On mixed super quasi-Einstein manifolds

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Abstract. In this paper, mixed super quasi-Einstein manifolds ($MS(QE)_n$) have been defined. The existence theorem and an example have been provided and the relations between the associated scalars have been established. As well, manifolds of mixed super quasi-constant curvature are defined, and it is shown that quasi conformally flat, conformally flat, conharmonically flat and projectively flat $MS(QE_n)$ are manifolds of mixed super quasi-constant curvature. Further, super quasi-umbilical hypersurfaces of Riemannian manifolds have been defined and it is proved that a super quasi-umbilical hypersurface of a Euclidean space is $MS(QE)_n$.

M.S.C. 2000: 53C25.
Key words: mixed super quasi-Einstein manifold, mixed super quasi-constant curvature, super quasi-umbilical hypersurfaces of a Riemannian manifold.

1 Introduction

The notion of quasi-Einstein manifold was introduced in [4] by M. C. Chaki and R. K. Maity. A non-flat Riemannian manifold (M^n, g), $(n \geq 3)$ is a *quasi-Einstein manifold* if its Ricci tensor S satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) \tag{1.1}$$

and is not identically zero, where a, b are scalars, $b \neq 0$ and A is a non-zero 1-form such that

$$g(X,U) = A(X), \ \ \forall X \ \in TM, \tag{1.2}$$

U being a unit vector field.

Here a and b are called the associated scalars, A is called the associated 1-form and U is called the generator of the manifold. Such an n-dimensional manifold will further be denoted by $(QE)_n$. As well, in [8], U.C. De and G.C. Ghosh defined the generalized quasi-Einstein manifolds. A non-flat Riemannian manifold is called *generalized quasi-Einstein manifold* if its Ricci-tensor is non-zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y), \tag{1.3}$$

Differential Geometry - Dynamical Systems, Vol.10, 2008, pp. 44-57.

where a, b and c are non-zero scalars and A, B are two 1-forms such that

$$g(X,U) = A(X) \text{ and } g(X,V) = B(X), \tag{1.4}$$

U and V being unit vectors which are orthogonal, i.e.,

$$g(U,V) = 0. \tag{1.5}$$

The vector fields U and V are called the generators of the manifold. This type of manifold will be denoted by $G(QE)_n$.

In [5], Chaki introduced the super quasi-Einstein manifold, denoted by $S(QE)_n$, where the Ricci tensor is not identically zero and satisfies the condition

$$\begin{aligned} S(X,Y) = \ & ag(X,Y) + bA(X)A(Y) + \\ & + c[A(X)B(Y) + A(Y)B(X)] + dD(X,Y), \end{aligned} \tag{1.6}$$

where a, b, c and d are scalars such that b, c, d are nonzero, A, B are two nonzero 1-forms defined as (1.4) and U, V are mutually orthogonal unit vector fields, D is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition

$$D(X,U) = 0, \ \ \forall X. \tag{1.7}$$

Here a, b, c, d are called the associated scalars, A, B are called the associated main and auxiliary 1-forms respectively, U, V are called the main and the auxiliary generators and D is called the associated tensor of the manifold.

Recently in [1], A. Bhattacharyya and T. De introduced the notion of mixed generalized quasi-Einstein manifold. A non-flat Riemannian manifold is called *mixed generalized quasi-Einstein manifold* if its Ricci tensor is non-zero and satisfies the condition

$$\begin{aligned} S(X,Y) = \ & ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) + \\ & + d[A(X)B(Y) + B(X)A(Y)], \end{aligned} \tag{1.8}$$

where a, b, c, d are non-zero scalars,

$$g(X,U) = A(X) \text{ and } g(X,V) = B(X), \tag{1.9}$$

$$g(U,V) = 0, \tag{1.10}$$

A, B are two non-zero 1-forms, U and V are unit vector fields corresponding to the 1-forms A and B respectively. If $d = 0$, then the manifold reduces to a $G(QE)_n$. This type of manifold is denoted by $MG(QE)_n$.

A Riemannian manifold is said to be a *manifold of generalized quasi-constant curvature* ([8]) if the curvature tensor $\mathcal{R}$ of type $(0,4)$ satisfies the condition

$$\begin{aligned} \mathcal{R}(X,Y,Z,W) = \ & p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & + q[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) \\ & + g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)] \\ & + s[g(X,W)D(Y)D(Z) - g(X,Z)D(Y)D(W) \\ & + g(Y,Z)D(X)D(W) - g(Y,W)D(X)D(Z)], \end{aligned} \tag{1.11}$$

where p, q, s are scalars, and T, D are non-zero 1-forms; ρ and $\bar{\rho}$ are unit orthogonal vector fields, such that

$$g(X,\rho) = T(X) \text{ and } g(X,\bar{\rho}) = D(X) \tag{1.12}$$

and

$$g(\rho,\bar{\rho}) = 0. \tag{1.13}$$

A Riemannian manifold is said to be a manifold of *mixed generalized quasi-constant curvature* ([1]) if the curvature tensor $\acute{R}$ of type $(0,4)$ satisfies the condition

(1.14)
$$\begin{aligned}\acute{R}(X,Y,Z,W) = \; & p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + q[g(X,W)A(Y)A(Z) \\ & -g(Y,W)A(X)A(Z) + g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)] \\ & +s[g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z) + g(Y,Z)B(X)B(W) \\ & -g(X,Z)B(Y)B(W) + t[\{A(Y)B(Z) + B(Y)A(Z)\}g(X,W) \\ & -\{A(X)B(Z) + B(X)A(Z)\}g(Y,W) + \{A(X)B(W) \\ & +B(X)A(W)\}g(Y,Z) - \{A(Y)B(W) + B(Y)A(W)\}g(X,Z)],\end{aligned}$$

where p, q, s, t are scalars, A, B are non-zero 1-forms, ρ and $\bar{\rho}$ are orthonormal unit vector fields corresponding to A and B, which are defined as (1.12) and (1.13) and

$$g(R(X,Y)Z,W) = \acute{R}(X,Y,Z,W). \tag{1.15}$$

This paper deals with mixed super quasi-Einstein manifold and mixed super quasi-constant curvature which are defined as follows

Definition 1. A non-flat Riemannian manifold (M^n, g), (n≥3) is called *mixed super quasi-Einstein manifold* if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$\begin{aligned}S(X,Y) = \; & ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \\ & +d[A(X)B(Y) + B(X)A(Y)] + eD(X,Y),\end{aligned} \tag{1.16}$$

where a, b, c, d, e are scalars of which $b \neq 0$, $c \neq 0$, $d \neq 0$, $e \neq 0$ and A, B are two non zero 1-forms such that

$$g(X,U) = A(X) \text{ and } g(X,V) = B(X), \;\; \forall \;\; X \in TM \tag{1.17}$$

U, V being mutually orthogonal unit vector fields, D is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition

$$D(X,U) = 0, \;\; \forall \;\; X \in TM. \tag{1.18}$$

Here a, b, c, d, e are called the associated scalars, A, B are called the associated main and auxiliary 1-forms, U, V are called the main and the auxiliary generators and

D is called the associated tensor of the manifold. We denote this type of manifold $MS(QE)_n$.

Definition 2. A Riemannian manifold is said to be a manifold of *mixed super quasi-constant curvature* if the curvature tensor $\acute{R}$ of type $(0,4)$ satisfies the condition (1.19)

$$\begin{aligned}\acute{R}(X,Y,Z,W) = \;& m[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + p[g(X,W)A(Y)A(Z)\\ & -g(Y,W)A(X)A(Z) + g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)]\\ & +q[g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z) + g(Y,Z)B(X)B(W)\\ & -g(X,Z)B(Y)B(W) + s[\{A(Y)B(Z) + B(Y)A(Z)\}g(X,W)\\ & -\{A(X)B(Z) + B(X)A(Z)\}g(Y,W) + \{A(X)B(W)\\ & +B(X)A(W)\}g(Y,Z) - \{A(Y)B(W) + B(Y)A(W)\}g(X,Z)]\\ & +t[g(Y,Z)D(X,W) - g(X,Z)D(Y,W)\\ & +g(X,W)D(Y,Z) - g(Y,W)D(X,Z)]\end{aligned}$$

where m, p, q, s, t are scalars. A, B are non-zero 1-forms defined in (1.17) and U, V are mutually orthogonal unit vector fields, D is a symmetric $(0,2)$ tensor defined in (1.18).

In this paper, the existence theorem of $MS(QE)_n$ has been proved with an example and it is shown that the scalars $a+b$ and $a+c+eD(V,V)$ are the Ricci curvatures in the directions of the vector fields U and V respectively.

We have also studied quasi conformally flat, conformally flat, conharmonically flat and projectively flat $MS(QE)_n$. Lastly we have shown that a super quasi umbilical hypersurface of a Euclidean space is $MS(QE)_n$.

2 Preliminaries

In an n-dimensional $(n > 2)$ Riemannian manifold the covariant quasi-conformal curvature tensor is defined as [3]

$$\begin{aligned}\tilde{\acute{C}}(X,Y,Z,W) = \;& \acute{a}\acute{R}(X,Y,Z,W) + \acute{b}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\\ & +g(Y,Z)g(QX,W) - g(X,W)g(QY,W)]\\ & -\frac{r}{n}[\frac{\acute{a}}{n-1} + 2\acute{b}][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\end{aligned} \tag{2.1}$$

where

$$g(\tilde{C}(X,Y)Z,W) = \tilde{\acute{C}}(X,Y,Z,W). \tag{2.2}$$

If $\acute{a} = 1$ and $\acute{b} = -\frac{1}{n-2}$ then (2.1) reduces to conformal curvature tensor, where

$$\begin{aligned}\acute{C}(X,Y,Z,W) = \;& \acute{R}(X,Y,Z,W) - \frac{1}{n-2}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\\ & +g(Y,Z)S(X,W) - g(X,W)S(Y,W)]\\ & +\frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].\end{aligned} \tag{2.3}$$

The conharmonic curvature tensor is denoted by $'H(X,Y,Z,W)$ and in a $V_n(n>2)$ it is defined as

$$\begin{aligned} 'H(X,Y,Z,W) = & \; 'R(X,Y,Z,W) - \tfrac{1}{n-2}[g(X,W)S(Y,Z) - g(X,Z)S(Y,W) \\ & +g(Y,Z)S(X,W) - g(Y,W)S(X,Z)] \end{aligned} \tag{2.4}$$

and L be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci-tensor S, where

$$g(LX,Y) = S(X,Y) \;\; \forall\; X,Y \in TM. \tag{2.5}$$

The projective curvature tensor is denoted by $'\tilde{P}(X,Y,Z,W)$ and in a $V_n(n>2)$ it is defined as

$$\begin{aligned} '\tilde{P}(X,Y,Z,W) = & \; 'R(X,Y,Z,W) \\ & -\tfrac{1}{n-1}[S(Y,Z)g(X,W) - S(Y,W)g(X,W)]. \end{aligned} \tag{2.6}$$

3 Existence theorem of a mixed super quasi-Einstein manifold

In this section we state and prove the existence theorem of a mixed super quasi-Einstein manifold.

Theorem 1. *If the Ricci-tensor S of a Riemannian manifold satisfies the relation*

$$\begin{aligned} S(X,W)S(Y,Z) - S(Y,W)S(X,Z) = & \; \mu[S(Y,W)g(Z,X) + S(Z,X)g(Y,W)] \\ & +\beta[g(X,W)g(Y,Z) - g(Y,W)g(Z,X)] \\ & +\gamma[g(Y,Z)D(X,W) - g(X,Z)D(Y,W) \\ & +g(X,W)D(Y,Z) - g(Y,W)D(X,Z)] \end{aligned} \tag{3.1}$$

where μ, β, γ are non-zero scalars and D is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition $D(X,U) = 0$, $\forall$ X then the manifold is mixed super quasi-Einstein manifold.

Proof. Let U be a vector field defined by

$$g(X,U) = T(X) \;\; \forall X \in TM. \tag{3.2}$$

Putting $X = W = U$ in (3.1), we obtain

$$\begin{aligned} S(U,U)S(Y,Z) - S(Y,U)S(U,Z) = & \; \mu[S(Y,U)g(Z,U) + S(Z,U)g(Y,U)] \\ & +\beta[g(U,U)g(Y,Z) - g(Y,U)g(Z,U)] \\ & +\gamma[g(Y,Z)D(U,U) - g(U,Z)D(Y,U) \\ & +g(U,U)D(Y,Z) - g(Y,U)D(U,Z)]. \end{aligned} \tag{3.3}$$

Now using (2.5) and (3.2) in the above equation , we get

$$\begin{aligned}\bar{\alpha}S(Y,Z) - T(LY)T(LZ) = \quad & \mu[T(LY)T(Z) + T(LZ)T(Y)] \\ & +\rho[|U|^2 g(Y,Z) - T(Y)T(Z)] \\ & +\lambda[|U|^2 D(Y,Z)]\end{aligned} \tag{3.4}$$

where $S(U,U) = \bar{\alpha}$ and $T(LY) = g(LY,U) = S(Y,U)$.

Therefore

$$\begin{aligned}S(Y,Z) = \quad & \alpha T(LY)T(LZ) + \mu\alpha[T(LY)T(Z) + T(LZ)T(Y)] \\ & +\rho\alpha[|U|^2 g(Y,Z) - T(Y)T(Z)] + \lambda\alpha[|U|^2 D(Y,Z)].\end{aligned} \tag{3.5}$$

Taking $\alpha = \frac{1}{\bar{\alpha}}$ and $T(LY) = P(Y)$ in (3.30), we get

$$\begin{aligned}S(Y,Z) = \quad & \rho\alpha[|U|^2 g(Y,Z)] + (-\rho\alpha)T(Y)T(Z)] + \alpha P(Y)P(Z) \\ & +\mu\alpha[P(Y)T(Z) + P(Z)T(Y)] + \lambda\alpha[|U|^2 D(Y,Z)].\end{aligned} \tag{3.6}$$

Therefore in view of (3.6) the manifold is mixed super quasi-Einstein manifold. □

4 Example of a mixed super quasi-Einstein manifold

A manifold of mixed super quasi-constant curvature defined by (1.19) is a mixed super quasi-Einstein manifold. Putting $X = W = e_i$ in (1.19), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i, 1≤i≤ n, we get

$$\begin{aligned}S(Y,Z) = \quad & \{(n-1)m + p + q + 2s + tr_1\}g(Y,Z) \\ & +p(n-2)A(Y)A(Z) + q(n-2)B(Y)B(Z) \\ & +s(n-2)\{A(Y)B(Z) + A(Z)B(Y)\} + t(n-2)D(Y,Z)\end{aligned} \tag{4.1}$$

where $r_1 = D(e_i, e_i)$.

¿From (4.1) we see that the manifold is $MS(QE)_n$.

5 The associated scalars of a $MS(QE)_n$ $(n \geq 3)$

In this section we consider $MS(QE)_n$ $(n \geq 3)$ with associated scalars a, b, c, d, e, associated main and auxiliary 1-forms A, B, main and auxiliary generators U, V and associated symmetric $(0,2)$ tensor D.

So (1.16), (1.17) and (1.18) will hold. Since U and V are mutually orthogonal unit vector fields, we have

$g(U,U) = 1$, $g(V,V) = 1$ and $g(U,V) = 0$.

Further,

$$traceD = 0 \tag{5.1}$$

$$D(X, U) = 0 \quad \forall \quad X \in TM. \tag{5.2}$$

In virtue of (1.17), $g(U, V) = 0$ can be expressed as

$$A(V) = B(U) = 0. \tag{5.3}$$

Now contracting X and Y in (1.16), we get

$$r = na + b + c \tag{5.4}$$

where r is the scalar curvature.

Again from (1.17) we have

$$S(U, U) = a + b \tag{5.5}$$

$$S(V, V) = a + c + eD(V, V) \tag{5.6}$$

and

$$S(U, V) = d. \tag{5.7}$$

If X is a unit vector field, then $S(X, X)$ is the Ricci-curvature in the direction of X. Hence from (5.5) and (5.6) we can state that $a + b$ and $a + c + eD(V, V)$ are the Ricci curvature in the directions of U and V respectively. ¿From (2.5) we have

$g(LX, Y) = S(X, Y)$.

We also consider

$$g(\ell X, Y) = D(X, Y). \tag{5.8}$$

Further, let d_1^2 and d_2^2 denote the squares of the lengths of the Ricci-tensor S and the associated tensor D. Then

$$d_1^2 = S(Le_i, e_i) \tag{5.9}$$

and

$$d_2^2 = D(\ell e_i, e_i) \tag{5.10}$$

where $\{e_i\}$, $i = 1, 2, ..., n$ is an orthonormal basis of the tangent space at a point of $MS(QE)_n$.Now from (1.16) we get

$$\begin{aligned} S(Le_i, e_i) = \; & (n-1)a^2 + a^2 + b^2 + c^2 \\ & +2ab + 2bc + ceD(V, V) + eS(\ell e_i, e_i). \end{aligned} \tag{5.11}$$

Now from (1.16), we have

$$S(\ell e_i, e_i) = eD(\ell e_i, e_i). \tag{5.12}$$

From (5.11) and (5.12) it follows that

$$\begin{aligned} S(Le_i, e_i) = \;& (n-1)a^2 + a^2 + b^2 + c^2 \\ & +2ab + 2bc + ceD(V,V) + e^2 D(\ell e_i, e_i). \end{aligned} \tag{5.13}$$

Hence

$$d_1^2 = (n-1)a^2 + (a+b+c)^2 - 2ca + ceD(V,V) + e^2(d_2)^2. \tag{5.14}$$

We consider $eD(V,V) = 2a$, i.e., $D(V,V) = \frac{2a}{e} = k$. From (5.14) we can write $d_1^2 - e^2(d_2)^2 = (n-1)a^2 + (a+b+c)^2 > 0$ and hence

$$e < d_1/d_2, \tag{5.15}$$

which conclude s the proof. $\square$.

From (5.15) we can state the following theorem

Theorem 2. *In $MS(QE)_n$ $(n \geq 3)$ the scalars $a+b$ and $a+c+eD(V,V)$ are the Ricci-curvature in the direction of the generators U and V respectively and the associated scalar e is less than the ratio which the length of Ricci-tensor S bears the length of the associated tensor D.*

6 Quasi conformally flat $MS(QE)_n$ $(n > 3)$

Let R be the curvature tensor of type $(1,3)$ of a quasi conformally flat MS$(QE)_n$ $(n \geq 3)$. Then

$$\begin{aligned} \acute{R}(X,Y,Z,W) = \;& \frac{r}{n\acute{a}}[\frac{\acute{a}}{n-1} + 2\acute{b}][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & -\frac{\acute{b}}{\acute{a}}[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ & +g(X,W)S(Y,Z) - g(Y,W)S(X,Z)] \end{aligned} \tag{6.1}$$

where $\acute{R}$ is defined in (1.15).

Using (1.16), in (6.1), we obtain

$$
\begin{aligned}
\acute{R}(X,Y,Z,W) = \ & \{\tfrac{r}{n\acute{a}}[\tfrac{\acute{a}}{n-1}+2\acute{b}] - \tfrac{2a\acute{b}}{\acute{a}}\}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)] \\
& - \tfrac{b\acute{b}}{\acute{a}}[g(Y,Z)A(X)A(W)-g(X,Z)A(Y)A(W) \\
& +g(X,W)A(Y)A(Z)-g(Y,W)A(X)A(Z)] \\
& -\tfrac{c\acute{b}}{\acute{a}}[g(Y,Z)B(X)B(W)-g(X,Z)B(Y)B(W) \\
& +g(X,W)B(Y)B(Z)-g(Y,W)B(X)B(Z)] \\
& -\tfrac{d\acute{b}}{\acute{a}}[g(Y,Z)\{A(X)B(W)+A(W)B(X)\} \\
& -g(X,Z)\{A(Y)B(W)+A(W)B(Y)\} \\
& +g(X,W)\{A(Y)B(Z)+A(Z)B(Y)\} \\
& -g(Y,W)\{A(X)B(Z)+A(Z)B(X)\}] \\
& -\tfrac{e\acute{b}}{\acute{a}}[g(Y,Z)D(X,W)-g(X,Z)D(Y,W) \\
& +g(X,W)D(Y,Z)-g(Y,W)D(X,Z)].
\end{aligned}
\tag{6.2}
$$

Now the equation (6.2) can be written as

$$
\begin{aligned}
\acute{R}(X,Y,Z,W) = \ & a_1[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)] \\
& +b_1[g(Y,Z)A(X)A(W)-g(X,Z)A(Y)A(W) \\
& +g(X,W)A(Y)A(Z)-g(Y,W)A(X)A(Z)] \\
& +c_1[g(Y,Z)B(X)B(W)-g(X,Z)B(Y)B(W) \\
& +g(X,W)B(Y)B(Z)-g(Y,W)B(X)B(Z)] \\
& +d_1[g(Y,Z)\{A(X)B(W)+A(W)B(X)\} \\
& -g(X,Z)\{A(Y)B(W)+A(W)B(Y)\} \\
& +g(X,W)\{A(Y)B(Z)+A(Z)B(Y)\} \\
& -g(Y,W)\{A(X)B(Z)+A(Z)B(X)\}] \\
& +e_1[g(Y,Z)D(X,W)-g(X,Z)D(Y,W) \\
& +g(X,W)D(Y,Z)-g(Y,W)D(X,Z)]
\end{aligned}
\tag{6.3}
$$

where

(6.4)

$$a_1 = \{\tfrac{r}{n\acute{a}}[\tfrac{\acute{a}}{n-1}+2\acute{b}] - \tfrac{2a\acute{b}}{\acute{a}}\}, \quad b_1 = -\tfrac{b\acute{b}}{\acute{a}}, \quad c_1 = -\tfrac{c\acute{b}}{\acute{a}}, \quad d_1 = -\tfrac{d\acute{b}}{\acute{a}} \text{ and } \ e_1 = -\tfrac{e\acute{b}}{\acute{a}}.$$

Let U^1 be the $(n-1)$ dimensional distribution, orthogonal to U, in a quasi conformally flat $MS(QE)_n$. Then $g(X,U)$=0 if $X \in U^1$. Hence from (6.3) we get the following properties of R

$$
\begin{aligned}
R(X,Y,Z) = \;& \lambda[g(Y,Z)X - g(X,Z)Y] \\
& + c_1[g(Y,Z)B(X) - g(X,Z)B(Y)]V \\
& + c_1[g(Z,V)g(Y,V)X - g(X,V)g(Z,V)Y] \\
& + d_1[g(Y,Z)B(X) - g(X,Z)B(Y)]U \\
& + e_1[D(Y,Z)X - D(X,Z)Y + g(Y,Z)\ell X - g(X,Z)\ell Y)]
\end{aligned} \tag{6.5}
$$

when $X, Y, Z \in U^1$ and

$$R(X,U,U) = \lambda X + c_1 g(X,V)V + e_1 \ell X \quad \text{when} \quad X \in U^1 \tag{6.6}$$

where $\lambda = \{\frac{na+b+c}{na'}[\frac{a'}{n-1} + 2b'] - \frac{2ab'}{a'}\}$ and c_1, d_1 and e_1 have values given by (6.4). □

We can therefore state

Theorem 3. *A quasi conformally flat mixed super quasi-Einstein manifold* $(n > 3)$ *is a manifold of mixed super quasi-constant curvature and the curvature tensor* R *of type* $(1,3)$ *satisfies the properties given by (6.5) and (6.6).*

Corollary 1. *A conformally flat mixed super quasi-Einstein manifold* $(n > 3)$ *is a manifold of mixed super quasi-constant curvature and the curvature tensor* R *of type* $(1,3)$ *satisfies the property* $R(X,U,U) = \lambda X + c_1 B(X)V + e_1 \ell X$ *when* $X \in U^1$, *where* $\lambda = \frac{(n-2)a-b-c}{(n-1)(n-2)}, c_1 = \frac{c}{n-2}$ *and* $e_1 = \frac{e}{n-2}$.

Corollary 2. *A conharmonically flat mixed super quasi-Einstein manifold* $(n > 3)$ *is a manifold of mixed super quasi-constant curvature and the curvature tensor* R *of type* $(1,3)$ *satisfies the property* $R(X,U,U) = \lambda X + c^1 B(X)V + e^1 \ell X$ *when* $X \in U^1$ *where* $\lambda = \frac{2a}{n-1}, c^1 = \frac{c}{n-2}$ *and* $e^1 = \frac{e}{n-2}$.

7 Projectively flat $MS(QE)_n$ $(n > 3)$

Let R be the curvature tensor of type $(1,3)$ of a projectively flat MS$(QE)_n$ (n≥3). Then

$$\acute{R}(X,Y,Z,W) = \frac{1}{n-1}[g(X,W)S(Y,Z) - g(Y,W)S(X,Z)] \tag{7.1}$$

where $\acute{R}$ is defined earlier.

From (1.16) and (7.1), we get

$$
\begin{aligned}
\acute{R}(X,Y,Z,W) = \;& \tfrac{a}{n-1}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\
& + \tfrac{b}{n-1}[g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)] \\
& + \tfrac{c}{n-1}[g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z)] \\
& + \tfrac{d}{n-1}[g(X,W)\{A(Y)B(Z) + A(Z)B(Y)\} \\
& - g(Y,W)\{A(X)B(Z) + A(Z)B(X)\}] \\
& + \tfrac{e}{n-1}[g(X,W)D(Y,Z) - g(Y,W)D(X,Z)].
\end{aligned} \tag{7.2}
$$

Let U^1 be the $(n-1)$ dimensional distribution, orthogonal to U, in a projectively flat $MS(QE)_n$. Then $g(X,U)$=0 if $X \in U^1$. Hence from (7.2) we get the following properties of R

$$\begin{aligned} R(X,Y,Z) = \; & a_2\,[g(Y,Z)X - g(X,Z)Y] \\ & + b_2\,[A(Y)A(Z)X - A(X)A(Z)Y] \\ & + c_2\,[B(Y)B(Z)X - B(X)B(Z)Y] \\ & + d_2\,[\{A(Y)B(Z) + A(Z)B(Y)\}X \\ & - \{A(X)B(Z) + A(Z)B(X)\}Y] \\ & + e_2\,[D(Y,Z)X - D(X,Z)Y] \end{aligned} \tag{7.3}$$

when $X, Y, Z \in U^1$, $a_2 = \frac{a}{n-1}, b_2 = \frac{b}{n-1}, c_2 = \frac{c}{n-1}, d_2 = \frac{d}{n-1}$ and $e_2 = \frac{e}{n-1}$. Also,

$$R(X,U,U) = (a_2 + b_2)X - b_2 g(X,U)U - d_2 g(X,V)U \tag{7.4}$$

for all $X \in U^1$. □

Thus we can state

Theorem 4. *In projectively flat $MS(QE)_n$ $(n > 3)$ the curvature tensor R of type $(1,3)$ satisfies the properties given by (7.3) and (7.4).*

8 Hypersurfaces of the Euclidean space

Let M^n be a hypersurface of the Euclidean space E^{n+1}, with the metric tensor $\tilde{g}$ of M^n induced by E^{n+1}.
The Gauss equation of M^n in E^{n+1} can be written as

$$\tilde{g}(\tilde{R}(X,Y)Z,W) = \tilde{g}(H(X,W),H(Y,Z)) - \tilde{g}(H(Y,W),H(X,Z)) \tag{8.1}$$

where $\tilde{R}$ is the Riemannian curvature tensor corresponding to the induced metric $\tilde{g}$. H is the second fundamental tensor of M^n (orthonormal to M^n) and X, Y, Z, W are vector fields tangent to M^n.

If A_ξ is the $(1,1)$ tensor corresponding to the normal valued second fundamental tensor H, then we have [6]

$$\tilde{g}(A_\xi(X),Y) = g(H(X,Y),\xi) \tag{8.2}$$

where ξ is the unit normal vector field and X, Y are tangent vector fields.
Let H_ξ be the symmetric $(0,2)$ tensor associated with A_ξ in the hypersurface defined by

$$\tilde{g}(A_\xi(X),Y) = H_\xi(X,Y). \tag{8.3}$$

A hypersurface of a Riemannian manifold (M^n, g) is called *quasi-umbilical* [6] if its second fundamental tensor has the form

$$H_\xi(X,Y) = \alpha\tilde{g}(X,Y) + \beta\omega(X)\omega(Y) \tag{8.4}$$

where ω is 1-form. The vector field corresponding to the 1-form ω is a unit vector field, and α, β are scalars. If $\alpha = 0$ (resp. $\beta = 0$ or $\alpha = \beta = 0$) holds then M^n is called cylindrical (resp. *umbilical or geodesic*).
A hypersurface of a Riemannian manifold (M^n, g) is called *generalized quasi-umbilical* [6] if its second fundamental tensor has the form

$$H_\xi(X,Y) = \alpha\tilde{g}(X,Y) + \beta\omega(X)\omega(Y) + \gamma\delta(X)\delta(Y) \tag{8.5}$$

where α, β, γ are scalars. The vector fields corresponding to 1-forms ω and δ are unit vector fields. If $\alpha = \beta = \gamma = 0, M^n$ is called geodesic. If $\alpha = \gamma = 0$ or $\alpha = \beta = 0, M^n$ is called cylindrical. Also M^n is called umbilical when $\beta = \gamma = 0$.

In this section, we define super quasi-umbilical hypersurface of a Riemannian manifold.

Definition 3. A hypersurface of a Riemannian manifold (M^n, g) is called *super quasi-umbilical* if its second fundamental tensor has the form

$$H_\xi(X,Y) = \alpha\tilde{g}(X,Y) + \beta\omega(X)\omega(Y) + \gamma\delta(X)\delta(Y) + \rho D(X,Y) \tag{8.6}$$

where α, β, γ are scalars. The vector fields corresponding to 1-forms ω and δ are unit vector fields. If $\alpha = \beta = \gamma = \rho = 0, M^n$ is called geodesic. If $\alpha = \gamma = \rho = 0$ or $\alpha = \beta = \rho = 0, M^n$ is called cylindrical. Also M^n is called umbilical when $\beta = \gamma = \rho = 0$.

Now from (8.2), (8.3) and (8.5), we get

$$\begin{aligned} g(H(X,Y),\xi) = \; & \alpha g(X,Y)g(\xi,\xi) + \beta\omega(X)\omega(Y)g(\xi,\xi) \\ & + \gamma\delta(X)\delta(Y)g(\xi,\xi). \end{aligned} \tag{8.7}$$

Since ξ is the only unit normal vector, (8.7) reduces to

$$H(X,Y) = \alpha g(X,Y)\xi + \beta\omega(X)\omega(Y)\xi + \gamma\delta(X)\delta(Y)\xi. \tag{8.8}$$

Let us assume that the hypersurface is super quasi-umbilical. Then in view of (8.8) we get

$$H(X,Y) = \alpha g(X,Y)\xi + \beta\omega(X)\omega(Y)\xi + \gamma\delta(X)\delta(Y)\xi + \rho D(X,Y)\xi. \tag{8.9}$$

From (8.1), (8.2) and (8.5)it follows that

$$
\begin{aligned}
\tilde{g}(\tilde{R}(X,Y)Z,W) = \quad & \alpha^2\{g(X,W)g(Y,Z) - g(Y,W)g(X,Z)\} \\
& +\alpha\beta\{g(X,W)\omega(Y)\omega(Z) + g(Y,Z)\omega(X)\omega(W) \\
& -g(Y,W)\omega(X)\omega(Z) - g(X,Z)\omega(Y)\omega(W)\} \\
& +\alpha\gamma\{g(X,W)\delta(Y)\delta(Z) + g(Y,Z)\delta(X)\delta(W) \\
& -g(Y,W)\delta(X)\delta(Z) - g(X,Z)\delta(Y)\delta(W)\} \\
& +\beta\gamma\{\omega(X)\omega(W)\delta(Y)\delta(Z) + \omega(Y)\omega(Z)\delta(X)\delta(W) \\
& -\omega(Y)\omega(W)\delta(X)\delta(Z) - \omega(X)\omega(Z)\delta(Y)\delta(W)\} \\
& +\alpha\rho[g(X,W)D(Y,Z) + g(Y,Z)D(X,W) \\
& -g(Y,W)D(X,Z) - g(X,Z)D(Y,W)] \\
& +\beta\rho\{D(X,W)\omega(Y)\omega(Z) + D(Y,Z)\omega(X)\omega(W) \\
& -D(Y,W)\omega(X)\omega(Z) - D(X,Z)\omega(Y)\omega(W)\} \\
& +\gamma\rho\{D(X,W)\delta(Y)\delta(Z) + g(Y,Z)\delta(X)\delta(W) \\
& -D(Y,W)\delta(X)\delta(Z) - D(X,Z)\delta(Y)\delta(W)\} \\
& +\{D(X,W)D(Y,Z) - D(Y,W)D(X,Z)\}.
\end{aligned}
\tag{8.10}
$$

On contraction to (8.10) we get

(8.11)

$$
\begin{aligned}
\overline{S}(Y,Z) = \quad & [\alpha^2(n-2) + \alpha\beta + \alpha\gamma + \alpha\rho r_1]g(Y,Z) + [(n-2)\alpha\beta + \beta\gamma + \beta\rho r_1]\omega(Y)\omega(Z) \\
& +[(n-2)\gamma\alpha + \beta\gamma + \gamma\rho r_1]\delta(Y)\delta(Z) - \beta\gamma[\omega(Y)\delta(Z) + \delta(Y)\omega(Z)] \\
& +[\rho^2(r_1-2) + \alpha\rho(n-2) + \beta\rho + \gamma\rho]D(Y,Z)
\end{aligned}
$$

which shows that the manifold is a mixed super quasi-Einstein manifold. □

Thus we can state the following theorem.

Theorem 5. *A super quasi-umbilical hypersurface of a Euclidean space is a mixed super quasi-Einstein manifold.*

Acknowledgement. This paper is dedicated to Late Prof. M. C. Chaki.

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