# On Modality of Lévy Processes Corresponding to Mixtures of Two Exponential Distributions 

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1. Introduction. By a Lévy process we mean a stochastically continuous stochastic process taking values in the real line $\boldsymbol{R}$, having stationary independent increments, and starting at the origin. In this paper we consider multimodality of some Lévy process. We use the following definition of strict $k$-modality given by Sato [5]. The restriction of a $\sigma$-finite measure $\mu$ on $\boldsymbol{R}$ to a Borel set $B$ is denoted by $\left.\mu\right|_{B}$.

Definition. (I) $A \sigma$-finite measure $\mu$ on $\boldsymbol{R}$ is said to be strictly unimodal with mode $a$ if it satis. fies the following:
(i) The support I of $\mu$ is an interval or a singleton, and $I$ contains $a$.
(ii) The measure $\left.\mu\right|_{I \backslash(a)}$ has a version $f(x)$ of the density which, is strictly increasing on $I \cap(-\infty, a)$ if $I \cap(-\infty, a) \neq \phi$ and strictly decreasing on $I \cap(a, \infty)$ if $I \cap(a, \infty) \neq \phi$.
(II) For $k \geq 2$, a $\sigma$-finite measure $\mu$ on $\boldsymbol{R}$ is said to be strictly $k$-modal if it satisfies the following:
(i) The support $I$ of $\mu$ is an interval.
(ii) There are disjoint sets $I_{1}, \ldots, I_{k}$ such that $I=$ $\cup_{i=1}^{k} I_{i}$, each $I_{i}$ is a singleton or an interval, and, for each $i,\left.\mu\right|_{I_{i}}$ is strictly unimodal.
(iii) If $l<k$, then there are no disjoint sets $J_{1}, \ldots$, $J_{l}$ such that $I=\cup_{j=1}^{l} J_{j}$, each $J_{j}$ is a singleton or an interval, and, for each $j,\left.\mu\right|_{J_{t}}$ is strictly unimodal.

Strictly 2 -modal is called strictly bimodal. The modes $a_{1}, a_{2}, \ldots, a_{k}$ of $\left.\mu\right|_{I_{1}},\left.\mu\right|_{I_{2}}, \ldots,\left.\mu\right|_{I_{k}}$ are called modes of $\mu$.

Brownian motion and stable processes, which are familiar examples of one-dimensional Lévy processes, are unimodal at any time (Yamazato [9]). But there are Lévy processes which have time evolution in modality. This fact is already known to Wolfe [8] and stressed by Sato [3] [4] [5] and Watanabe [7]. Examples show that there are Lévy processes which change from unimodal to non-unimodal, or from non-unimodal to unimodal, or from unimodal to non-unimodal and again to unimodal as time passes. There are Lévy
processes which change between unimodal and non-unimodal infinitely many times. Among these examples, we have few Lévy processes whose evolution in modality is completely known. One of such examples is Wolfe's (see [5] and [8]) and another is a compound Poisson process $\left\{X_{t}: t \geq\right.$ $0\}$ whose distribution at $t=1$ is
(1) $\mu=p \delta_{0}+(1-p) a e^{-a x} I_{(0, \infty)}(x) d x$, where $\delta_{0}$ stands for the delta distribution at 0,0 $<p<1,0<a$, and $I_{(0, \infty)}(x)$ stands for the indicator function of the interval ( $0, \infty$ ). Sato [5] proved that the distribution of $X_{t}$ is strictly unimodal for $t \leq(1+p) /(1-p)$, and strictly bimodal for $t>(1+p) /(1-p)$. The distributions of these two examples have point mass at the origin. Hence, when they are strictly unimodal, they have modes at 0 and, when they are strictly bimodal, one of their two modes is located at 0 .

We would like to find out examples which do not have point mass and are strictly $k$-modal at some $t$ and whose time evolution in modality can be analyzed for all time. In order to consider this problem for $k=2$, we shall investigate modality of the Lévy process $\left\{X_{t}: t \geq 0\right\}$ that has the following distribution $\mu$ at $t=1$ :
(2) $\mu=(1-p) a e^{-a x} I_{(0, \infty)}(x) d x+p b e^{-b x} I_{(0, \infty)}(x) d x$, where $0<p<1$ and $0<a<b$. The distributions (1) and (2) are infinitely divisible by the result of Goldie [1], and $X_{t}$ is unimodal with mode 0 for $0<t<1$ by the result of Steutel [6]. It is difficult to analyze modality of $X_{t}$ for noninteger $t>1$, but we can analyze it for integer $t=n$.
2. Results. From now on $\left\{X_{t}\right\}$ is the Lévy process that has distribution $\mu$ of (2) at $t=1$. We shall obtain the following theorem. Denote the set of all positive integers by $\boldsymbol{N}$.

Theorem. The distribution of $X_{n}, n \in \boldsymbol{N}$, is either strictly unimodal or strictly bimodal. Furth-
ermore it is strictly unimodal if $n \geq \frac{b}{b-a}\left(1+\frac{b}{a}\right.$ $\left.\frac{p}{1-p}\right)$.

Remark. Sato points out that, if $t>(1+$ $p) /(1-p)$, then, for any $a>0$, the distribution of $X_{t}$ is non-unimodal for any sufficiently large $b$, because, as $b \rightarrow \infty, X_{t}$ converges to the Lévy process that has distribution (1) at $t=1$. Our theorem shows that, if $t$ is an integer, then non-unimodality of the distribution of $X_{t}$ implies strictly bimodal.

Before proceeding to the proof of theorem we shall state important two lemmas. In counting the number of changes of sign of a finite sequence $a_{0}, a_{1}, a_{2}, \ldots, a_{m}$, or an infinite sequence $a_{0}, a_{1}$, $a_{2}, a_{3}, \ldots$, we disregard zero terms (see [2], p. 36).

We can find the following lemma in [2], p. 41.

Lemma 1 (Extension of Descartes' rule of signs to power series). Let the radius of converg. ence of the power series $\sum_{l=0}^{\infty} A_{l} x^{l}$ be $\rho$. Then the number of its zeros in $(0, \rho)$ does not exceed the number of changes of sign of its coefficients. Here we count the zeros according to their multiplicity.

We can find a proposition including the following lemma in [2], p. 41.

Lemma 2. Suppose that

$$
\sum_{l=0}^{m} a_{l} \frac{\lambda^{l}}{(\alpha-\lambda)^{l}}=\sum_{l=0}^{\infty} A_{l} \lambda^{l}
$$

with $\alpha>0$. Then the number of changes of sign of $\left\{A_{l}\right\}_{l \geq 0}$ does not exceed the number of changes of sign of $\left\{a_{l}\right\}_{l=0,1, \cdots, m}$.

The distribution of $X_{n}$ has the following density $f_{n}(x)$ :

$$
\begin{aligned}
& f_{n}(x)=e^{-a x}\left[(1-p)^{n} a^{n} \frac{x^{n-1}}{(n-1)!}\right. \\
& +\sum_{l=0}^{n-2} x^{l} \frac{(n-2-l)!}{(b-a)^{n-l-1}} \sum_{j=l+1}^{n-1}\binom{n}{j} p^{n-j}(1-p)^{j} \\
& \left.\quad \times \frac{a^{j} b^{n-j}}{(j-1)!(n-j-1)!}(-1)^{j-1-l}\binom{j-1}{l}\right] \\
& \quad+e^{-b x}\left[p^{n} b^{n} \frac{x^{n-1}}{(n-1)!}+\sum_{l=0}^{n-2} x^{l} \frac{(n-2-l)!}{(a-b)^{n-l-1}}\right.
\end{aligned}
$$

$$
\times \sum_{j=l+1}^{n-1}\binom{n}{j}(1-p)^{n-j} p^{j}
$$

$$
\left.\times \frac{b^{j} a^{n-j}}{(j-1)!(n-j-1)!}(-1)^{j-1-l}\binom{j-1}{l}\right]
$$

This is proved by induction. We denote by $g^{(l)}(x), l \in \boldsymbol{N}$, the $l$-th derivative of a function $g(x)$.

Proof of Theorem. Set

$$
F_{n}(s)=\int_{0}^{\infty} e^{-\frac{x}{s}} f_{n}(x) d x
$$

Then

$$
F_{n}(s)=\left((1-p) \frac{a}{a+(1 / s)}+p \frac{b}{b+(1 / s)}\right)^{n}
$$

We have

$$
\begin{aligned}
& F_{n}\left(\frac{\lambda}{b(1-\lambda)}\right) \\
& \quad=\frac{\lambda}{b(1-\lambda)}\left(f_{n}(0)+\int_{0}^{\infty} e^{-\frac{b}{\lambda} x} e^{b x} f_{n}^{\prime}(x) d x\right)
\end{aligned}
$$

Here we used the form of $f_{n}(x)$. Set $h_{n}(x)=e^{b x}$ $f_{n}^{\prime}(x)$. In order to study modality of the distribution of $X_{n}$, we look at the number of zeros of $h_{n}(x)$ in $(0, \infty)$. Since $h_{n}(x)$ is analytic, Lemma 1 says that it is enough to look at the number of changes of sign of $\left\{h_{n}^{(l)}(0)\right\}_{l \geq 0}$. Now we consider the power series

$$
(1-\lambda) F_{n}\left(\frac{\lambda}{b(1-\lambda)}\right)=\sum_{l=0}^{\infty} A_{l} \frac{\lambda^{l}}{l!}
$$

Use integration by parts repeatedly. Then,

$$
\begin{gathered}
\sum_{l=0}^{\infty} A_{l} \frac{\lambda^{l}}{l!}=\frac{\lambda}{b} f_{n}(0)+\left(\frac{\lambda}{b}\right)^{2} h_{n}(0)+ \\
\cdots+\left(\frac{\lambda}{b}\right)^{l} h_{n}^{(l-2)}(0)+\left(\frac{\lambda}{b}\right)^{l} \int_{0}^{\infty} e^{-\frac{b}{\lambda} x} h_{n}^{(l-1)}(x) d x
\end{gathered}
$$

Differentiate both sides $l$ times, and let $\lambda \rightarrow 0$. Then

$$
\begin{aligned}
A_{l}=\frac{l!}{b^{l}} h_{n}^{(l-2)}(0)+ & \lim _{\lambda \rightarrow 0} \frac{d^{l}}{d \lambda^{l}}\left(\left(\frac{\lambda}{b}\right)^{l} \int_{0}^{\infty} e^{-\frac{b}{\lambda} x} h_{n}^{(l-1)}(x) d x\right) \\
& =\frac{l!}{b^{l}} h_{n}^{(l-2)}(0)
\end{aligned}
$$

Here we used the form of $h_{n}^{(l-1)}(x)$. Hence the number of changes of sign of $\left\{h_{n}^{(l)}(0)\right\}_{l \geq 0}$ is equal to the number of changes of sign of $\left\{A_{l}\right\}_{l \geq 2}$. On the other hand,

$$
\begin{aligned}
(1-\lambda) & F_{n}\left(\frac{\lambda}{b(1-\lambda)}\right) \\
& =\lambda^{n}(1-\lambda)\left(\frac{(1-p) a}{b-(b-a) \lambda}+p\right)^{n} \\
& =p^{n} \lambda^{n}(1-\lambda)\left(\frac{\beta}{\alpha-\lambda}+1\right)^{n}
\end{aligned}
$$

where $\alpha=\frac{b}{b-a}, \beta=\frac{(1-p) a}{p(b-a)}$. Now notice that
$(1-\lambda)\left(\frac{\beta}{\alpha-\lambda}+1\right)^{n}=\left(1-\frac{\lambda}{\alpha}\right)$

$$
\begin{aligned}
& \times\left(1-(\alpha-1) \frac{\lambda}{a-\lambda}\right)\left(\frac{\beta}{\alpha}+1+\frac{\beta}{\alpha} \frac{\lambda}{\alpha-\lambda}\right)^{n} \\
&=\left(1-\frac{\lambda}{\alpha}\right)\left[\left(\frac{\beta}{\alpha}+1\right)^{n}+\sum_{l=1}^{n}\left(\frac{\beta}{\alpha}+1\right)^{n-l}\right. \\
& \times\left(\frac{\beta}{\alpha}\right)^{l-1}\left(\frac{\lambda}{\alpha-\lambda}\right)^{l} \frac{n!}{(n-l+1)!l!} \\
& \times\left\{(n-l+1) \frac{\beta}{\alpha}-(\alpha-1) l\left(\frac{\beta}{\alpha}+1\right)\right\} \\
&=\left(\frac{\beta}{\alpha}+1\right)^{n}+\lambda\left[\left(\frac{\beta}{\alpha}+1\right)^{n-1}\left(\frac{\beta}{\alpha}\right)^{n}\left(\frac{\lambda}{\alpha-\lambda}\right)^{n+1}\right] \\
& \times \frac{1}{\alpha}\left\{-\alpha\left(\frac{\beta}{\alpha}+1\right)+n \frac{\beta}{\alpha}\right\} \\
& \times \sum_{l=1}^{n-1}\left(\frac{\beta}{\alpha}+1\right)^{n-l-1}\left(\frac{\beta}{\alpha}\right)^{l} \frac{n!}{(n-l)!(l+1)!} \frac{1}{\alpha} \\
& \times\left\{(n-l) \frac{\beta}{\alpha}-(\alpha-1)(l+1)\left(\frac{\beta}{\alpha}+1\right)\right\} \\
&\left.\times\left(\frac{\lambda}{\alpha-\lambda}\right)^{l}-\frac{\alpha-1}{\alpha}\left(\frac{\beta}{\alpha}\right)^{n}\left(\frac{\lambda}{\alpha-\lambda}\right)^{n}\right] .
\end{aligned}
$$

Since $\alpha>1$ and $\beta>0$, the coefficients of $\left(\frac{\lambda}{\alpha-\lambda}\right)^{l}, l=0,1, \ldots, n$, in the brackets in the last expression change sign at most twice. Now we can apply Lemma 2. We see that $A_{l}=0$ for 0 $\leq l \leq n-1, A_{n}>0$, and the sequence $\left\{A_{l}\right\}_{l \geq}$ ${ }_{n+1}$ changes sign at most twice. Therefore, $f_{n}^{\prime}(x)$ has at most three zeros in ( $0, \infty$ ). Hence $X_{n}$ is either strictly unimodal or strictly bimodal.

$$
\text { Suppose that } n \geq \frac{b}{b-a}\left(1+\frac{b}{a} \frac{p}{1-p}\right)
$$ which is equivalent to

$$
-\alpha\left(\frac{\beta}{\alpha}+1\right)+n \frac{\beta}{\alpha} \geq 0
$$

Then we see that $A_{n+1} \geq 0$ and that $\left\{A_{l}\right\}_{l \geq n+1}$ changes sign at most once. We see that, for $n \geq$ $2, f_{n}(0)=0$ and $f_{n}^{\prime}(x)$ has only one zero in $(0, \infty)$. This completes the proof of Theorem.

Corollary. The distribution of $X_{n}$ is strictly unimodal for every $n \in \boldsymbol{N}$ if $\frac{a(b-2 a)}{b^{2}} \geq \frac{p}{1-p}$.

Proof. By the latter half of Theorem, the distribution of $X_{n}$ is strictly unimodal for every $n \geq 2$, if

$$
2 \geq \frac{b}{b-a}\left(1+\frac{b}{a} \frac{p}{1-p}\right)
$$

This condition is equivalent to $\frac{a(b-2 a)}{b^{2}} \geq$ $\frac{p}{1-p}$.

Remark. If $n \leq \frac{1}{b-a}\left(a+b+2 b \frac{p}{1-p}\right)$, then the distribution of $X_{n}$ is strictly unimodal. In fact, in this case $(n-1) \frac{\beta}{\alpha}-(\alpha-1) 2$ $\left(\frac{\beta}{\alpha}+1\right) \leq 0$ and $\left\{A_{l}\right\}_{l \geq n}$ changes sign at most once.

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