ON MIXING AND PARTIAL MIXING

BY

N. A. FRIEDMAN AND D. S. ORNSTEIN[1]

1. Introduction

Let $(X, \mathfrak{A}, m)$ denote the unit interval with Lebesgue measure, and let τ be an invertible ergodic measure preserving transformation on X. τ is mixing if

$$\lim_n m(A \cap \tau^n B) = m(A)m(B), \qquad A \text{ and } B \text{ in } \mathfrak{A}. \tag{1.1}$$

Given $\alpha > 0$, τ is partially mixing for α if

$$\lim_n \inf m(A \cap \tau^n B) \geqq \alpha m(A)m(B), \qquad A \text{ and } B \text{ in } \mathfrak{A}. \tag{1.2}$$

In [3], a transformation τ is constructed such that τ is partially mixing for $\alpha = \frac{1}{8}$ but τ is not mixing. It is easily verified that τ is mixing if and only if τ is partially mixing for $\alpha = 1$.

The results in this paper are in two parts. The first result is concerned with mixing transformations. Let τ be mixing, $f \in L_1$, and let (k_n) be an increasing sequence of positive integers. Define f_n and $E(f)$ as

$$f_n(x) = (1/n) \sum_{i=1}^{n} f(\tau^{k_i}(x)), \qquad E(f) = \int f\, dm.$$

In [1], Blum and Hanson proved that f_n converges to $E(f)$ in the mean. In §4, we construct an example such that for a. e. x, $f_n(x)$ does not converge pointwise.

The second result concerns partial mixing transformations. In §5, it is shown that given $\alpha \in (0, 1)$, there is an explicit construction of a transformation τ such that τ is partially mixing for α but τ is not partially mixing for any $\alpha + \varepsilon$, $\varepsilon > 0$.

Both of the above results are based on a construction given in §3. Some preliminary results are given in §2. We shall utilize notation and terminology n [2].

2. Preliminaries

In [2], [3], the S operator was defined for a tower with columns of equal width. The definition will now be extended to the case where the columns generally have unequal widths. Let

$$T = \{C_j : 1 \rightarrow j \rightarrow q\} \quad \text{where } C_j = (I_{j,k} : 1 \rightarrow k \rightarrow h_j).$$

The intervals in C_j have the same width $w_j(T)$. The top of T is

$$A(T) = \textstyle\bigcup_{j=1}^{q} I_{j,h_j},$$

Received June 1, 1969.

[1] Research supported by a National Science Foundation grant.

and the base of T is

$$B(T) = \bigcup_{j=1}^{q} I_{j,1};$$

hence,

$$m(A(T)) = m(B(T)) = \sum_{j=1}^{q} w_j(T). \tag{2.1}$$

A subtower T_* of T is a *copy* of T if there exists $\alpha \in (0, 1]$ such that $w_j(T_*) = \alpha w_j(T)$, $1 \leq j \leq q$, and the h_j are the same. In this case, we also denote T_* as αT. Note that given $\alpha \in (0, 1)$, T can be decomposed into two disjoint copies αT and $(1 - \alpha)T$.

We shall now define $S(T)$ where T is as above. The transformation τ_T will be extended so as to map a subinterval of the top interval of each column onto a subinterval of the base interval of each column where the length of each subinterval is proportional to the corresponding widths. Let $p_j = w_j(T)/m(B(T))$, $1 \leq j \leq q$. Hence (2.1) implies

$$\sum_{j=1}^{q} p_j = 1.$$

We decompose the left half of I_{j,h_j} into q disjoint subintervals $E_{j,l}$ where $m(E_{j,l}) = p_l w_j/2$, $1 \leq j, l \leq q$. We also decompose the right half of $I_{l,1}$ into q disjoint subintervals $F_{l,j}$ where $m(F_{l,j}) = p_l w_j/2$, $1 \leq j, l \rightarrow q$. $E_{j,l}$ is now mapped linearly onto $F_{l,j}$, $1 \leq j, l \leq q$. The extension is measure preserving since $m(E_{j,l}) = m(F_{l,j})$, $1 \rightarrow j, l \rightarrow q$. We also have

$$\sum_{j=1}^{q} p_l w_j/2 = w_j/2, \qquad \sum_{j=1}^{q} p_l w_j/2 = w_l/2.$$

Thus, τ_T is extended to half of $A(T)$ and τ_T^{-1} is extended to half of $B(T)$. Let the corresponding tower be denoted by $S(T)$. As in [2], $S(T)$ consists of a bottom copy T_0 of T and a copy of T above each column in T_0.

We denote $\tau(T) = \lim_n \tau_{S^n(T)}$. As in [2], it follows that $\tau(T)$ is an ergodic measure preserving transformation on T'. If T is an M-tower, then $\tau(T)$ is mixing. (T' is the union of the intervals in T.)

Given a tower T and $\alpha \in (0, 1)$, let αT denote a copy of T as above. Note that if A is a union of intervals in T and $B = (\alpha T)'$, then

$$m(A \cap B) = m(A)\alpha = m(A)m(B)/m(T'). \tag{2.2}$$

Thus if $T' = X$, then A and B are independent sets.

Given disjoint towers T_1 and T_2, let $T_1 \cup T_2$ denote the tower consisting of the columns in T_1 and the columns in T_2. We do not require that the columns have the same width.

Let T_1 and T_2 be towers with q columns. We say the towers are similar if there exists $\alpha > 0$ such that $w_j(T_1) = \alpha w_j(T_2)$, $h_j(T_1) = h_j(T_2)$, $1 \leq j \leq q$. In particular, a copy of T is similar to T. However, a tower similar to T need not be a copy of T since it may not be a subtower of T. We note that if T_1 is similar to T_2, then $S^n(T_1)$ is similar to $S^n(T_2)$, $n = 1, 2, \cdots$.

The following result follows from the definition of the S operator.

(2.3) LEMMA. *Let T_1 and T_2 be similar towers, and let $T_3 = T_1 \cup T_2$. Let $\tau_1 = \tau(T_1)$ and $\tau_2 = \tau(T_3)$. Let I and J be intervals in T_1. Then*

$$m(\tau_2^n I \cap J) \geq m(\tau_1^n I \cap J) - 2m(T_2').$$

Let T_1 be a tower, and let C be a column. We shall utilize C to form a tower $T_1(C)$ such that $T_1(C)$ is similar to T_1. Furthermore, $\tau_{T_1(C)}$ will be an extension of τ_C and $T_1(C)$ will be unique up to similarity. Let T_1 have q columns with heights H_j and widths W_j, $1 \leq j \leq q$. Let h denote the height of C and $H = \min_{1 \leq j \leq q} H_j$. We assume there exists a positive integer K such that $H > Kh$. Let n_j denote the largest positive integer such that $n_j h \leq H_j$, $1 \leq j \leq q$. Thus $n_j \geq K$, $1 \leq j \leq q$. Define w_i as

$$w_i = w(C)W_i / \textstyle\sum_{j=1}^{q} n_j W_j, \qquad 1 \leqq i \leqq q, \tag{2.4}$$

where $w(C)$ denotes the width of C. Now (2.4) implies

$$\textstyle\sum_{j=1}^{q} n_j w_j = w(C), \tag{2.5}$$

$$w_i/w_j = W_i/W_j, \qquad 1 \leq i, j \leq q. \tag{2.6}$$

By (2.5), we can decompose C into $\sum_{j=1}^{q} n_j$ columns where n_j columns have width w_j, $1 \leq j \leq q$. We stack the columns of width w_j to form a single column of height $n_j h$. If $n_j h < H_j$, then we add $H_j - n_j h$ additional intervals of width w_j to obtain a column c_j of height H_j and width w_j. Let $T_1(C) = \{c_j : 1 \leq j \leq q\}$. $T_1(C)$ is similar to T_1 by (2.6). Let μ denote the total amount of additional measure needed to form $T_1(C)$. Then (2.5) implies

$$\mu < h \textstyle\sum_{j=1}^{q} w_j \leq hw(C)/K. \tag{2.7}$$

Let T be a tower, and let C be a column. We can choose p sufficiently large so that if $T_1 = S^p(T)$, then μ in (2.7) can be made arbitrarily small.

Let T be an M-tower, and let $\delta > 0$. Since $\tau = \tau(T)$ is mixing, there exists a positive integer $N(T)$ such that

$$m(\tau^n I \cap J) \geq (1 - \delta)m(I)m(J)/m(T'), \qquad n \geq N(T), \tag{2.8}$$

where I and J are intervals in T.

Let T be a tower, and let I be an interval. Given $\alpha \in (0, 1)$, we say αI is in T if there exists a set A consisting of a union of intervals in T such that $A \subset I$ and $m(A) = \alpha m(I)$.

3. Construction

Let V_1 be an M-tower, $\alpha \in (0, 1)$, $\delta_1 > 0$, $\delta_2 > 0$, and $\eta > 0$. Decompose V_1 into disjoint copies $V_2 = \alpha V_1$ and $V_3 = (1 - \alpha)V_1$. Assume V_1 has q columns with rational widths and α is rational. Then V_2 and V_3 each have q columns with rational widths. Denote the columns of V_3 as C_j with widths a_j/b, $1 \leq j \leq q$ (a_j and b are integers). Then $S_{a_j}(C_j)$ is a column with width $1/b$, $1 \leq j \leq q$. (We form $S_{a_j}(C_j)$ by dividing C_j into a_j copies and stacking them.) We then stack the $S_{a_j}(C_j)$ to form a single column V_4; hence

$$V_4 = \prod_{j=1}^{q} S_{a_j}(C_j) \qquad \text{(note that } V_3' = V_4'\text{)}.$$

Let r be a positive integer, and let $V_5 = S_r(V_4)$. Let $V_6 = S^p(V_2)$, and let $V_7 = V_6(V_5)$ as defined in §2. Note that p can be chosen sufficiently large with respect to r and η so that if μ denotes the measure added to form V_7, then $\mu < \eta$. Let $V_8 = V_6 \cup V_7$. V_7 is similar to V_6, and V_7 is an M-tower. Thus V_8 is an M-tower, and the columns in V_8 have rational widths.

Let $N_1 = N(V_1, \delta_1)$. (See (2.8).) If $\tau_1 = \tau(V_1)$, then I and J in V_1 imply

$$m(\tau_1^n I \cap J) \geq (1 - \delta_1) m(I) m(J)/m(V_1'), \qquad n \geq N_1. \tag{3.1}$$

Since αI and αJ are in V_2 and V_2 is a copy of V_1, it follows that if $\tau_2 = \tau(V_2)$, then

$$m(\tau_2^n I \cap J) \geq (1 - \delta_1)\alpha m(I) m(J)/m(V_1'), \qquad n \geq N_1. \tag{3.2}$$

Since $V_6 = S^p(V_2)$, we have $\tau_2 = \tau(V_2) = \tau(V_6)$. Let $\tau_3 = \tau(V_8)$. Hence Lemma 2.3 implies that if E and F are intervals in V_6, then

$$m(\tau_3^n E \cap F) \geq m(\tau_2^n E \cap F) - 2m(V_7'). \tag{3.3}$$

Since αI and αJ are in V_6, (3.2) and (3.3) imply

$$m(\tau_3^n I \cap J) \geq (1 - \delta_1)\alpha m(I) m(J)/m(V_1') - 2m(V_7'), \qquad n \geq N_1. \tag{3.4}$$

Let $N_1^* = N(V_8, \delta_2)$ where we also assume $N_1^* > N_1$. Since $\tau_3 = \lim_{t \to \infty} \tau_{S^t V_8}$, we can choose t sufficiently large so that if $V_9 = S^t V_8$ and $\tau = \tau_{V_9}$, then

$$m(\tau^n I \cap J) \geq (1 - \delta_1)\alpha m(I) m(J)/m(V_1') - 2m(V_7'), \quad N_1 \leq n \leq N_1^*. \tag{3.5}$$

In (3.5), I and J are in V_1.

4. Mixing

We shall now construct a mixing transformation in stages utilizing the construction in §3 inductively. At each stage most of the space is mixed. However, at the n^{th} stage, the transformation is defined on a small part of the space B_n so that certain Cesaro averages oscillate.

Let T_1 be an M-tower, and let (α_n), (ε_n) and (η_n) be sequences of positive numbers such that $\alpha_n \uparrow 1$, $\varepsilon_n \downarrow 0$, $\sum_{n=1}^{\infty} \eta_n < \infty$, and $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$. Let $V_1 = T_1$ in §3, and let $T_{1,i} = V_i$, $2 \leq i \leq 9$, correspond to $\alpha = \alpha_1$, $\delta_1 = \varepsilon_1$, $\delta_2 = \varepsilon_2$, and $\eta = \eta_1$. Let $\tau = \tau_{T_{1,9}}$, $N_1 = N(T_1, \varepsilon_1)$ and $N_1^* = N(T_{1,8}, \varepsilon_2)$. Thus (3.6) implies that if I and J are in T_1, then

$$m(\tau^n I \cap J) \geq (1 - \varepsilon_1)\alpha_1 m(I) m(J)/m(T_1') - 2m(T_{1,7}'), \quad N_1 \leq n \leq N_1^*. \tag{4.1}$$

Let $T_2 = T_{1,9}$ and $N_2 = N(T_2, \varepsilon_2)$. Consider $V_1 = T_2$ in §3, and let $T_{2,i} = V_i$, $2 \leq i \leq 9$, correspond to α_2, ε_2, ε_3 and η_2. Note that $T_{2,6} = S^{p_2}(T_{2,2})$ where we can choose p_2 arbitrarily large. Now $T_{2,2} = \alpha_2 T_2$

and $T_2 = S^{t_1}(T_{1,8})$ for some positive integer t_1. Thus we can choose p_2 sufficiently large so that if $\tau = \tau_{T_{2,6}}$ and I and J are intervals in $T_{1,8}$, then

$$(4.2) \qquad m(\tau^n I \cap J) \leq (1 - \varepsilon_2)\alpha_2\, m(I)m(J)/m(T'_{1,8}), \qquad N_1^* \leq n \leq N_2 .$$

Note that (4.2) also holds for I and J in T_1.

Let $N_2^* = N(T_{2,8}, \varepsilon_3)$ and $\tau = \tau_{T_{2,9}}$. Thus (3.5) implies that if I and J are in T_2, then

$$(4.3) \quad m(\tau^n I \cap J) \geq (1 - \varepsilon_2)\alpha_2\, m(I)m(J)/m(T'_2) - 2m(T'_{2,7}), \quad N_2 \leq n \leq N_2^*$$

Let us now consider we have $T_1, \cdots, T_{k-1}$. For each i, $1 \leq i \leq k - 2$, we have

$$(4.4) \quad \begin{aligned} &m(\tau^n I \cap J) \geq (1 - \varepsilon_{i+1})\alpha_{i+1}\, m(I)m(J)/m(T'_{i,8}), \\ &\qquad\qquad N_i^* \leq n \leq N_{i+1},\ I \text{ and } J \text{ in } T_i,\ \tau = \tau_{T_{i+1,6}} \end{aligned}$$

$$(4.5) \quad \begin{aligned} &m(\tau^n I \cap J) \geq (1 - \epsilon_{i+1})\alpha_{i+1}\, m(I)m(J)/m(T'_{i+1}) - 2m(T'_{i+1,7}), \\ &\qquad\qquad N_{i+1} \leq n \leq N_{i+1}^*,\ I \text{ and } J \text{ in } T_{i+1},\ \tau = \tau_{T_{i+1,9}} \end{aligned}$$

Let $T_k = T_{k-1,9}$ and $N_k = N(T_k, \varepsilon_k)$. Consider $V_1 = T_k$ in §3, and let $T_{k,i} = V_i$, $2 \leq i \leq 9$, correspond to α_k, ε_k, ε_{k+1} and η_k. Note that $T_{k,6} = S^{p_k}(T_{k,2})$ where we can choose p_k arbitrarily large. Now $T_{k,2} = \alpha_k\, T_k$ and $T_k = S^{t_{k-1}}(T_{k-1,8})$ for some positive integer t_{k-1}. Thus we can choose p_{k-1} sufficiently large so that if $\tau = \tau_{T_{k,6}}$ and I and J are intervals in $T_{k-1,8}$, then

$$(4.6) \quad m(\tau^n I \cap J) \geq (1 - \varepsilon_k)\alpha_k\, m(I)m(J)/m(T'_{k-1,8}), \quad N_{k-1}^* \leq n \leq N_k .$$

Let $N_k^* = N(T_{k,8}, \varepsilon_{k+1})$ and $\tau = \tau_{T_{k,9}}$. Thus (3.5) implies that if I and J are in T_k, then

$$(4.7) \quad m(\tau^n I \cap J) \geq (1 - \varepsilon_k)\alpha_k\, m(I)m(J)/m(T'_k) - 2m(T'_{k,7}), \quad N_k \leq n \leq N_k^*.$$

Thus (4.6) and (4.7) imply (4.4) and (4.5) hold for k. Hence the induction step is complete.

We thus obtain a sequence of towers (T_k) such that τ_{T_k} extends $\tau_{T_{k-1}}$. The construction implies

$$m(T'_k) \leq m(T'_1) + \textstyle\sum_{i=1}^{k-1} \eta_i .$$

Since $\sum_{n=1}^{\infty} \eta_n < \infty$, we can consider $X = \bigcup_{k=1}^{\infty} T'_k = [0, 1)$. We define τ as $\tau = \lim_k \tau_{T_k}$. The properties of (ε_k) and (α_k) imply that

$$\lim_k\, (1 - \varepsilon_k)\alpha_k/m(T'_k) = 1.$$

Also, $\lim_k(1 - \alpha_k) + \eta_k = 0$ implies $\lim_k m(T'_{k,7}) = 0$. Thus (4.4) and (4.5) imply that if I and J are intervals in T_k for some k, then

$$\lim_n \inf m(\tau^n I \cap J) \geq m(I)m(J).$$

Since the intervals in T_k, $k = 1, 2, \cdots$, generate $\mathcal{a}$, an approximation argument implies

$$\lim_n \inf m(\tau^n A \cap B) \geq m(A)m(B), \qquad A \text{ and } B \text{ in } \mathfrak{A}. \tag{4.8}$$

Thus (4.8) implies τ is partially mixing for $\alpha = 1$, hence τ is mixing.

We now consider the column $T_{n,4}$ which is formed from $T_{n,3} = (1 - \alpha_n)T_n$. Let $B_n = T'_{n,3} = T'_{n,4}$. (Thus $m(B_n) = (1 - \alpha_n)m(T'_n)$.)

Now for a fixed integer k, (2.2) implies

$$m[B_{n+1} \cap (T'_n - \cup_{j=k}^{n} B_j)] = (1 - \alpha_{n+1})m(T'_n - \cup_{j=k}^{n} B_j).$$

For fixed k, the sets $B_{n+1} \cap (T'_n - \cup_{j=k}^{n} B_j)$ are disjoint. Hence their measure tends to 0. Since $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$, $\lim_{n\to\infty} m(T'_n - \cup_{j=k}^{n} B_j) = 0$. Since this happens for each fixed k and since $m(T'_n) \to 1$, we get that $m(\lim_n \sup B_n) = 1$.

Let I denote a base interval in a column in T_1 which has at least two intervals. Thus $x \in I$ implies $\tau(x) \notin I$. Let f denote the characteristic function of I. We shall define an increasing sequence of positive integers

$$k_{1,1}, k_{1,2}, \cdots, k_{1,a_1}, k_{2,1}, k_{2,2}, \cdots, k_{2,a_2}, \cdots, k_{n,1}, k_{n,2}, \cdots, k_{n,a_n}, k_{n+1,1}, \cdots$$

such that the average of f along the sequence oscillates for a.e. x.

In order to define $k_{n,1}, k_{n,2}, \cdots k_{n,a_n}$ we consider the column $T_{n,4}$. The interval I chosen above is scattered in T_n as certain intervals in certain columns in T_n. Now $T_{n,4}$ was formed from $T_{n,3}$ which is a copy of T_n. Thus some intervals in $T_{n,4}$ are contained in I and some are not. Let p_n be the number of intervals in $T_{n,4}$ *not* contained in I.

Let $d_n = \sum_{i=1}^{n-1} a_i$ and choose $j_{n,1}$ so that

$$d_n/j_{n,1} < 1/n.$$

Proceed to choose $j_{n,i}$, $2 \leqq i \leqq p_n + 2$, inductively so that

$$(d_n + j_{n,1} + \ldots + j_{n,i})/j_{n,i+1} < 1/n, \qquad 1 \leqq i \leqq p_n + 1.$$

Let K_n be the height of $T_{n,4}$; hence we can write

$$T_{n,4} = (I_{n,1}, I_{n,2}, \cdots, I_{n,K_n}).$$

There are p_n intervals in $T_{n,4}$ that are not contained in I and we list these as

$$I_{n,i_l}, \qquad 1 \leqq l \leqq p_n.$$

Let $x \in I_{n,i_l}$ and let t_l be the smallest positive integer such that $\tau^{t_l}(x) \in I_{n,i} \subset I$. If $i_l = K_n$, then take x such that $\tau(x) \in I_{n,1}$. This is possible since we shall have $T_{n,5} = S_{r_n}(T_{n,4})$ where r_n is chosen below. Hence most of I_{n,K_n} is mapped to $I_{n,1}$. By definition of τ on $T_{n,4}$, t_l is well defined and satisfies $t_l < K_n$, $1 \leqq l \leqq p_n$.

Choose u_n so that

$$u_n K_n > k_{n-1,a_{n-1}}$$

and let $j_0 = 1$. Define

$$k_{n,j} = (j + u_n)K_n + t_1, \qquad j_0 \leqq j \leqq j_1$$

and

$$k_{n,j} = (j + u_n)K_n + t_l, \ \sum_{i=1}^{l-1} j_i < j \leqq \sum_{i=1}^{l} j_i, \qquad 2 \leqq l \leqq p_n.$$

We also define

$$k_{n,j} = (j + u_n)K_n + 1, \quad \sum_{i=1}^{p_n} j_i < j \leqq \sum_{i=1}^{p_n+1}$$

$$k_{n,j} = (j + u_n)K_n, \quad \sum_{i=1}^{p_n+1} j_i < j \leqq \sum_{i=1}^{p_n+2} j_i.$$

Let $a_n = \sum_{i=1}^{p_n+2} j_i$ and $b_n = a_n + u_n$. Let $T_{n,4}$ have width w_n. Choose r_n so large that $b_n w_n/r_n < 2^{-n}$. Let C_n denote the subcolumn $(1 - b_n/r_n)T_{n,4}$ of $T_{n,4}$, chosen so that $C'_n \cap I_{n,1}$ is an interval with the same left endpoint as $I_{n,1}$. In the following we consider only $x \in C'_n$. Thus if $x \in I_{n,i}$ then $\tau^{jK_n}(x) \in I_{n,i}$, $1 \leqq j \leqq b_n$, $1 \leqq i \leqq K_n$.

Now f is the characteristic function of the chosen interval I and f_v is the v^{th} Cesaro average along the sequence

$$k_{1,1}, \cdots, k_{1,a_1}, k_{2,1}, \cdots k_{n,1}, \cdots k_{n,a_n}.$$

Let $x \in I_{n,i_1}$. The choice of j_1 guarantees that

$$f_v(x) > 1 - 1/n \tag{4.9}$$

for $v = d_n + j_1$. In general, let $x \in I_{n,i_l}$, $1 \leqq l \leqq p_n$. The choice of $j_1, \cdots, j_l$ guarantees that (4.9) holds for $v = d_n + j_1 + \cdots + j_l$.

Now consider $x \in I_{n,i} \subset I$. Recall that $x \in I$ implies $\tau(x) \notin I$. Hence the choice of j_{p_n+1} guarantees that

$$f_v(x) < 1/n \tag{4.10}$$

for $v = d_n + j_1 + \cdots + j_{p_n+1}$. Lastly, the choice of j_{p_n+2} guarantees that for

$$v = d_{n+1} = \sum_{i=1}^{n} a_i = d_n + \sum_{i=1}^{p_n+2} j_i$$

we have (4.9) satisfied for $x \in I_{n,i} \subset I$ and (4.10) satisfied for $x \in I_{n,i} \not\subset I$.

The choice of r_n implies $m(B_n - E_n) < 2^{-n} m(B_n)$, where $B_n = T'_{n,4}$ and $E_n = C'_n$. We have already shown that $m(\limsup B_n) = 1$; hence $m(\limsup E_n) = 1$. If $x \in \limsup E_n$, then the above construction implies

$$\limsup f_n(x) = 1 \quad \text{and} \quad \liminf f_n(x) = 0.$$

Thus $f_n(x)$ does not converge a.e.

5. Partial mixing

Let τ be partially mixing for some $\alpha > 0$. We define an invariant $\bar{\alpha}(\tau)$ as

$$\bar{\alpha}(\tau) = \sup \{\alpha : \tau \text{ is partially mixing for } \alpha\}.$$

It follows at once that τ is partially mixing for $\bar{\alpha}(\tau)$. Thus τ is mixing if and only if $\bar{\alpha}(\tau) = 1$.

Given $\alpha \in (0, 1)$, we shall construct τ so that $\bar{\alpha}(\tau) = \alpha$. The construction is a slight modification of the construction in §3. At the n^{th} stage, we have an M-tower T_n. We decompose T_n into disjoint copies ${}_1T_{n,2} = \alpha_n T_n$ and $U_{n,j} = (1 - \alpha_n)/nT_n$, $1 \leq j \leq n$. The method is to mix ${}_1T_{n,2}$ and unmix $U_{n,1}$. Then mix ${}_1T_{n,2}$ with $U_{n,1}$ and unmix $U_{n,2}$. Then mix ${}_1T_{n,2}$, $U_{n,1}$ and $U_{n,2}$, and unmix $U_{n,3}$, etc. Since $m(U'_{n,j}) \leq 1/n$, the perturbation due to $U_{n,j}$ is small. We proceed to describe the construction as follows.

Let T_1 be an M-tower, and let (α_n), (ε_n) and (η_n) be sequences of positive numbers such that α_n is rational and $\lim_n \alpha_n = \alpha$, $\varepsilon_n \downarrow 0$ and $\sum_{n=1}^{\infty} \eta_n < \infty$. At the n^{th} stage, we have an M-tower T_n, and let ${}_1T_{n,2} = \alpha_n T_n$ and $U_{n,j} = (1 - \alpha_n)/nT_n$, $1 \leq j \leq n$ as above. Let $V_2 = {}_1T_{n,2}$ and $V_3 = U_{n,1}$. Let ${}_1T_{n,i} = V_i$, $4 \leq i \leq 9$, corresponding to $\delta_1 = \delta_2 = \varepsilon_n$ and $\eta = \eta_n/n$. Now let $V_2 = {}_1T_{n,9}$ and $V_3 = U_{n,2}$. Let ${}_2T_{n,i} = V_i$, $4 \leq i \leq 9$, corresponding to $\delta_1 = \delta_2 = \varepsilon_n$ and $\eta = \eta_n/n$. Proceeding inductively, let $k < n$, and let $V_2 = {}_kT_{n,9}$ and $V_3 = U_{n,k+1}$. Let ${}_{k+1}T_{n,i} = V_i$, $4 \leq i \leq 9$, corresponding to $\delta_1 = \delta_2 = \varepsilon_n$ and $\eta = \eta_n/n$. Let $T_{n+1} = {}_nT_{n,9}$.

At the n^{th} stage, we repeat the construction in §3 n times. This requires choosing positive integers $r_{n,k}$ and $p_{n,k}$, $1 \leq k \leq n$ where $r_{n,k}$ and $p_{n,k}$ are utilized in forming ${}_kT_{n,i}$, $4 \leq i \leq 9$, corresponding to $\delta_1 = \delta_2 = \varepsilon_n$ and $\eta = \eta/n$. Thus the amount of measure added at the n^{th} stage is less than $n\eta_n/n = \eta_n$. Hence the total measure added is finite. Thus we may consider $X = \bigcup_{n=1}^{\infty} T'_n = [0, 1)$ and $\tau = \lim_n \tau_{T_n}$.

Utilizing the same technique as in §4, we can choose the parameters $r_{n,k}$ and $p_{n,k}$, $1 \leq k \leq n$, so that

$$\begin{aligned} \liminf m(\tau^n \cap J) &\geq \liminf\, (1 - \varepsilon_n) \frac{\alpha_n m(I)m(J)}{m(T'_n)} - 2\eta_n/n \\ &= \alpha m(I)m(J) \end{aligned} \tag{5.1}$$

where I and J are intervals in T_l for some l. Since these intervals generate $\mathfrak{A}$, (5.1) implies τ is partially mixing for α. On the other hand, we can guarantee τ is not partially mixing for $\alpha + \varepsilon$, $\varepsilon > 0$, by simply choosing $r_{n,k} = n$, $1 \leq k \leq n$.

Added in proof. U. Krengel has shown examples as in §4 hold for general mixing transformations.

References

1. J. R. Blum and D. L. Hanson, *On the mean ergodic theorem for subsequences*, Bull. Amer. Math. Soc., vol. 66 (1960), pp. 308–311.
2. N. A. Friedman and D. S. Ornstein, *On induced transformations*, Technical report, Univ. of New Mexico.
3. ———, *On partially mixing transformations*, J. Math. Mech., vol. 20 (1971), pp. 767–775.

State University of New York
Albany, New York
Stanford University
Stanford, California