The Annals of Probability
1978, Vol. 6, No. 2, 325–331

ON MIXING AND STABILITY OF LIMIT THEOREMS

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Rényi introduced and developed the ideas of limit theorems which are mixing or stable. These concepts are a strengthening of the idea of weak convergence of random variables. In this expository note we point out some equivalent definitions of mixing and stability and discuss the use of these concepts in several contexts. Further, we show how a recent central limit theorem for martingales can be obtained directly using stability. Though the results are not new, the proofs seem substantially simpler than those previously given.

Throughout this note we talk only of rv's defined on some fixed probability space $(\Omega, \mathscr{F}, P)$ and taking values in R^1. The extension to the metric-spaced-valued situation is straightforward.

Stable limit theorems. If $\{Y_n\}$ is a sequence of rv's with distribution functions F_{Y_n}, then Y_n is said to converge in distribution to Y, a rv with distribution function F_Y, if for a countable, dense set of points x,

$$\lim_{n\to\infty} F_{Y_n}(x) = F_Y(x) .$$

We shall write this as $F_{Y_n} \Rightarrow F_Y$.

Given that $F_{Y_n} \Rightarrow F_Y$, the convergence is said to be *stable* (written $Y_n \Rightarrow F_Y$ (stably)) if, for every $\mathscr{F}$-measurable set B, and for a countable, dense set of points x,

$$\lim_{n\to\infty} P(Y_n \leqq x, B) \quad \text{exists.}$$

In other words, a limit theorem is stable if for all events B such that $P(B) > 0$, the distribution of Y_n, conditional on B, converges in law to some distribution which may depend on B and which must, as the $\{Y_n\}$ are tight, be proper.

Unlike convergence in distribution, stable convergence in distribution is a property of the sequence of rv's $\{Y_n\}$ rather than of the corresponding sequence of distribution functions. For example, let X and X' be independent with common, nondegenerate distribution. Let

$$\begin{aligned} Z_n &= X\,, &&\text{for } n \text{ odd} \\ &= X'\,, &&\text{for } n \text{ even;} \end{aligned}$$

then it is not true that $Z_n \Rightarrow F_X$ (stably).

Despite this dependence on the sequence $\{Y_n\}$, the requirement that a limit theorem be stable is quite weak. Most known limit theorems are in fact stable

Received June 22, 1977.

AMS 1970 *subject classifications.* Primary 60F05; Secondary 60F01, 60G45.

Key words and phrases. Mixing limit theorems, stable limit theorems, martingales, central limit theorem.

and if a limit theorem is not stable one can choose a subsequence along which it will be stable.

The following proposition gives a number of equivalent definitions of stability.

PROPOSITION 1. *Suppose that* $F_{Y_n} \Rightarrow F_Y$. *The following conditions are equivalent*:

(A) $Y_n \Rightarrow F_Y$ (*stably*);

(B) *For all fixed* $\mathscr{F}$*-measurable* rv's σ, *the sequence of random vectors* (Y_n, σ) *converges* jointly *in distribution*;

(C) *For each fixed real* t, *the sequence of* (*complex-valued*) rv's $\{e^{itY_n}\}$ *converges weakly in* L_1;

(D) *For all fixed* k *and* $B \in \sigma(Y_1, \cdots, Y_k), P(B) > 0$, $\lim_{n\to\infty} P(Y_n \leqq x \mid B)$ *exists for a countable dense set of points* x.

(A sequence $\{\xi_n\}$ of L_1 rv's is said to converge *weakly in* L_1 to ξ, also in L_1, if for all bounded $\mathscr{F}$-measurable rv's η

$$\lim_{n\to\infty} E\xi_n\eta = E\xi\eta .$$

An equivalent condition is that for all $\mathscr{F}$-measurable events B, $P(B) > 0$,

$$\lim_{n\to\infty} E\xi_n I_B = E\xi I_B .$$

This is the concept of convergence in the $\sigma(L_1, L_\infty)$ topology, familiar to functional analysts, and which we denote by $\rightarrow_{w-L_1}$.)

PROOF. (A) $\Rightarrow$ (B) by taking the testing sets $B = I_{[\sigma \leqq z]}$ and (B) $\Rightarrow$ (A) by choosing $\sigma = I_B$.

That (A) $\Leftrightarrow$ (C) is equally trivial. As $\{Y_n\}$ converges in distribution, $\{Y_n\}$ is tight and, a fortiori, so is $\{Y_n I_B\}$ for each $\mathscr{F}$-measurable set B. If (A) is true, then the distribution of Y_n, conditional on B ($P(B) > 0$), converges and hence

$$\lim_{n\to\infty} \int_B e^{itY_n}\, dP \quad \text{exists.}$$

But the $\{e^{itY_n}\}$ are uniformly bounded and hence weak-L_1 sequentially compact. This is sufficient to ensure (C). On the other hand, if (C) is true, set the weak-L_1 limit of $\{e^{itY_n}\}$ to be $\phi(t, \omega)$. By the tightness of $\{Y_n I_B\}$, for each $B(P(B) > 0)$ there exists a subsequence $\{n_k\}$ such that

$$\lim_{k\to\infty} E(e^{itY_{n_k}} \mid B) = E(I_B \phi(t, \omega))/P(B)$$

is a characteristic function. Hence the original sequence $\{Y_n\}$, conditional on B, converges in distribution—i.e., (A) is true.

Finally, that (A) $\Leftrightarrow$ (D) was proved by Rényi (1963). Alternatively, letting $\mathscr{F}_n = \sigma(Y_1, \cdots, Y_n)$, it is easy to see that (D) $\Rightarrow$ (C) as the existence of

$$\lim_{n\to\infty} \int_B e^{itY_n}\, dP$$

for all $B \in \bigcup_{n=1}^\infty \mathscr{F}_n$ implies by the usual approximation argument that the limit exists for all $B \in \mathscr{F}_\infty = \bigvee_{n=1}^\infty \mathscr{F}_n$ and for general $B \in \mathscr{F}$, one first conditions on $\mathscr{F}_\infty$.

Some of the above conditions are more useful in a given situation than the others. So, for example, in testing a particular limit theorem for stability (D) is usually used. In order to obtain theoretical consequences of stability, (A) has usually been applied, though in fact (B) is often much simpler to use. As an illustration of this fact, we prove the following theorem of Kátai and Mogyoródi.

THEOREM 1. *Suppose that* $Y_n \Rightarrow F_Y$ *(stably). Let* $g(x, y)$ *be a continuous function of two variables and let* σ *be any* $\mathscr{F}$*-measurable* rv. *Then* $g(Y_n, \sigma)$ *converges stably also.*

PROOF. Let η be another $\mathscr{F}$-measurable rv. Then by (B), (Y_n, σ, η) converges jointly and, by the continuous mapping theorem, so does

$$(g(Y_n, \sigma), \eta) .$$

Of course, knowledge that a limit theorem exists is hardly useful if one cannot determine the limit distribution. If $\{Y_n\}$ converges stably,

$$\lim_{n\to\infty} P(Y_n \leqq x, B) = Q(x, B)$$

exists for all $\mathscr{F}$-measurable B and a countable number of x; it is clear that for each such x, $Q(x, \cdot)$ is a measure, absolutely continuous with respect to P. A version, $\alpha(x, \omega)$, of $dQ(x, \cdot)/dP$, can be chosen so that $\alpha(x, \omega)$ is a.s. a distribution function. (The random distribution function α is called the *local density* of the stable sequence $\{Y_n\}$.)

One would like to construct a rv Y^*, say, which is distributed as the limit variable Y and which is such that

$$P(Y^* \leqq x \mid \mathscr{F})(\omega) = \alpha(x, \omega) \quad \text{a.s.}$$

One way to achieve this is to replace the original probability space $(\Omega, \mathscr{F}, P)$ by the new space $(\Omega \times I, \mathscr{F} \times \mathscr{A}, P \times \lambda)$, where $(I, \mathscr{A}, \lambda)$ is the unit interval with Lebesgue measure. Then for each ω we can define $Y^*(\omega, t)$ as the inverse distribution function of $\alpha(\cdot, \omega)$.

From now on, when we have a stably converging sequence $\{Y_n\}$ we shall write

$$Y_n \Rightarrow Y^* \quad \text{(stably)}$$

with the convention that the distribution of Y^*, conditional on $\mathscr{F}$, is given by the local density of $\{Y_n\}$. With this convention, conditions (B) and (C) of Proposition 1 become

(B′) For all fixed $\mathscr{F}$-measurable rv's σ,

$$(Y_n, \sigma) \Rightarrow (Y^*, \sigma) ;$$

(C′) For all fixed real t,

$$e^{itY_n} \rightarrow_{w-L_1} E(e^{itY^*} \mid \mathscr{F}) .$$

When the local density (and so the dependence of Y^* on $\mathscr{F}$) is known, either (B′) or (C′) can be used to obtain the limit distributions. Thus we can improve Theorem 1 to:

THEOREM 1′. *Suppose that* $Y_n \Rightarrow Y^*$ *(stably), that* σ *is any fixed* $\mathscr{F}$*-measurable* rv *and that* $g(x, y)$ *is a continuous function of two variables. Then*

$$g(Y_n, \sigma) \Rightarrow g(Y^*, \sigma) \quad (stably).$$

PROOF. The proof is immediate from (B′) and the continuous mapping theorem.

The point of the above proof is that the result is trivial when looked at in the correct way (cf. the original proof in Kátai and Mogyoródi (1967)).

REMARK. The conditions of Theorem 1 can be relaxed in a number of obvious ways. Instead of a fixed $\mathscr{F}$-measurable random variable σ we could have taken any sequence $\sigma_n \to_p \sigma$. We could also have taken σ to be an $\mathscr{F}$-measurable random vector and $g(x, y)$ need only be a.s.-continuous with respect to (Y^*, σ).

Mixing limit theorems. If in a stable limit result the limit rv Y^* can be taken to be independent of $\mathscr{F}$, then the limit theorem is said to be *mixing* (with density F_Y, where F_Y is the distribution function of Y^*). We will write $Y_n \Rightarrow F_Y$ (mixing). Note that a limit result is mixing if, and only if, the very same limit result is true, conditional on any $\mathscr{F}$-measurable set B, $P(B) > 0$. In this special case Proposition 1 becomes:

PROPOSITION 2. *Suppose that* $F_{Y_n} \Rightarrow F_Y$. *The following conditions are equivalent*:

(A″) $Y_n \Rightarrow F_Y$ *(mixing)*;

(B″) *For all fixed* $\mathscr{F}$*-measurable* rv's σ, *the sequence of random vectors* (Y_n, σ) *converges jointly in distribution to* (Y^*, σ), *where* Y^* *is distributed as* Y, *independently of* $\mathscr{F}$;

(C″) *For all fixed real* t,

$$e^{itY_n} \to_{w-L_1} E(e^{itY})\,;$$

(D″) *For all fixed* k *and* $B \in \sigma(Y_1, \cdots, Y_k)$, $P(B) > 0$,

$$\lim_{n\to\infty} P(Y_n \leqq x \mid B) = F_Y(x)\,,$$

for a countable dense set of points x.

Many of the classical limit theorems are mixing. Eagleson (1976) used (C″) to derive some simple sufficient conditions for limit theorems to be mixing. His results are typically for limit theorems of normalized sums of rv's and the conditions are in terms of the tail or invariant σ-fields of the summands.

Applications. The knowledge that a limit theorem is mixing can be useful for many reasons. For example:

(a) If a limit theorem is mixing, then the theorem remains true under an absolutely continuous change of measure (Rényi (1950), Rényi and Révész (1958)).

(b) If a limit theorem is mixing, one can randomly normalize and still obtain a limit theorem (Smith (1945), Takahashi (1951 (a) and (b)).

(c) If a limit theorem is mixing, then it can be extended to random indices (Smith (1945), Takahashi (1951 (a) and (b)), Rényi (1960)). For a review of mixing results and random sum central limit theorems, see Csörgö and Fischler (1973), Fischler (1976).

(d) If a limit theorem is mixing, then the range of $\{Y_n\}$ is almost surely dense in the support of the limit distribution (Rootzén (1976)).

However, most consequences of mixing require, in fact, only the weaker hypothesis of stable convergence though, of course, in order to identify limit distributions one must know the local density. Thus we have:

(a) If $Y_n \Rightarrow Y^*$ (stably), then $\{Y_n\}$ still converges stably under an absolutely continuous change of measure. (Use condition (A) of Proposition 1. If the new measure has a Radon–Nikodym derivative, $\gamma(\omega)$, with respect to P, then the new local density is $\gamma(\omega)\alpha(x, \omega)$.)

(b) If a limit theorem is stable, one can randomly normalize and still obtain a limit theorem. (This is just Theorem 1′ with a particular g.)

(c) Where mixing has been used as a condition in random indices limit theorems, it may be replaced by stability (Aldous (1978)), and the limit distribution remains the same. Fischler (1976) proves a functional limit theorem for random indices by using the idea of a random time change as in Billingsley (1968, Section 17), but fails to identify the limit distribution.

(d) While (d) itself does not generalize to stable limit theorems there does exist an analogue which may be deduced from results in Aldous (1977); in our notation, if $Y_n \Rightarrow Y^*$ (stably) with local density $\alpha(x, \omega)$, then for almost all ω the points $Y_n(\omega)$ are dense in the support of the measure generated by the distribution function $\alpha(x, \omega)$.

The martingale central limit theorem. As an example of the use of condition (C′) to check stability and to identify the limit rv, we consider a recent central limit theorem for normalized sums of martingale differences of Hall (1977). It is easy to show from Hall's work that his central limit theorem for single martingale sequences is stable, using condition (C). Alternatively, McLeish's (1974) proof of the central limit theorem can be adapted to obtain the following result.

THEOREM 2. *Let* $\{X_n, \mathscr{F}_n; n \geqq 1\}$ *be a sequence of martingale differences on* $(\Omega, \mathscr{F}, P)$, $\{b_n\}$ *a sequence of positive norming constants,* $b_n \nearrow \infty$, *and set* $S_n = b_n^{-1} \sum_{j=1}^n X_j$. *If*

(a) $$\max_{j\leq n} |X_j|/b_n \to_p 0 \,,$$

(b) $$b_n^{-2} \sum_{j=1}^n X_j^2 \to_p \eta^2 \,, \quad \textit{an a.s. finite random variable},$$

(c) $$\sup_n E(\max_{j\leq n} X_j^2/b_n^2) < \infty \,,$$

then

$$e^{itS_n} \to e^{-\frac{1}{2}t^2\eta^2} \qquad \textit{weakly in} \quad L_1\,, \quad \textit{for all real} \quad t\,.$$

That is, $S_n \Rightarrow F$ *(stably), where* F *is a distribution function with characteristic function* $E(e^{-\frac{1}{2}t^2\eta^2})$.

The same result under a slightly stronger condition than (c) was obtained by Chatterji (1974).

To understand this result, let Y be a standard normal variable, independent of $\mathscr{F}$. Set $Y^* = Y\eta$. Then

$$E(e^{itY^*} \mid \mathscr{F}) = e^{-\frac{1}{2}t^2\eta^2}\,,$$

so by condition (C′),

$$S_n \Rightarrow Y\eta \quad \text{(stably)}.$$

Thus, by condition (B′), for all fixed $\mathscr{F}$-measurable σ,

$$(S_n, \sigma) \Rightarrow (Y\eta, \sigma)$$

and so, if $P(\sigma = 0) = 0$,

$$S_n/\sigma \Rightarrow Y\eta/\sigma\,.$$

Taking $\sigma = \eta$ (if $\eta^2 > 0$ a.s.),

$$\frac{S_n}{\eta} \Rightarrow Y \sim N(0, 1)\,.$$

Hence, under the conditions of Theorem 2,

$$S_n/(\textstyle\sum_1^n X_j^2)^{\frac{1}{2}} \Rightarrow N(0, 1)\,.$$

Thus the possibility of randomly normalizing the martingale central limit theorem and so transforming the limit law from a mixture of normals to the standard normal, is a direct consequence of the fact that the original limit theorem was stable.

Of course, from Theorem 1′, even more is true. If g is a continuous function of two variables and σ is any $\mathscr{F}$-measurable rv, then

$$g(S_n, \sigma) \Rightarrow g(Y\eta, \sigma) \quad \text{(stably)},$$

where $Y \sim N(0, 1)$, independently of $\mathscr{F}$.

REMARK. There are many results, Chatterji (1974), Gaposhkin (1972), Morgenthaler (1955) etc., where it is shown that some sequence of random variables satisfies the hypothesis of Theorem 1′. Of course, Theorem 1′ can be applied to these results as well.

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