H^1(\mathfrak{n},\ \boldsymbol{C}) &\cong V(2,0)\,, & w &= w_2\,.\\
H^2(\mathfrak{n},\ \boldsymbol{C}) &\cong V(2,1)\,, & w &= w_2\circ w_1\,.\\
H^3(\mathfrak{n},\ \boldsymbol{C}) &\cong V(0,1)\,, & w &= w_2\circ w_1\circ w_2\,.
\end{aligned}
$$

As a consequence of 6.5, given $w \in W^P$, one can now determine the list of irreducible unitary $({}^0\mathfrak{m},\ K_0^M)$-modules H_σ with non-zero cohomology $H^*({}^0\mathfrak{m}, K_0^M, H_\sigma \otimes F_{\mu_{w,\lambda}})$.

If $w \in W^P$ is the longest element, i.e., $l(w) = 3$, the modules D_0, $V(0,0)$ and $V(0,1)$ exhaust this list up to equivalence. One determines the corresponding cohomology spaces by 6.5 and obtains

$$
H^q({}^0\mathfrak{m},\ K_0^M;\, D_0 \otimes V(0,1)) = \begin{cases} \boldsymbol{C} & q = 1\,,\\ 0 & \text{otherwise}\,,\end{cases}
$$

$$
H^q({}^0\mathfrak{m},\ K_0^M;\, V(0,0) \otimes V(0,1)) = \begin{cases} \boldsymbol{C} & q = 2\,,\\ 0 & \text{otherwise}\,,\end{cases}
$$

$$
H^q({}^0\mathfrak{m},\ K_0^M;\, V(0,1) \otimes V(0,1)) = \begin{cases} \boldsymbol{C} & q = 0\,,\\ 0 & \text{otherwise}\,.\end{cases}
$$

If $w = w_2 \circ w_1$ is in the element of length 2 in W^P, the module D_2 is (up to equivalence) the only irreducible unitary $({}^0\mathfrak{m},\ K_0^M)$-module with non-zero cohomology with respect to the coefficients system $F_{\mu_w,0} \cong V(2,1)$. One has

$$
H^q({}^0\mathfrak{m},\ K_0^M;\, D_2 \otimes V(2,1)) = \begin{cases} \boldsymbol{C} & q = 1\,,\\ 0 & \text{otherwise}\end{cases}
$$

in this case.

These results allow us to conclude the discussion in Sections 6.3 and 6.4. As already noted, the boundary $\partial S(\boldsymbol{C})$ of the Borel-Serre compactification $\bar{S}(\boldsymbol{C})$ has dimension 5. Thus its cohomology $H^q(\partial S(\boldsymbol{C}),\ E)$ vanishes in degrees $q > 5$. We enumerate the possibe types (π, ω) (in the sense of the definition given after 6.3.(3)) of virtual non-trivial cohomology classes in $H^q(\partial S(\boldsymbol{C}),\ E)$ for the relevant other degrees. We confine our discussion to the case when (τ, E) equals to the trivial coefficient system.

In degree $q = 5$, the cohomology $H^5(\partial S(\boldsymbol{C}), \boldsymbol{C})$ is built up by cohomology classes of type (π, w) with $l(w) = 3$ and π_∞ the trivial $({}^0\mathfrak{m},\ K_0^M)$-module.

In degree $q = 4$, the only possible types are of the form (π, w) with $l(w) = 2$ and $\pi_\infty \cong D_0$, the discrete series representation of loweset K_0^M-type 2.

In degree $q = 3$, there are two possible types of classes, to be distinguished by the length $l(w)$ of the element $w \in W^P$. First, if $l(w) = 3$, then a possible type (π, w) has to satisfy the requirement $\pi_\infty \cong V(0,1)$. Second, if $l(w) = 2$, then the only corresponding type has the form (π, w) with $\pi_\infty \cong D_2$, the discrete series representation of lowest K_0^M-type 4. Accordingly, there is a decomposition of $H^3(\partial S_K(\boldsymbol{C}))$ into the corresponding subspaces determined by the type, that is,

$$H^3(\partial S_K(\boldsymbol{C})) = \left[\bigoplus_{l(w)=3,\pi_\infty=\det} A(\pi, w)\right] \oplus \left[\bigoplus_{l(w)=2,\pi_\infty=D_2} A(\pi, w)\right].$$

With this framework in place one obtains the following structural description of the Eisenstein cohomology. Its structure is related to the analytic properties of certain Euler products attached to cuspidal automorphic representations of GL_2 or inner forms thereof. These automorphic L-functions naturally appear in the constant terms of the Eisenstein series under consideration.

THEOREM 6.10. *For $q = 4, 5$, the Eisenstein cohomology $H^q_{\mathrm{Eis}}(S_K(\boldsymbol{C}), \boldsymbol{C})$ is built up by regular Eisenstein cohomology classes. It restricts isomorphically onto the image I^q of the restriction map $r^q : H^q(S_K(\boldsymbol{C})) \to H^q(\partial S_K(\boldsymbol{C}))$. Note that $H^5(S_K(\boldsymbol{C}), \boldsymbol{C}) \cong H^5_{\mathrm{Eis}}(S_K(\boldsymbol{C}), \boldsymbol{C})$, and codim $I^5 = 1$. The map r^4 is surjective, that is, one obtains $H^4_{\mathrm{Eis}}(S_K(\boldsymbol{C}), \boldsymbol{C}) \xrightarrow{\sim} I^4 = H^4(\partial S_K(\boldsymbol{C}), \boldsymbol{C})$.*

PROOF. In degree 5, the Eisenstein series to be considered is attached to a class of type (π, w), $w \in W^Q$, $Q \in \{P\}$, $l(w) = 3$ and π an automorphic representation with π_∞ = trivial representation. The point of evaluation is, by 6.3., $\Lambda_w = \rho_Q$. If π is not trivial the Eisenstein series is holomorphic at this point, and the result follows from 6.5. If π is trivial, the associated Eisenstein series has a pole at Λ_w and $[\mathrm{Res}_{\Lambda=\Lambda_w} E_Q(\varphi, \Lambda)]$ is a residual class in $H^0(S(\boldsymbol{C}), \boldsymbol{C})$.

In degree 4, the Eisenstein series are attached to classes of type (π, w) for $w \in W^Q$ and $l(w) = 3$, and π an automorphic representation of M_Q with $\pi_\infty = D_0$.

THEOREM 6.11. *The Eisenstein cohomology in degree* 3 *decomposes into two subspaces*

$$H^3_{\mathrm{Eis}}(S_K(\boldsymbol{C}), \boldsymbol{C}) = \mathrm{Eis}^3[\det] \oplus \mathrm{Eis}^3[D_2]\,;$$

it consists of regular Eisenstein cohomology classes. The first summand is built up by Eisenstein series attached to classes in $A(\pi, w)$, $l(w) = 3$, $\pi_\infty = \det$, and restricts isomorphically under the restriction map r^3 onto the first summand in the decomposition of $H^3(\partial S_K(\boldsymbol{C}))$ given at the end of Section 6.9. *The second summand contains all* (*regular*) *Eisenstein cohomology classes attached to classes in $A(\pi, w)$ with $l(w) = 2$ and $\pi_\infty = D_2$ which satisfy one of the following conditions.*

(1) *The central character χ_π of π is non-trivial.*

(2) *If χ_π is trivial, the partial Langlands L-function $L_S(\pi, r, s)$ attached to π* (*compare* [19]) *for S large enough vanishes at $s = 1/2$.*

PROOF. One proves the assertion regarding the first summand as the analogous statement in degree 4 in Theorem 6.10. With regard to the second summand we have to study the analytic behaviour of the Eisenstein series $E_Q(\varphi, \Lambda)$ attached to a class of type (π, w) with $l(w) = 2$ and $\pi_\infty = D_2$ at the point $\Lambda_w = (1/3)\rho_Q$. An analysis of the global intertwining operator occurring in the constant Fourier coefficient of this Eisenstein series shows that, if one of the conditions (1) or (2) is satisfied, the Eisenstein series is holomorphic at this point. This argument runs parallel to the one in [34, Section 3].

REMARK 6.12. (1) Suppose that π is an automorphic representation of $M/\boldsymbol{Q}$ with $\pi_\infty = D_2$ and trivial central character χ_π. Via the Jacquet-Langlands correspondence [17], there exists a cuspidal automorphic representation Π of $Gl_2(\boldsymbol{Q})$ so that locally $\pi_v = \Pi_v$ for all finite places outside the ramifications sets of D and π, and $\Pi_\infty = \pi_\infty = D_2$. The representation Π is uniquely determined up to equivalence. Then for S large enough, that is, $S \supset V_\infty \cup \mathrm{Ram}(D) \cup \mathrm{Ram}(\pi)$, the partial L-function attached to π coincides with $L_S(\Pi, \rho_2, s)$ where ρ_2 denotes the standard representation of $GL_2(\boldsymbol{C})$.

(2) Note that the conditions χ_π trivial and $L_S(\Pi, \rho_2, s)$ does not vanish at $s = 1/2$ are not sufficient to ensure that the Eisenstein series in question does have a pole at the corresponding value Λ_w. Thus, also classes of this type might contribute to $\mathrm{Eis}^3[D_2]$. This can happen if the pole of the ratio of partial L-functions occurring in the relevant global intertwining operator is compensated for by a zero of one of the local intertwining operators at the places $v \in S$. However, if there is a pole for the Eisenstein series, then the residue gives rise to a non-trivial class in $H^2(S(\boldsymbol{C}), \boldsymbol{C})$. We have the following

THEOREM 6.13. *These residual Eisenstein cohomology classes* $[\mathrm{Res}_{\Lambda=\Lambda_w} E(\pi, w)]$, *attached to a class of type* (π, w) *with* $l(w) = 2, \pi_\infty = D_2, \chi_\pi$ *trivial and* $L_S(\Pi, \rho_2, s)$ *does not vanish at* $s = 1/2$, *are square-integrable and make up a subspace* $H^2_{\mathrm{res}}(S(\boldsymbol{C}), \boldsymbol{C})$ *in* $H^2(S(\boldsymbol{C}), \boldsymbol{C})$ *that is complementary to the interior cohomology* $H^2_!(S(\boldsymbol{C}), \boldsymbol{C})$.

7. Residual Eisenstein cohomology classes in the degree 2. In this section we investigate more closely the Hodge structure on $H^2_{\mathrm{res}}(\Gamma\backslash X, \boldsymbol{C})$, that is, on the space of residual Eisenstein cohomology classes of degree two. Recall that the cohomology $H^2(\Gamma\backslash X, \boldsymbol{C})$ is spanned by square integrable classes and decomposes

$$H^2(\Gamma\backslash X, \boldsymbol{C}) = H^2_!(\Gamma\backslash X, \boldsymbol{C}) \oplus H^2_{\mathrm{res}}(\Gamma\backslash X, \boldsymbol{C})$$

into the interior cohomology and the space of residual Eisenstein cohomology classes. First we prove the following

PROPOSITION 7.1. *Let* κ *be the descent of the* $G(\boldsymbol{R})$*-invariant Kähler form on* X *to its quotient* $\Gamma\backslash X$. *Then* κ *represents a cohomology class in the interior cohomology* $H^2_!(\Gamma\backslash X, \boldsymbol{Q})$, *that is, it is an element of the kernel of the restriction map* $r^2 : H^2(\Gamma\backslash X, \boldsymbol{C}) \to H^2(\partial(\Gamma\backslash X), \boldsymbol{C})$.

PROOF. The $G(\boldsymbol{R})$-invariant Kähler form κ_X on X is the Chern form of the $G(\boldsymbol{R})$-linearized line bundle L_X on X corresponding to the automophy factor which is given by

$$j(g, Z) = \det(CZ + D) \quad \left(g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z \in X\right).$$

Let L be the descent of L_X to $\Gamma\backslash X$. Then κ is the Chern form of L. Here we recall the following fact.

LEMMA 7.2. *The line bundle L is extendable to a line bundle $\tilde{L}$ on a smooth toroidal compactification $\tilde{V} = \Gamma\backslash\tilde{X}$ of $V = \Gamma\backslash X$ which is trivial on a neighbourhood of the boundary divisors D_p in $\Gamma\backslash\tilde{X}$ associated with each cusp p.*

PROOF OF LEMMA. We have a neighbourhood of U of D_p in $\Gamma\backslash\tilde{X}$ such that $U - D_p \cong \pi(U) - \{p\}$ is isomorphic to the quotient $(P_p \cap \Gamma)\backslash(X)_{p,d}$. Here $(X)_{p,d}$ is the neighbourhood of the cusp p with 'distance' in X. Since the automorphy factor j is trivial on the unipotent radical N_p of the parabolic subgroup P_p, the restriction of L to $U - D_p$, which is also the descent of L_X on $(X)_{p,d}$ with trivial N_p-linearization, is a trivial line bundle. Hence it is extendable to U, trivial on each D_p. This amounts to say that the first Chern class $\kappa = c_1(L) \in H^2(\Gamma\backslash X, \boldsymbol{C})$ is mapped to zero by the restrction map r in the exact sequence

$$\to H^2(\Gamma\backslash\tilde{X} \bmod D_p, \boldsymbol{C}) \to H^2(\Gamma\backslash\tilde{X}, \boldsymbol{C}) \xrightarrow{r} \bigoplus_p H^2(D_p, \boldsymbol{C}) \to \cdots .$$

Therefore L comes from

$$H^2(\Gamma\backslash\tilde{X} \bmod D_p, \boldsymbol{C}) \cong H^2(\Gamma\backslash X^* \bmod \{p\}, \boldsymbol{C}) \cong H^2_c(\Gamma\backslash X, \boldsymbol{C}) .$$

This proves our claim. □

PROPOSITION 7.3. *The subspace $H^2_{\mathrm{res}}(\Gamma\backslash X, \boldsymbol{C})$ in $H^2(\Gamma\backslash X, \boldsymbol{C})$, spanned by residual Eisenstein cohomology classes, is a rational sub-Hodge structure of $H^2(\Gamma\backslash X, \boldsymbol{Q}) \otimes \boldsymbol{C}$, consisting only of type $(1, 1)$ elements. Thus, by the Lefschetz $(1, 1)$-type criterion, $H^2_{\mathrm{res}}(\Gamma\backslash X, \boldsymbol{C})$ consists entirely of algebraic cycles.*

PROOF. In general the interior cohomology

$$H^2_!(\Gamma\backslash X, \boldsymbol{Q}) := \mathrm{Image}\{H^2_c(\Gamma\backslash X, \boldsymbol{Q}) \to H^2(\Gamma\backslash X, \boldsymbol{Q})\}$$

is a pure Hodge structure of weight 2 (cf. [8, Section 3]), which is a rational sub-Hodge structure of $H^2(\Gamma\backslash\tilde{X}, \boldsymbol{Q})$. For cohomology groups with complex coefficients, we have a natural decompostion

$$H^2(\Gamma\backslash X, \boldsymbol{C}) = H^2_!(\Gamma\backslash X, \boldsymbol{C}) \oplus H^2_{\mathrm{res}}(\Gamma\backslash X, \boldsymbol{C}) .$$

The point is to define a sub-Hodge structure H^2_{comp} in $H^2(\Gamma\backslash X, \boldsymbol{Q})$ which is complementary to $H^2_!(\Gamma\backslash X, \boldsymbol{Q})$.

Let $H^2_{\mathrm{prim}}(\Gamma\backslash\tilde{X}, \boldsymbol{Q})$ be the primitive part of the second cohomology group with some choice of polarization λ of $\Gamma\backslash\tilde{X}$. By the semisimplicity of the polarized Hodge structure,

$$H^2_{!,\,\mathrm{prim}}(\Gamma\backslash X, \boldsymbol{Q}) := H^2_!(\Gamma\backslash X, \boldsymbol{Q}) \cap H^2_{\mathrm{prim}}(\Gamma\backslash\tilde{X}, \boldsymbol{Q})$$

is also a polarized Hodge structure. As an ample class λ we may take a sum $a\tilde{\kappa} - b\sum_p D_p$ with a, b integers large enough where $\tilde{\kappa}$ denotes the Chern class of the extension $\tilde{L}$ of the line bundle L introduced in the lemma above. Then [29, Theorem 8.4] shows that this is an ample class.

Then κ is not contained in the orthogonal complement $H^2_{\mathrm{prim}}(\Gamma\backslash\tilde{X}, \boldsymbol{Q})$, a fortiori not in $H^2_{!,\,\mathrm{prim}}(\Gamma\backslash X, \boldsymbol{Q})$. Then $H^2_{!,\,\mathrm{prim}}(\Gamma\backslash\tilde{X}, \boldsymbol{Q}) \oplus \boldsymbol{Q}\kappa$ is canonically isomorphic to $H^2_!(\Gamma\backslash X, \boldsymbol{Q})$

as a rational Hodge structure and we can equip a new intersection form with sign change on $\boldsymbol{Q}\kappa$. Then it is a polarized rational sub-Hodge structure of

$$H^2_{\mathrm{prim}}(\Gamma\backslash X, \boldsymbol{Q}) \oplus \boldsymbol{Q}\lambda \cong H^2(\Gamma\backslash\tilde{X}, \boldsymbol{Q})$$

similarly defined. Thus we can define the complement rational Hodge structure as the orthogonal complement of $H^2_!(\Gamma\backslash X, \boldsymbol{Q})$ in $H^2(\Gamma\backslash\tilde{X}, \boldsymbol{Q})$ with respect this new intersection form. The extension of coefficients from $\boldsymbol{Q}$ to $\boldsymbol{C}$ of this space is canonically isomorphic to $H^2_{\mathrm{res}}(\Gamma\backslash X, \boldsymbol{C})$, as desired.

Now we have to show the second part of our proposition. Recall that any non-trivial residual Eisenstein cohomology class ω in H^2_{res} is cohomologically generated by construction out of the contribution of the $(\mathfrak{g}, K)$-module $J^{(1,1)}$ which is given as the Langlands quotient in the following sequence of $(\mathfrak{g}, K)$-modules

$$0 \to D^{(2,1)} \oplus D^{(1,2)} \to \mathrm{Ind}^G_{Q_2}(D_2 \otimes 1_N) \to J^{(1,1)} \to 0\,.$$

The $(\mathfrak{g}, K)$-module $D^{(2,1)} \oplus D^{(1,2)}$ denotes the sum of the two discrete series representations of $G(\boldsymbol{R})$ that are non-holomorphic. The first one has only cohomology in bidegree $(2, 1)$, the latter one in bidegree $(1, 2)$. Since this cohomological representation $J^{(1,1)}$ has only non-vanishing relative Lie algebra cohomology $H^{(i,j)}(\mathfrak{g}, K; J^{(1,1)})$ in bidgree (i, i) with $i = 1,\ 2$, the Hodge type of ω in $H^2_{\mathrm{res}}(\Gamma\backslash X, \boldsymbol{C})$ should be of $(1, 1)$-type in the theory of harmonic integrals.

But for the case of isolated singularities, the combination of Saper [29] and Zucker [38] confirms that the Hodge type via transcendental theory and the one via algebraic theory coincide. Hence $H^2_{\mathrm{res}}(\Gamma\backslash X, \boldsymbol{Q})$ is of $(1, 1)$-type as the sub-Hodge structure of $H^2(\Gamma\backslash\tilde{X}, \boldsymbol{Q})$. Thus we can apply the $(1, 1)$-type criterion of Lefschetz to conclude that this space is generated by algebraic cycles.

7.4. The dimension of the second residual cohomology group. As shown in 4.6, the interior cohomology in degree 3 coincides with the space $W_3H^3(\Gamma\backslash X, \boldsymbol{C})$. It follows from the very construction of the residual Eisenstein cohomology classes in $H^2_{\mathrm{res}}(\Gamma\backslash X, \boldsymbol{C})$ that the dimension of this space can be extracted from the sum

$$\dim H^2_{\mathrm{res}}(\Gamma\backslash X, \boldsymbol{C}) + \dim_{\boldsymbol{C}} W_4H^3(\Gamma\backslash X, \boldsymbol{C})/W_3H^3(\Gamma\backslash X, \boldsymbol{C})\,,$$

which is equal the number of equivalence classes of cusps with respect to Γ. Recall that the second summand

$$W_4H^3(\Gamma\backslash X, \boldsymbol{C})/W_3H^3(\Gamma\backslash X, \boldsymbol{C})\,,$$

is defined (see 4.5) as the first cohomology group of a complex of Gysin homomorphisms

$$H^i(D^{[k]}, \boldsymbol{C}) \to H^{i+2}(D^{[k-1]}, \boldsymbol{C}) \to H^{i+4}(D^{[k-2]}, \boldsymbol{C})$$

for $i = 0$ and $k = 2$ within the context of the Leray spectral sequence associated to the open immersion $j : V \to V^*$. More precisely, it is the term $E_3^{2,1}$.

This provides a geometric interpretation of the dimension of $H^2_{\mathrm{res}}(\Gamma\backslash X, \boldsymbol{C})$. The latter entity is related to the vanishing or non-vanishing of certain special values of automorphic L-functions on Shimura curves as explained in 6.11–6.13.

7.5. A conjecture on the second cuspidal cohomology group. In this section we formulate a conjecture regarding the Hodge type of the cuspidal part $H^2_!(\Gamma\backslash X, \boldsymbol{Q})$, namely, that it consists of algebraic cycles. In doing so, we have to discuss some technical prerequisites.

First we recall the construction of an infinite number of modular surfaces in $\Gamma\backslash X$, to be called Humbert surfaces because these surfaces originate in the classical work of G. Humbert. For this purpose, we have to realize the adjoint group of our group $G/\boldsymbol{Q}$ as an orthogonal group of 5 variables over $\boldsymbol{Q}$.

We consider the $\boldsymbol{Q}$-vector space of quaternionic Hermitian forms of two variables given as

$$\left\{ M = \begin{pmatrix} a & \beta \\ {}^{\iota}\beta & c \end{pmatrix} \;\middle|\; a \in \boldsymbol{Q},\ \beta \in D,\ c \in \boldsymbol{Q} \right\}.$$

Then the given group G acts on this space by the adjoint action defined by $g \circ M := g \cdot M \cdot {}^{\iota}g$.

Since $g \in G$ stabilizes the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, it also stabilizes the 5 dimensional $\boldsymbol{Q}$-vector space

$$\begin{aligned} T &= \left\{ M = \begin{pmatrix} a & \beta \\ {}^{\iota}\beta & c \end{pmatrix} \;\middle|\; \mathrm{tr}\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & \beta \\ {}^{\iota}\beta & c \end{pmatrix} \right\} = 0 \right\} \\ &= \left\{ M = \begin{pmatrix} a & \beta \\ {}^{\iota}\beta & c \end{pmatrix} \;\middle|\; \beta \in D \text{ pure quaternion} \right\}. \end{aligned}$$

The group G stabilizes the bilinear form $T \times T \to \boldsymbol{Q}$ defined by the assignment

$$(M_1, M_2) \mapsto \mathrm{tr}(M_1 \cdot {}^{t\iota}M_2)\,, \quad M_1, M_2 \in T\,.$$

This is a quadratic form of signature $(2+, 3-)$.

Choose any $M \in T$ such that $\mathrm{tr}(M \cdot {}^{t\iota}M) < 0$. Then the stabilizer $S(M)$ of M in $G_{\boldsymbol{Q},ad}$ is an algebraic $\boldsymbol{Q}$-group whose group of real points is an orthogonal group of signature $(2+, 2-)$ over the real number field. Take a maximal compact subgroup $K \subset G(\boldsymbol{R})$ so that $K \cap S(M)(\boldsymbol{R})$ is also a maximal compact subgroup in $S(M)(\boldsymbol{R})$. Then we have an holomorphic embedding

$$S_M := \Gamma \cap S(M)\backslash S(M)(\boldsymbol{R})/K \cap S(M)(\boldsymbol{R}) \stackrel{j_M}{\to} \Gamma\backslash X\,.$$

Similarly, as in our former paper [24], the union of subdomains

$$\bigcup_M S(M)(\boldsymbol{R})/S(M)(\boldsymbol{R}) \cap K$$

defines a dense subset in X. We have the following stability argument.

LEMMA 7.6. *Let $\Gamma_!(\Gamma\backslash X, \Omega^2)$ be the $(2,0)$-part of the Hodge structure $H^2_!(\Gamma\backslash X, \boldsymbol{Q}) \otimes_{\boldsymbol{Q}} \boldsymbol{C}$. Then there exists a finite number of surfaces S_{M_i}, $1 \le i \le m$, such that the sum of* *pull-back maps* $j^*_{M_i}$:

$$\bigoplus_{i=1}^{m} j^*_{M_i} : \ \Gamma_!(\Gamma\backslash X, \Omega^2) \to \bigoplus_{i=1}^{m} \Gamma(S_{M_i}, \ \Omega^2_{M_i})$$

is injective.

PROOF. Firstly we see that $\bigcup_M S_M$ with $M \in T$ satisfying $\mathrm{tr}(M \cdot {}^{t\iota}M) < 0$ is dense in a very strong sense in $\Gamma\backslash X$. Suppose that M_i, $i \in I := \{1, 2, 3\}$, are three linearly independent elements in T. Then the intersection $S_{M_1} \cap S_{M_2} \cap S_{M_3}$ is a non-empty finite set consisting of special points, i.e., CM points in the theory of complex multiplication. The union of these intersections is a dense subset in $\Gamma\backslash X$ in the classical topology. Let p be a point in $S_{M_1} \cap S_{M_2} \cap S_{M_3}$. Then each intersection $S_{M_i} \cap S_{M_j}$, $i, j \in I, i \neq j$ defines a curve along p. Let z_k, $k \in I, k \neq i, j$, be the local coordinate at p tangential to this curve at p. Then any holomorphic 2-form ω is locally written in the form

$$\omega = f_3(z)dz_1 \wedge dz_2 + f_2(z)dz_1 \wedge dz_3 + f_1(z)dz_2 \wedge dz_3 \,.$$

If the restriction of ω to any S_{M_i} is zero then the $f_i(z)$, $i \in I$, are all zero locally around p. Therefore a holomorpic 2-form ω in the kernel of the pull-back map

$$\bigoplus_{\text{all } M} j^*_{M} : \ \Gamma_!(\Gamma\backslash X, \Omega^2) \to \prod_{\text{all } M} \Gamma(S_{M}, \ \Omega^2_{M_i})$$

has to vanish identically. Moreover the space on the left hand side of the arrow is of finite dimension. Hence it is enough to take into account only a finite number of M's to assure the injectivity of the map.

An immediate consequence of the lemma is the following

PROPOSITION 7.7. *The kernel of the sum of the pull-back homomorphism of pure Hodge structures of weight* 2

$$\bigoplus_{i=1}^{m} j^*_{M_i} : \ H^2_!(\Gamma\backslash X, \boldsymbol{Q}) \to \bigoplus_{i=1}^{m} W_2 H^2(S_{M_i}, \boldsymbol{Q})$$

is generated by algebraic cycles.

PROOF. The kernel is a rational sub-Hodge structure of $H^2_!(\Gamma\backslash X, \boldsymbol{Q})$ without $(2, 0)$-type component by the lemma above. By Hodge symmetry the $(0, 2)$-type component is zero as well. Hence it is a rational Hodge structure purely of $(1, 1)$-type. By applying the Lefschetz $(1, 1)$-criterion, we obtain the assertion.

Let H^2_{ker} denote the kernel of the homomorphism

$$\bigoplus_{i=1}^{m} j^*_{M_i} : \ H^2_!(\Gamma\backslash X, \boldsymbol{Q}) \to \bigoplus_{i=1}^{m} W_2 H^2(S_{M_i}, \boldsymbol{Q}) \,.$$

Since the restriction of the Kähler form κ on $\Gamma\backslash X$ to each surface S_M defines a Kähler form on S_M the space H^2_{ker} does not contain κ. With regard to the structure of the quotient $H^2_!(\Gamma\backslash X, \boldsymbol{Q})/H^2_{\mathrm{ker}}$ there is evidence for the following conjecture.

CONJECTURE 7.8. *The rational Hodge structure* $H^2_!(\Gamma\backslash X, \boldsymbol{Q})$ *is purely of* $(1, 1)$-*type*, *i.e.*, *it consists of algebraic cycles.*

We give some heuristic justification. It is enough to show that the image or coimage of the morphism

$$\bigoplus_{i=1}^{m} j^*_{M_i} : H^2_{!,\ \mathrm{prim}}(\Gamma\backslash X, \boldsymbol{Q}) \to \bigoplus_{M} \{W_2 H^2(S_M, \boldsymbol{Q})\}_{\mathrm{prim}}$$

is purely of $(1, 1)$-type. We can delete any finite number of M's by the density argument used above. The primitive part $\{W_2 H^2(S_M, \boldsymbol{Q})\}_{\mathrm{prim}}$ of the pure Hodge structure of the Humbert surfaces S_M corresponds to a direct sum associated with cusp forms and one component $\boldsymbol{Q}(-1)$ (see [25, Theorem 1.12]).

Here we may pass from Hodge structures to the corresponding Galois representations on the l-adic cohomology groups of the Shimura varieties corresponding to $\Gamma\backslash X$ and the Humbert surfaces S_M. Then we can consider the analogous Galois version of the morphism in question with the set of M's replaced by a smaller one whose discriminant (i.e., the set of bad primes) is supported by a finite set S_1. Then the image space of the Galois representations is ramified only over $S_1 \cup \mathrm{Supp}(n)$, where n is a natural number such that $\Gamma(n) \subset \Gamma$. We can replace S_1 by another set S_2 which is disjoint to S_1. Then the image is ramified only over $\mathrm{Supp}(n)$. Thus we might apply an argument of Abrashkin-Fontaine type to assure that the image is a multiple of the cyclotomic character χ^{-1} of weight 2.

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GRADUATE SCHOOL OF
MATHEMATICAL SCIENCES
UNIVERSITY OF TOKYO
3–8–1 KOMABA, TOKYO
JAPAN

E-mail address: Takayuki@ms.u-tokyo.ac.jp

FACULTY OF MATHEMATICS
UNIVERSITY OF VIENNA
NORDBERGSTRASSE 15
A-1090 VIENNA
AUSTRIA

E-mail address: joachim.schwermer@univie.ac.at