On Models of Default Risk

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Abstract

We first discuss some mathematical tools used to compute the intensity of a single jump process, in its canonical filtration. In the second part, we try to clarify the meaning of default and the links between the default time, the asset's filtration, and the intensity of the default time. We finally discuss some examples.

1 Introduction

The problem of modeling a default time is well represented in the literature. There are two main approaches: either the default time τ is a stopping time in the asset's filtration, or is a stopping time in a larger filtration (see Cooper and Martin (1996) for a comparison between these approaches). The payoff ζ which is a positive random variable is promised at fixed time T in a default free framework. The defaultable payoff is ζ if the default has not appeared before payment time T and 0 otherwise.

In the first approach, the so-called structural form, pionered by Merton (1974), the default time τ is a stopping time in the filtration of the prices. Therefore, valuing the defaultable claim reduces to the pricing problem of the claim $\zeta \mathbb{1}_{T<\tau}$ which is measurable with respect to the prices' filtration taken at time T. We do not address this problem here.

In the second case, the idea is also to compute the value of the defaultable claim $\zeta \mathbb{1}_{T<\tau}$. However it may happen that this claim is not measurable with respect to the σ-algebra generated by prices up to time T. In this case, it is generally assumed that the defaultable market is complete, which means that the defaultable claim is hedgeable. In order to compute the expectation of $\zeta \mathbb{1}_{T<\tau}$ under the risk-neutral probability, it is fruitful to introduce the notion of intensity of the default. Then, under some assumptions, the intensity of the default time acts as a change of the spot interest rate in the pricing formula.

We proceed in a different way, and try to understand the links between a "default-free" world and a defaultable one. We recall some well known, though perhaps forgotten, tools to compute this expectation and simplify most of the proofs in the mathematical finance literature. We make precise the relation between the default time and the price's filtration.

In the first part we recall that if the information is only the time when the default appears, the computation of the expectation of a defaultable payoff involves the intensity of the default process $\mathbb{1}_{\tau \le t}$ which can be explicitly defined in terms of the distribution function of τ. We discuss a result of Duffie and Lando and give a simpler form of the intensity of the hitting time.

In a second part, we assume that the information of the agent at time t consists of knowledge of the behavior of the prices up to time t as well as the default time. We show that, in this case, the results depend strongly on the stochastic link between the asset process and the default time. In particular, we show that the intensity does not provide sufficient information about this stochastic link. We use some tools from the theory of enlargement of filtrations to compute the intensity of the default time when

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it exists. In a final section, we give an example where the usual assumptions made in the literature are not satisfied and where the value of a default claim is not obtained by a change of spot rate. The appendix gives a new proof of the Duffie and Lando result quoted in section 2.

A detailed version of this paper including all the proofs is available upon request to M. Jeanblanc.

In this paper, we make precise the choice of the filtration and we shall say that, for a given filtration $\mathbf{J} = (\mathcal{J}_t, t \geq 0)$, the $\mathbf{J}$-adapted nonnegative process $(\lambda_s, s \geq 0)$ is the $\mathbf{J}$-intensity of τ if $(1\!\!1_{\tau \leq t} - \int_0^{\tau \wedge t} \lambda_s ds\,,\, t \geq 0)$ is a $\mathbf{J}$-martingale. There is no uniqueness of the intensity after τ: in fact, there is no meaning for the "intensity " after τ, even if it is sometimes mentioned.

2 The single jump process in its own filtration

2.1 An elementary martingale

We start with some well known facts established for the first time in Dellacherie (1970). Suppose that τ is an $\overline{I\!\!R}_+^*$-valued random variable on some probability space $(\Omega, \mathcal{G}, P)$. Let $F(t) = P(\tau \leq t)$ be the right-continuous distribution function of τ. We denote by $F(t-) = P(\tau < t)$ the left limit of $F(s)$ as s approaches t. We assume that $F(0) = 0$ and $F(t) < 1, \forall t > 0$. Denote by $(N_t\,; t \geq 0)$ the default process defined as the right-continuous increasing process $N_t = 1\!\!1_{\tau \leq t}$ and by $\mathbf{H} = (\mathcal{H}_t)$ its natural filtration $\mathcal{H}_t = \sigma(N_u, u \leq t)$, completed as usual with negligeable sets. The σ-algebra $\mathcal{H}_t$ is generated by the sets $\{\tau \leq s\}$ for $s \leq t$ and the atom $\{t < \tau\}$; hence $\mathbf{H}$ is the smallest filtration satisfying the usual hypotheses such that τ is an $\mathbf{H}$-stopping time. Any random variable H which is $\mathcal{H}_t$-measurable is of the form $H = h(\tau) 1\!\!1_{\tau \leq t} + \widetilde{h} 1\!\!1_{t < \tau}$, where h is a Borel function defined on $[0, t]$ and $\widetilde{h}$ a constant. We borrow from Dellacherie the following results:

Proposition 1 *If X is any integrable, $\mathcal{G}$-measurable random variable, then*

$$E(X|\mathcal{H}_t) = 1\!\!1_{\tau \leq t} E(X|\mathcal{H}_\infty) + 1\!\!1_{t < \tau} \frac{E(X 1\!\!1_{t<\tau})}{P(t < \tau)}\,.$$

In particular,

$$E(X|\mathcal{H}_t) 1\!\!1_{t<\tau} = 1\!\!1_{t<\tau} \frac{E(X 1\!\!1_{t<\tau})}{P(t < \tau)}\,, \tag{1}$$

and if X is $\sigma(\tau)$-measurable, i.e., $X = h(\tau)$, then $E(X|\mathcal{H}_t) = 1\!\!1_{\tau \leq t} h(\tau) + 1\!\!1_{t<\tau} \frac{E(h(\tau) 1\!\!1_{t<\tau})}{P(t < \tau)}$.
The process $(M_t\,; t \geq 0)$, where

$$M_t \stackrel{def}{=} N_t - \int_{]0, \tau \wedge t]} \frac{dF(s)}{1 - F(s-)}\,, \tag{2}$$

is an $(\mathcal{H}_t)$-martingale.

If F is differentiable, then τ admits a density $f = F'$ and $P(\tau \leq s) = 1 - \exp[-\Lambda(s)]$, where $\Lambda(s) = \int_0^s \lambda(u) du$. Here, λ is the deterministic nonnegative function $\lambda(s) = \frac{f(s)}{1 - F(s)}$. Moreover, the process $M_t = N_t - \Lambda(t \wedge \tau)$ is a martingale, the $\mathbf{H}$-intensity of τ is equal to λ and $E(1\!\!1_{T<\tau}|\mathcal{H}_t) = 1\!\!1_{t<\tau} \exp - \int_t^T \lambda(s) ds$.

Let us study the case $\tau = \inf(\tau_1, \tau_2)$ where $(\tau_i, i = 1, 2)$ are independent and admit an $\mathbf{H}_i$-intensity equal to λ_i. The probability distribution of τ is $F(t) = 1 - P(\tau > t) = 1 - (1 - F_1(t))(1 - F_2(t))$. Therefore, the $\mathbf{H}$-intensity of τ is equal to $\frac{f_1(t)}{1 - F_1(t)} + \frac{f_2(t)}{1 - F_2(t)} = \lambda_1(t) + \lambda_2(t)$. If the τ_i are not independent, the intensity of τ cannot be given in an explicit form from the law of each τ_i. Our result differs from Duffie's (1998), since we are not working in the same filtration. See Jeanblanc-Rutkowski (1999) for details.

In the general case, when F is not absolutely continuous, we introduce the hazard function $\Gamma(t) = -\ln(1 - F(t))$. From (1), for $t < T$ for any Borel function h

$$E(h(\tau)\mathbb{1}_{T<\tau}|\mathcal{H}_t) = \mathbb{1}_{t<\tau} \int_t^T h(u) \exp(\Gamma(t) - \Gamma(u))\, d\Gamma(u)\,. \tag{3}$$

In particular, $E(\mathbb{1}_{T<\tau}|\mathcal{H}_t) = \mathbb{1}_{t<\tau} \exp(-[\Gamma(T) - \Gamma(t)])$. We have also obtained the decomposition of the **H**-submartingale N as $N_t = M_t + A_t$, where M is a **H**-martingale and $(A_t = \int_0^{\tau\wedge t} \frac{dF(s)}{1 - F(s-)}\,;\, t \geq 0)$ is a **H**-predictable increasing process, also called the **H**-compensator of N. We shall call a continuous increasing process Λ such that $N_t - \Lambda(t \wedge t)$ is a **H**-martingale the **H**-generalized intensity of τ. If F is continuous, the generalized intensity equals the hazard function. This is not the case if F is discontinuous (see Rutkowski (1999) for a discussion).

2.2 Defaultable claims

Suppose now that, in a financial market, there exists a deterministic spot rate $r(s)$. The present value of a zero-coupon, which pays 1 at time T, is $\exp - \int_0^T r(s)ds$. We assume that some institution has issued a bond whose payoff is obtained at maturity if and only if a default has not appeared. We suppose that the random time τ has a **H**-intensity λ. The t-time expected value of the defaultable zero-coupon which pays $\mathbb{1}_{T<\tau}$ at time T, for an agent who has the information $\mathcal{H}_t$, is

$$E\big(\exp(-\int_t^T r(s)\, ds)\, \mathbb{1}_{T<\tau}|\, \mathcal{H}_t\big) = \mathbb{1}_{t<\tau} \exp\big(-\int_t^T (r(s) + \lambda(s)\,)\, ds\big)\,. \tag{4}$$

This is not really a price, since it is not possible to hedge this default, the only tradeable asset being the riskless one. This explains why we do not compute the expectation under a "risk-neutral" probability, which would be meaningless. Moreover, any change of probability will induce a change of intensity. However, this formula is attractive, since it indicates that the default acts as a change of interest rate. We shall show in section 3 that this is not always the case.

The t-time expected discounted value of the defaultable claim X is $E(X\mathbb{1}_{T<\tau} \exp(-\int_t^T r(s)ds)\,|\mathcal{H}_t)$. If X is independent of τ, this equals $\mathbb{1}_{t<\tau} \exp(-\int_t^T (r(u) + \lambda(u))du)\, E(X)$, which is the value of X in a model where the interest rate is $r + \lambda$. Suppose that the rebate $h(\tau)$ is paid at default time if the default appears before maturity (here, h is a Borel function). From (3), in the case when the default has not appeared at time t, the t-time value of the rebate part is

$$E(h(\tau)\mathbb{1}_{\tau<T} \exp(-\int_t^\tau r(s)ds)\,|\mathcal{H}_t)\mathbb{1}_{t<\tau} = \mathbb{1}_{t<\tau} \int_t^T du \left[h(u)\lambda(u) \exp - \left(\int_t^u (r(s) + \lambda(s))\, ds\right)\right]\,.$$

2.3 Duffie and Lando's result

Consider Duffie-Lando's model (1999) who assume that $\tau = T_0 = \inf\{t\,:\, V_t = 0\}$, where V satisfies

$$dV_t = \mu(t, V_t)dt + \sigma(t, V_t)dW_t\,;\, V_0 = v > 0\,, \tag{5}$$

and where W is a Brownian motion. We assume that V is a strong solution, hence the time T_0 is a stopping time with respect to the Brownian filtration $\mathcal{F}_t = \sigma(W_s, s \leq t)$. Therefore, it is predictable in that filtration and admits no **F**-intensity. We shall discuss this point later.

Here we suppose, as in Duffie and Lando, that the agent will observe default when it happens but will have no knowledge of V before default has occured. In this case, when the default has not yet appeared, section 2.2 proves that the value of a zero-coupon is given in terms of the generalized intensity of N as $\exp -[\Gamma_T - \Gamma_t]$, where $d\Gamma_s = dF(s)/(1 - F(s))$ and $F(s) = P(\tau \leq s)$, as soon as F is continuous.

Duffie and Lando have shown that the intensity of N is $\lambda(t) = \frac{1}{2}\sigma^2(t,0)\frac{\partial \varphi}{\partial x}(t,0)$, where $\varphi(t,x)$ is the conditional density of V_t when $t < T_0$, i.e., the derivative with respect to x of $\frac{P(V_t \leq x, t < T_0)}{P(t < T_0)}$. The equivalence between Duffie-Lando's and our result is obvious. In fact, Duffie and Lando wrote that

$$\lambda_t = \lim_{h \to 0} \frac{1}{hP(t < T_0)} \int_0^\infty P(V_t \in dx, t < T_0) P_x(T_0 < h) = \frac{1}{2}\sigma^2(t,0)\frac{\partial \varphi}{\partial x}(t,0)\,. \tag{6}$$

The middle term of (6) equals $\frac{1}{hP(t < T_0)} P(t < T_0 \leq t+h)$ and tends to $\frac{f(t)}{1 - F(t)}$ as h tends to infinity, which is our result. In fact, the proof that the limit in (6) is $\frac{1}{2}\sigma^2(t,0)\frac{\partial \varphi}{\partial x}(t,0)$ is quite complicated. Duffie and Lando proved this result for Brownian motion, then for the Örnstein-Uhlenbeck process, and finally for more general processes. We establish in the appendix a direct proof for homogeneous diffusions.

3 Stochastic intensity

We now assume that, in addition to τ, a Brownian motion B lives on the space $(\Omega, \mathcal{G}, P)$. We denote by $\mathcal{F}_t = \sigma(B_s, s \leq t)$ its canonical filtration. Let T be a fixed horizon and X be an $\mathcal{F}_T$-measurable integrable random variable.

We are mainly concerned with the general case, where τ is a random time on $(\Omega, \mathcal{G}, P)$, i.e., a positive random variable. We write as in the previous section $N_t = 1\!\!1_{\tau \leq t}$, $\mathcal{H}_t = \sigma(N_s\,;\, s \leq t)$ and $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ the enlarged filtration generated by the pair (B, N). (We denote by $\mathbf{F} = (\mathcal{F}_t)$ the original **F**iltration and by $\mathbf{G} = (\mathcal{G}_t)$ the enlar**G**ed one). We assume that the filtrations are completed and we use their right-continuous version. The choice of **G** as the information filtration is due to the fact that, as in the previous section, we assume that the agent will observe default when it happens. Here, the agent observes also the behavior of the prices and τ is a **G**-stopping time. See Rutkowski (1999) for different models.

In this general setting, the **F**-Brownian motion B is no longer a Brownian motion in the **G** filtration and is not always a **G**-semimartingale[1]. However, $(B_{t \wedge \tau}, t \geq 0)$ remains a semimartingale and we are able, without additional hypotheses, to give the decomposition of any **F**-martingale stopped at time τ as a **G**-semimartingale. We shall give some examples in the next section. Nevertheless, the conditional expectation with respect to $\mathcal{G}_t$ is easy to compute from the expectation with respect to $\mathcal{F}_t$, as long as we restrict our attention to times before τ.

3.1 Conditional expectation

Let $\mathcal{G}_\infty = \mathcal{F}_\infty \vee \sigma(\tau)$, so $\mathcal{G}_t \subset \{A \in \mathcal{G}_\infty \,|\, \exists A_t \in \mathcal{F}_t, A \cap \{t < \tau\} = A_t \cap \{t < \tau\}\ \}$.

Lemma 1 *Let X be a $\mathcal{G}_T$-measurable, integrable random variable. Then, for any $t < T$,*

$$E(X|\mathcal{G}_t) 1\!\!1_{t<\tau} = \frac{E(X 1\!\!1_{t<\tau}|\mathcal{F}_t)}{E(1\!\!1_{t<\tau}|\mathcal{F}_t)} 1\!\!1_{t<\tau}\,. \tag{7}$$

PROOF: This result is obvious, from the remarks on the $\mathcal{G}_t$-mesurable sets. In particular, we obtain $E(X 1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{t<\tau} \frac{E(X 1\!\!1_{T<\tau}|\mathcal{F}_t)}{E(1\!\!1_{t<\tau}|\mathcal{F}_t)}$. It is easy to check that $E(X 1\!\!1_{T<\tau}|\mathcal{G}_t)$ is equal to 0 on the set $\{\tau \leq t\}$. Indeed,

$$E(X 1\!\!1_{T<\tau}|\mathcal{G}_t) 1\!\!1_{\tau \leq t} = E(X 1\!\!1_{T<\tau} 1\!\!1_{\tau \leq t}|\mathcal{G}_t) = 0\,.$$

Note that formula (7) is a simple generalization of formula (1). It is not easy to compute $E(X|\mathcal{G}_t)$ after time τ, except in the case where for any $G_t \in \mathcal{G}_t$ there exists F_t and $\widetilde{F}_t$ both belonging to $\mathcal{F}_t$ such that $G_t = (F_t \cap \{t < \tau\}) \cup (\widetilde{F}_t \cap \{\tau \leq t\})$. This case occurs when τ is an "honest" random-time (see Jeulin, (1980), page 73).

[1] This fact was alluded to in Hull and White (1995), footnote 4 :" When we move from the vulnerable world to a default free world, the stochastic processes followed by the underlying state variables may change."

3.2 Hazard process

Let us introduce the **F**-optional hazard process Γ defined as $\Gamma_t = -\ln(1 - F_t)$ where F is the submartingale $F_t = P(\tau \le t|\mathcal{F}_t)$. We assume in that paper that $F_t < 1$, for all t. This process contains all the information we need. Indeed, from (7) we obtain easily

Proposition 2 *For any $\mathcal{F}_T$-measurable integrable random variable X, for $t < T$,*

$$E(X 1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{t<\tau} E(X e^{\Gamma_t - \Gamma_T}|\mathcal{F}_t)$$

In particular, $E(1\!\!1_{T<\tau}|\,\mathcal{G}_t) = 1\!\!1_{t<\tau}\, E(\exp[\Gamma_t - \Gamma_T]\,|\,\mathcal{F}_t)\,.$

We emphazise that, in general, Γ is not an increasing process, and not even a bounded variation process. It can be useful to note that, if there exists an increasing **F**-adapted process Δ, such that $E(X 1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{t<\tau} E(X e^{\Delta_t - \Delta_T}|\mathcal{F}_t)$ for any pair $t < T$ and any $X \in \mathcal{F}_T$, then $\Gamma = \Delta$.

3.3 Intensity

Our aim is now to study the **G**-intensity of τ and to give explicit formulae to compute it. The Doob-Meyer decomposition theorem states that there exists a unique **G**-predictable increasing process A (the **G**-compensator) such that $(N_t - A_t, t \ge 0)$ is a **G**-martingale. This implies in particular that A is constant after time τ.

In this paper, we assume that A is continuous. The existence of a continuous compensator makes the choice of τ as an **F**-stopping time impossible. In this case, τ would be a predictable stopping time, as is any stopping time in a Brownian filtration; therefore N would be a predictable process and its predictable compensator would be N itself. Clearly, N is not continuous. Moreover, it is proved in Dellacherie (1972), page 111 that if the **G**-compensator A is continuous, then τ is totally inaccessible in the **G** filtration.

Let us assume that τ admits a **G**-intensity, that is if A is of the form $A_t = \int_0^{t\wedge\tau} \widetilde{\lambda}_s ds$ where $\widetilde{\lambda}$ is a **G**-predictable process. It is well known (Jeulin (1980), p. 63) that if $(\widetilde{k}_t)$ is any **G**-predictable bounded process, then there exists a **F**-predictable bounded process (k_t) such that $\widetilde{k}_t 1\!\!1_{t\le\tau} = k_t 1\!\!1_{t\le\tau}$. Therefore, there exists an **F**-predictable process λ, such that $\widetilde{\lambda}_t 1\!\!1(t \le \tau) = \lambda_t 1\!\!1(t \le \tau)$ and $(N_t - \int_0^{t\wedge\tau} \lambda_s ds,\, t \ge 0)$ is a **G**-martingale. We would like to emphasize that the process λ here is, without any supplementary condition, mesurable with respect to the initial filtration **F**, and is unique up to time τ.

In most examples, the continuous compensator of N is not absolutely continuous with respect to the Lebesgue measure. In the case where $N_t - \Lambda(t \wedge \tau)$ is a **G**-martingale with Λ an **G**-adapted continuous increasing process, we shall say that $d\Lambda$ is the **G**-generalized intensity of N.

The filtration **G** is the filtration **F** enlarged, in a progressive way, by $\sigma(\tau \wedge t)$. Therefore, the process

$$(1\!\!1_{\tau\le t} - \int_{]0,t\wedge\tau]} \frac{1}{Z^\tau_{s-}} dA^\tau_s,\, t \ge 0) \tag{8}$$

is a **G**-martingale (see Yor (1994)). Here A^τ is the **F**-dual predictable projection of the increasing process $N_t = 1\!\!1_{\tau\le t}$, i.e., the **F**-predictable increasing process such that $(E(N_t|\mathcal{F}_t) - A^\tau_t, t \ge 0)$ is a **F**-martingale and Z^τ is the **F**-supermartingale $Z^\tau_t = P(t < \tau|\mathcal{F}_t)$. Therefore, we obtain the following lemma :

Lemma 2 *If A^τ is continuous, the* **G***-generalized intensity of τ is the* **F***-adapted process*

$$d\Lambda_s = \frac{1}{Z^\tau_{s-}} dA^\tau_s = \frac{1}{1 - F_{s-}} dA^\tau_s\,. \tag{9}$$

Let us remark that formula (9) is very similar to the formula (2), and that when $\mathcal{F}_t$ is the trivial σ-algebra for each t, the formula (9) reduces to (2). In the particular case when τ is independent of **F**, we obtain that the **G**-intensity of τ is deterministic and defined as in section 2.

Lemma 3 *If A^τ is constant after τ, then $A_t = \int_0^{t\wedge\tau} \frac{1}{Z^\tau_{s-}} dA^\tau_s$ is equal to A^τ_t.*

Proof: It suffices to prove that the process $(M_t = 1\!\!1_{\tau \le t} - A^\tau_t, t \ge 0)$ is a **G**-martingale. Let H be any **G**-predictable bounded process, and h the **F**-predictable bounded process such that $H_s 1\!\!1_{s\le\tau} = h_s 1\!\!1_{s\le\tau}$. Then,

$$\begin{aligned} E(\int_0^\infty H_s dM_s) &= E(H_\tau - \int_0^\infty H_s dA^\tau_s) = E(H_\tau - \int_0^\tau H_s dA^\tau_s) \\ &= E(h_\tau - \int_0^\tau h_s dA^\tau_s) = E(h_\tau - \int_0^\infty h_s dA^\tau_s) = 0 \end{aligned}$$

Working in the **G** filtration is possible, because the decomposition of any **F**-martingale in this filtration is known up to time τ. For example, if B is an **F**-Brownian motion, its decomposition in the **G** filtration up to time τ is

$$B_{t\wedge\tau} = \beta_{t\wedge\tau} + \int_0^{t\wedge\tau} \frac{d < B, Z^\tau >_s}{Z^\tau_{s-}} ,$$

where $(\beta_{t\wedge\tau} \ , \ t \ge 0)$ is a continuous **G**-martingale with increasing process $(t\wedge\tau)$. If the dynamics of an asset S are given by $dS_t = S_t(r_t dt + \sigma_t dB_t)$ in a default free framework, where B is a Brownian motion under the EMM, its dynamics will be

$$dS_t = S_t(r_t dt + \sigma_t \frac{d < B, Z^\tau >_t}{Z^\tau_{t-}} + \sigma_t d\beta_t)$$

in the default filtration, if we restrict our attention to time before default. Therefore, the default will act as a change of drift term on the prices.

In some examples, Z^τ is a continuous decreasing process. In this case, the bracket $< B, Z^\tau >$ is equal to zero, and the **F**-Brownian motion B remains a Brownian motion in the **G** filtration up to time τ. Therefore, any **F**-martingale is equal to a **G**-martingale up to time τ. Moreover, $A^\tau_t = 1 - Z^\tau_t$ and the hazard process and the intensity process are equal. We shall give some examples below.

3.4 Rebate

The theory of dual predictable projection proves that the process A^τ enjoys the property that for any **F**-predictable bounded process h, $E(h_\tau) = E\left(\int_0^\infty h_s dA^\tau_s\right)$. This property lead to the computation of the rebate's value : $E(1\!\!1_{t<\tau\le T} h_\tau | \mathcal{G}_t) = 1\!\!1_{t<\tau} \frac{1}{Z^\tau_t} E\left(\int_t^T h_s dA^\tau_s | \mathcal{F}_t\right)$. In the very particular case where the hazard process is increasing and absolutely continuous, i.e., $Z^\tau_t = \exp -\Gamma_t$, where $\Gamma_t = \int_0^t \gamma_s ds$, if h is any **F**-predictable process,

$$E(h_\tau 1\!\!1_{t<\tau<T} | \mathcal{G}_t) = 1\!\!1_{t<\tau} E\left(\int_t^T h_s \gamma_s \, e^{\Gamma_t - \Gamma_s} \, ds\right) . \qquad (10)$$

3.5 Examples of stochastic intensity

An example of default time with stochastic intensity is a single jump Cox process, that is, a process $N_t = 1\!\!1_{t\le\tau}$ such that there exists an **F**-predictable process f with $P(\tau \le t | \mathcal{F}_\infty) = \int_0^t f_s ds \stackrel{def}{=} F_t$. In this case it is easy to check that $N_t - \int_0^{t\wedge\tau} \lambda_s ds$ is a **G**-martingale with $\lambda_s = \frac{f_s}{1 - F_s}$ and that $P(\tau \le t | \mathcal{F}_t) = \exp - \int_0^t \lambda_u du$.

As we mentioned above, the default is often modelled from a given non-negative **F**-adapted process $(\lambda_t, t \ge 0)$. In order to construct a process N with intensity λ an easy way is to set

$$\tau = \inf\{t \,;\, \int_0^t \lambda_s ds \ge \Theta\} \qquad (11)$$

where Θ is a random variable with an exponential law of parameter 1, independent of the Brownian motion (constructed on an auxiliary space).

However, such modeling is very restrictive. If the **G**-intensity of the random time τ can be computed, τ is not necessarily defined as in (11), even if it is equal in law to such a random time. For example, random times as in (11) are never $\mathcal{F}_\infty$-measurable. We reinforce this remark by the fact that the intensity does not contain all the information about default time. More precisely, the dependence of τ with respect to **F** is not contained in the intensity process.

3.6 (H)-hypothesis and generalization

Here we assume, as in Kusuoka (1999), the following hypothesis:
(H) Every **F**-square integrable martingale is a **G**-square integrable martingale.

This hypothesis implies that the **F**-Brownian motion remains a Brownian motion in the enlarged filtration. This hypothesis is quite natural, despite its technical form. It was studied in Dellacherie and Meyer (1978), Bremaud-Yor (1978) and Mazziotto-Szpirglas (1979), who proved that it is equivalent to one of the following hypothesis

(H1) $\forall t$, the σ-algebras $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$.
(H2) $\forall F \in \mathcal{F}_\infty, \forall G_t \in \mathcal{G}_t,\ E(FG_t|\mathcal{F}_t) = E(F|\mathcal{F}_t)\, E(G_t|\mathcal{F}_t)$.
(H3) $\forall t, \forall G_t \in \mathcal{G}_t, E(G_t|\mathcal{F}_\infty) = E(G_t|\mathcal{F}_t)$.
(H4) $\forall t, \forall F \in \mathcal{F}_\infty, E(F|\mathcal{G}_t) = E(F|\mathcal{F}_t)$.
(H5) $\forall s \le t,\ P(\tau \le s|\mathcal{F}_\infty) = P(\tau \le s|\mathcal{F}_t)$.

Lemma 4 *Suppose that (H) holds, $F_t = P(\tau \le t|\mathcal{F}_t)$ is continuous and $F_t < 1$. Then, the hazard process Γ is increasing and is the* **G***-generalized intensity of τ, i.e., the process $N_t - \Gamma(t \wedge \tau)$ is a* **G***-martingale.*

Indeed, in this case, $Z_t^\tau = E(t < \tau|\mathcal{F}_\infty)$ is a nonincreasing predictable process and is equal to A_t^τ.

The hypothesis (H5) appears in many papers on default risk, often without any reference to the (H)-hypothesis. For example, in the Madan-Unal paper(1998), the main theorem follows from the fact that (H5) holds (See the proof of B9 in the appendix of their paper). Note also that, if τ is $\mathcal{F}_\infty$-mesurable, and if (H5) holds, then τ is an **F**-stopping time (and does not admit intensity).

It is easy to generalize and prove

Proposition 3 *Suppose that $F_t = P(\tau \le t|\mathcal{F}_t)$ is a continuous increasing process and $F_t < 1$ for any t. Then, the hazard process Γ is increasing and is the* **G***-generalized intensity of τ.*

It seems that all the cases where the hazard function is increasing reduce to the model (11), as is made precise in the following proposition:

Proposition 4 *Suppose that $(F_t, t \ge 0)$ is a continuous increasing process, with $F_t < 1, \forall t$. Let X be any predictable* **F** *process and X^τ the killed process, i.e. $X_t^\tau = X_t 1\!\!1_{t<\tau}$. Define $\tilde{\tau}$ as $\tau = \inf\{t\,;\,\Gamma_t \ge \Theta\}$ on an extension of Ω as in (11) . Then $(X_T^\tau) \overset{law}{=} (X_T^{\tilde{\tau}})$.*

Proof: Let H be any predictable process. Then, from (10)

$$E(H_\tau) = E\int_0^\infty H_s d_s(1\!\!1_{\tau \le s}) = -E\int_0^\infty H_s dF_s = E(H_{\tilde{\tau}})\,.$$

In fact, models such that Γ is increasing are close to (H) hypothesis :

Proposition 5 *Suppose that F is a continuous function, and $F_t < 1, \forall t$. The two following conditions are equivalent*
(a) The process $E(1\!\!1_{\tau \le t}|\mathcal{F}_t)$ is increasing.

(b) If $(Y_t, t \geq 0)$ is an **F***-martingale, then $(Y_{t\wedge\tau}, t \geq 0)$ is a* **G***-martingale.*
In this case, for any **F***-predictable process h*

$$E(h_\tau|\mathcal{G}_t) = 1\!\!1_{\tau\leq t} h_\tau + 1\!\!1_{t<\tau}\, E(\int_t^\infty h(u)\exp[\Gamma(t) - \Gamma(u)]d\Gamma(u)\,|\mathcal{F}_t).$$

PROOF: If (a) holds, then the process $Z_t^\tau = E(1\!\!1_{t<\tau}|\mathcal{F}_t)$ is decreasing. Therefore, the **F**-Brownian motion B remains a **G**-Brownian motion up to time τ, and (b) holds. If (b) holds, the bracket $< B, Z^\tau >$ is equal to zero. This implies that the martingale part of Z^τ is equal to zero, therefore Z^τ is a non-increasing process and Γ is increasing.

Duffie et al. (1996) assume that λ is the **J**-intensity of the $\mathcal{J}$-stopping time τ and that X is an integrable $\mathcal{J}_T$-measurable random variable. Under the hypothesis that there exists an extension λ^* of λ (i.e., $\lambda_s^* 1\!\!1_{s<\tau} = \lambda_s 1\!\!1_{s<\tau}$) such that the process $E(X\exp - \int_t^T \lambda_u^* du|\mathcal{J}_t)$ is continuous at time τ, they establish that

$$E(X 1\!\!1_{T<\tau}|\mathcal{J}_t) = 1\!\!1_{t<\tau} E(X\exp - \int_t^T \lambda_u^* du|\mathcal{J}_t)\,. \tag{12}$$

This result differs from ours, since in (12) the σ-algebra is the same in both sides, whereas we work with the enlarged filtration and the initial one.

3.7 Information

Suppose that the market is $\mathcal{F}_T$ complete, i.e., any $\mathcal{F}_T$ random variable is hedgeable and the price of any contingent claim $\zeta \in \mathcal{F}_T$ is $E_Q(\zeta|\mathcal{F}_t)$, where Q is the E.M.M.. If the default time is such that $\{T < \tau\} \in \mathcal{F}_T$ (see the following section), we are led to distinguish two kinds of agents. The uninformed agent does not know the time when the default occurs and will price the defaultable contingent claim as $C_t^{un} = E_Q(\zeta 1\!\!1_{T<\tau}|\mathcal{F}_t)$. If the informed agent, who observes the default time, prices the defaultable claim as $C_t^{in} = E_Q(\zeta 1\!\!1_{T<\tau}|\mathcal{G}_t)$, the spread between the two prices is, if the default has not yet appeared

$$C_t^{in} - C_t^{un} = E_Q(\zeta 1\!\!1_{T<\tau}|\mathcal{F}_t)\left(\frac{1}{E_Q(1\!\!1_{t<\tau}|\mathcal{G}_t)} - 1\right)$$

which is non-negative. This can be interpreted as the value of the information. However, in such a model the informed agent acts as an insider as soon as the default appears.

4 Examples: Last passage times

Our aim is to study some particular examples where the hazard function is not increasing and to give tools to compute this function and the intensity. Other examples can be found in Kusuoka (1999) and in Jeanblanc (1999).

We suppose that the dynamics of the value of a firm are

$$dV_t = \mu(V_t)dt + \sigma(V_t)dB_t,\ V_0 = v > 0\,. \tag{13}$$

The filtration of the Brownian motion B will be denoted by **F**. We recall that a scale function for the diffusion V is a function s such that $s(V)$ is a local martingale.

4.1 Value of defaultable claims

For simplicity, we assume that the interest rate r is equal to 0. Our aim is to compute the value of a contingent claim with payoff $G(V_T)$, where T is a fixed time, if the default has not appeared before time T. This payoff is made at time T or later (see below). The case where a payment of h_τ is made at time τ if the default has appeared before T and where h is some given **F**-predictable process is also taken into account. In this setting, the value of the defaultable claim is the expectation of

$$G(V_T)1\!\!1_{T<\tau} + h_\tau 1\!\!1_{\tau\leq T}\,.$$

The value of the defaultable claim for a **G** informed agent is made of two parts. The value of the terminal payoff is

$$E(G(V_T)\mathbb{1}_{T<\tau}|\mathcal{G}_t) = \mathbb{1}_{t<\tau}E(G(V_T)e^{\Gamma_t-\Gamma_T}|\,\mathcal{F}_t)\,.$$

In the particular case where $e^{-\Gamma_t} = Z_t^\tau = \psi(V_t)$, we obtain, thanks to Markov property $E(G(V_T)e^{-\Gamma_T}|\mathcal{F}_t) = \Psi(V_t, T-t)$ with $\Psi(x,u) = E(G(V_u^x)\psi(V_u^x))$ where V^x is the solution of (13) with initial value x. The value of the rebate part is, if the default has not appeared before t

$$E(h_\tau \mathbb{1}_{\tau<T}|\,\mathcal{G}_t)\mathbb{1}_{t<\tau} = \mathbb{1}_{t<\tau}e^{\Gamma_t}E(\int_t^T h_s dA_s^\tau|\mathcal{F}_t)$$

In many examples, the process A^τ involves a local time $L^a(V)$. In the particular case $h_s = h(V_s)$, the computation of the rebate part is easy, because $\int_0^T h(V_s)dL_s^a(V) = h(a)L_T^a(V)$.

4.2 Last passage at a fixed level

Let V be a transient diffusion and s a scale function such that $s(-\infty) = 0$ and $s(x) > 0$. Let $\tau = \gamma_a \stackrel{def}{=} \sup\{t : V_t = a\}$ the last passage at the level a.

Lemma 5 *We have* $Z_t^\gamma = P_v(t < \gamma_a|\mathcal{F}_t) = \frac{s(V_t)}{s(a)} \wedge 1$ *and the dual predictable projection of* $N_t = \mathbb{1}_{\gamma_a \le t}$ *is* $A_t^\gamma = \frac{1}{2s(a)}L_t^{s(a)}$ *where* L *is the local time of the continous semimartingale* $s(V)$.

These results are well known (see for example, Pitman-Yor (1980), or Yor (1997) p.48).

We make explicit the computation of the value of a defaultable claim in the particular case where $V_t = v + \mu t + \sigma W_t$, with $\mu < 0$. In this case, the scale function is $s(x) = \exp -\frac{2\mu x}{\sigma^2}$.
Note that $z \wedge 1 = 1 - (1-z)^+$. Then, $E(G(V_T)Z_T^\tau|\mathcal{F}_t) = \Psi(V_t, T-t)$, where

$$\begin{aligned}\Psi(x,u) &= E\left[G(x+\mu u+\sigma W_u)\left(1-(1-\frac{1}{s(a)}s(x+\mu u+\sigma W_u))^+\right)\right]\\ &= E[G(x+\mu u+\sigma W_u)] - E\left[G(x+\mu u+\sigma W_u)\left(1-\frac{1}{s(a)}\exp-\frac{2\mu}{\sigma}(x+\mu u+\sigma W_u)\right)^+\right]\end{aligned}$$

In the case of a defaultable zero coupon with zero recovery, the computation reduces to a European claim case. Indeed, from the above computations

$$E(\mathbb{1}_{T<\gamma}|\mathcal{G}_t) = \mathbb{1}_{t<\gamma}\frac{s(a) - E((s(a)-s(V_T))^+|\mathcal{F}_t)}{s(V_t)\wedge s(a)}$$

and the value of $E((s(a)-s(V_T))^+|\mathcal{F}_t) = E((s(a) - \exp(-\frac{2\mu}{\sigma^2}(v+\mu T+\sigma B_T)^+|\mathcal{F}_t)$ is given as a put's price. In the other cases, even though the computations are involved, they require only the law of a normal variable.

4.3 Last passage at a fixed level before bankruptcy

We suppose that $\tau = g_{T_0}^a(V) = \sup\{t \le T_0 : V_t = a\}$ with $v > a$. Consider the case when $\mu = 0$ and σ is a non-negative constant. Then, $E(g_{T_0}^a(V) \le t|\mathcal{F}_t) = P(d_t^\alpha(W) > T_0(W)|\mathcal{F}_t)$ on the set $\{t < T_0(W)\}$, where $\alpha = \frac{v-a}{\sigma}$. It is easy to prove that $P(d_t^\alpha(W) < T_0(W)|\mathcal{F}_t) = \Phi(\sigma W_{t\wedge T_0(W)})$ where

$$\Phi(x) = P_x(T_\alpha(W) < T_0(W)) = \frac{x}{\alpha} \text{ for } 0 < x < \alpha$$

and $\Phi(x) = 1$ for $\alpha < x$; $\Phi(x) = 0\,\text{for}\; x < 0$.

Then, on the set $t < T_0(V)$, we have $Z_t^\tau = E(g^a_{T_0} \le t|\mathcal{F}_t) = \frac{(\alpha - B_{t\wedge T_0})^+}{\alpha} = \frac{(\alpha - B_t)^+}{\alpha}$. We deduce that the dual predictable projection of the process $N = 1\!\!1_{\{g^\alpha_{\tau_0} \le t\}}$is $A_t^g = \frac{1}{2\alpha} L_t^\alpha(W)$.

4.4 Last passage at a fixed level before maturity

This firm operates until time $T + \varepsilon = \theta$, where T is a fixed time, and promises to pay to the investors at date θ the amount $G(V_T)$. If the value of the firm remains above a level a, where $a < v$ between T and θ, the firm defaults and the payment of $G(V_T)$ at time θ is not made. In this setting, the value of the defaultable claim with a rebate given by the process h is the expectation of

$$G(V_T) 1\!\!1_{T<\tau} + G(V_T) 1\!\!1_{V_T \ge a} 1\!\!1_{\tau \le T} + h_\tau 1\!\!1_{V_T \le a} 1\!\!1_{\tau \le T}$$

where $\tau = \sup\{t \le \theta : V_t = a\} = g^a_\theta(V)$. If $V_t > a, \forall t \le \theta$, we set $\tau = \theta$. In what follows, we set $\theta = 1$.

If the time ε, which plays the role of a delay, is equal to 0, the payoff is an $\mathcal{F}_1$-measurable claim equal to $G(V_1) 1\!\!1_{V_1 > a} + h_g 1\!\!1_{V_1 < a}$, and the computation is easy, at least for the terminal payoff part. We shall explain below how to compute the rebate part.

We present here only the Brownian motion case, i.e., when $V_t = v + B_t$. The more general computation, in case where the process V is a Brownian motion with constant drift are more involved. They are available upon request and are published in Jeanblanc (1999).

The computation of the **H**-intensity is easy, since we need only the probability distribution of

$$\tau = \sup\{t \le 1\,;\, V_t = a\} = g^\alpha \stackrel{def}{=} \sup\{t \le 1\,;\, B_t = \alpha\}$$

where $\alpha = a - v$. It is well known that g^0 follows a arcsine law, and the probability density of g^α given in (Yor (1995), formula (3.b)), is $P(g^\alpha \in du) = \frac{du}{\pi\sqrt{u(1-u)}} \exp -\frac{\alpha^2}{2(1-u)}$. Note that the right-hand side is a subprobability, and that the missing mass is $P(g^\alpha = 1) = P(T_\alpha \ge 1) = P(|N| \le \alpha)$. The intensity of g^α in the **H** filtration follows from proposition 1.

For the computation of the **G**-intensity, we need the following (see Yor 1994 and 1997)

Lemma 6 *The dual predictable projection of* $N_t = 1\!\!1_{g^\alpha \le t}$ *is* $dA_t^g = \sqrt{\frac{2}{\pi}} \frac{dL_t^\alpha}{\sqrt{1-t}}$ *where* L^α *is the local time of the Brownian motion at level* α*,and* $E(1\!\!1_{g \le t}|\mathcal{F}_t) = \Phi\left(\frac{|\alpha - B_t|}{\sqrt{1-t}}\right)$ *where* $\Phi(x) = \sqrt{\frac{2}{\pi}} \int_0^x du \exp(-\frac{u^2}{2})$

We present now the computation of the value of the terminal payoff, which is given in terms of $E(G(B_T) 1\!\!1_{T<g}|\mathcal{F}_t)$, for $t < T < 1$. After first conditioning with respect to $\mathcal{F}_T$, we obtain

$$E(G(B_T) 1\!\!1_{T<g}|\mathcal{F}_t) = E(G(B_T)|\mathcal{F}_t) - E\left(G(B_T)\Phi\left(\frac{|\alpha - B_T|}{\sqrt{1-T}}\right)|\mathcal{F}_t\right)$$

then computation can be done using the Markov property: $E(G(B_T)|\mathcal{F}_t) = \widehat{G}(B_t)$ where $\widehat{G}(y) = E(G(B_{T-t} + y))$ and $E\left(G(B_T)\Phi\left(\frac{|\alpha - B_T|}{\sqrt{1-T}}\right)|\mathcal{F}_t\right) = \widehat{G\Phi}(B_t)$, where

$$\widehat{G\Phi}(y) = E\left(G(B_{T-t} + y)\Phi\left(\frac{|B_{T-t} + y - \alpha|}{\sqrt{1-T}}\right)\right) = \frac{1}{\sqrt{2\pi(T-t)}} \int G(u)\Phi\left(\frac{|u - \alpha|}{\sqrt{1-T}}\right) \exp -\frac{(u-y)^2}{2(T-t)} du\,.$$

The computation for the rebate part, in the particular case $h_s = h(V_s)$ follows from

$$E(\int_t^T h_s dA_s^g|\mathcal{F}_t) = h(a)E(A_T^g - A_t^g|\mathcal{F}_t) = h(a)[E(Z_T^g|\mathcal{F}_t) - Z_t^g]$$

which can be done using Markov property.

4.5 Information

Suppose that the value of the firm is an asset, or that there is an asset such that the market is $\mathcal{F}_T$ complete, i.e. any $\mathcal{F}_T$ random variable is hedgeable. Suppose that, as in the last example, the default time is $\mathcal{F}_T$ measurable. In this case, if the information of an agent is the filtration $\mathcal{G}_t$, this agent will be an inside trader. In an obvious way, if the agent observes the last passage time at a fixed level, as soon as this time is revealed, he will know that the price will stay below (or above) this level. Investigating this kind of information is work in progress.

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5 Appendix

We give here the details of the alternative proof mentioned in Duffie and Lando's paper (1999). Two other approaches to this result can be found in Kusuoka (1999) and Song (1997).
In fact, we obtain a more general result than Duffie and Lando's, and we establish in particular the following proposition

Proposition 6 *For any differentiable and integrable function f, such that $f(0) = 0$,*

$$\lim_{h\to 0} \frac{1}{h}\int_0^\infty dx f(x)P_x(T_0 < h) = -\frac{1}{2}f'(0)$$

where $T_0 = \inf\{t|X_t = 0\}$, and P_x denotes the family of laws of a diffusion satisfying $X_t = x + B_t + \int_0^t du\mu(X_u)$, i.e., X is a diffusion with unit diffusion coefficient and drift μ.

We begin by establishing a lemma in a general setting. Consider a pair of diffusions and associated probability measures $((X_t, t \geq 0), P_x)$ and $((R_t, t \geq 0), \widehat{P}_x)$ such that

(i) under P_x, $(X_{T_0-t}, t \leq T_0)$ is distributed as $(R_t, t \leq \gamma_x)$ under $\widehat{P}_0$, where $\gamma_x = \sup\{t : R_t = x\}$

(ii) $R_t \overset{t\to\infty}{\to} \infty$

(iii) $\widehat{E}_x [F(R_u, u \leq t)] = \dfrac{1}{s(x)}E_x [F(X_u, u \leq t)s(X_{t\wedge T_0})]$ where s is a scale function for X, so that $r = -1/s$ is a scale function for R.

Consider the infinitesimal generators of (X) and (R) written in their Feller form

$$L = \frac{1}{2}\frac{d}{dm}\left(\frac{d}{ds}\right), \; \widehat{L} = \frac{1}{s}L(s\,\cdot) = \frac{1}{2}\frac{d}{d\widehat{m}}\left(\frac{d}{dr}\right),$$

where we have used the notation

$$\frac{d}{d\alpha}f(x) = \lim_{\epsilon\to 0}\frac{f(x+\epsilon) - f(x)}{\alpha(x+\epsilon) - \alpha(x)}$$

In the particular case where $dX_t = \sigma(X_t)dB_t + \mu(X_t)dt$ the infinitesimal generator can be written as

$$\begin{aligned} Lf(x) &= \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x) \\ &= \frac{1}{2m'(x)\,s'(x)}f''(x) - \frac{s''(x)}{2m'(x)\,(s')^2(x)}f'(x) \end{aligned}$$

Therefore $s'(x) = c \exp -2 \int_0^x dy \frac{\mu}{\sigma^2}(y)$ and $m'(x)\, s'(x) = \frac{1}{\sigma^2(x)}$.

By identification, we obtain $d\widehat{m}(x) = s^2(x)\, dm(x)$.

Lemma 7 *Let F be a positive functional. Then,*

$$\begin{aligned} \int_0^\infty d\widehat{m}(x) \widehat{E}_0(F(R_u, u \le \gamma_x)) &= \int_0^\infty d\widehat{m}(x) E_x(F(X_{T_0 - u}, u \le T_0)) \\ &= \int_0^\infty dt \widehat{E}_0\left(-\frac{1}{2r(R_t)} F(R_u, u \le t)\right) \end{aligned} \tag{14}$$

In particular, one has

$$\int_0^\infty d\widehat{m}(x) f(x) E_x(\phi(T_0)) = -\frac{1}{2}\int_0^\infty dt \phi(t) \widehat{E}_0(\frac{f(R_t)}{r(R_t)})\,.$$

Proof: From our time reversal assumption

$$\int_0^\infty d\widehat{m}(x) E_x[F(X_{T_0-u}, u \le T_0)] = \int_0^\infty d\widehat{m}(x) \widehat{E}_0[F(R_u, u \le \gamma_x)]$$

Then (14) will be a consequence of the following result (e.g., Yor (1994), 4.5)

$$\widehat{E}_0[F(R_u, u \le \gamma_x)] = \int_0^\infty dt \left(\frac{-1}{2r(x)} \widehat{p}_t(0,x) \widehat{E}_0[F(R_u, u \le t) | R_t = x]\right) \tag{15}$$

where $\widehat{p}_t(0,x)$ denotes the density of $\widehat{P}_t(0,dx)$ with respect to $\widehat{m}(dx)$: $\widehat{P}_t(0,dx) = \widehat{p}_t(0,x)\widehat{m}(dx)$. Then, from the above result

$$\begin{aligned} \int_0^\infty d\widehat{m}(x) E_x(F(X_{T_0-u}, u \le T_0)) &= \int_0^\infty d\widehat{m}(x) \int_0^\infty dt \left(-\frac{1}{2r(x)}\right) \widehat{p}_t(0,x)\, \widehat{E}_0(F(R_u, u \le t) | R_t = x) \\ &= \int_0^\infty dt \int_0^\infty \widehat{P}_t(0,dx) \left(-\frac{1}{2r(x)}\right) \widehat{E}_0(F(R_u, u \le t)|R_t = x) \\ &= \int_0^\infty dt \widehat{E}_0 \left(F(R_u, u \le t)\left(-\frac{1}{2r(R_t)}\right)\right) \end{aligned}$$

In particular, for $F(X_u, u \le t) = \phi(t) f(X_t) s(X_t)$

$$\int_0^\infty dm(x) s(x) f(x) \widehat{E}_0(\phi(\gamma_x)) = \int_0^\infty dm(x) s(x) f(x) E_x(\phi(T_0)) = -\frac{1}{2}\int_0^\infty dt \phi(t) \widehat{E}_0(f(R_t))\,.$$

It follows that $\widehat{E}_0(\phi(\gamma_x)) = -\frac{1}{2r(x)}\int_0^\infty dt \phi(t) \widehat{p}_t(0,x)$ and the law of γ_x is $\widehat{P}_0(\gamma_x \in dt) = dt \frac{-1}{2r(x)} \widehat{p}_t(0,x)$.

Now we consider the case where X is a diffusion of the form $dX_t = dB_t + \mu(X_t)dt$ with B a Brownian motion. We denote by s a scale function of X such that $s(0) = 0, s(+\infty) = \infty$. The properties of the scale function enable us to define the probability measure

$$\widehat{P}_x|_{\mathcal{F}_t} = \frac{s(X_{t\wedge T_0})}{s(x)} P_x|_{\mathcal{F}_t} \tag{16}$$

where $T_0 = \inf\{t > 0\,;\, X_t = 0\}$. From Girsanov's theorem, under $\widehat{P}_x$, the process X satisfies

$$X_t = x + \beta_t + \int_0^t du\, c(X_u) \tag{17}$$

where $c(x) = (\frac{s'}{s} + \mu)(x)$. Moreover, the definition (16) implies that a scale function of X under the $\widehat{P}_x$ family is $r(x) = -\frac{1}{s(x)}$, and $m's' = 1$. For clarity, we now switch back to our notation R to denote the process under $\widehat{P}_x$ and X for the process under P_x. In this general setting, the time reversal property holds, i.e., under P_x, the process $(X_{T_0-t}, t \leq T_0)$ is distributed as the process $(R_t, t \leq \gamma_x)$ under $\widehat{P}_0$ (D. Williams time reversal theorem). From (14), we obtain

$$\widehat{E}_0 \left[\int_0^\infty dx H_{\gamma_x} \right] = -\frac{1}{2} \widehat{E}_0 \left[\int_0^\infty du \, \frac{r'}{r}(R_u) \, H_u \right] . \tag{18}$$

Therefore, in the case where $H_t = \phi(t) f(R_t)$ and with ϕ and f two deterministic functions, the left-hand side of (18) is

$$f(x) \widehat{E}_0 \left[\phi(\gamma_x) \right] = f(x) E_x(\phi(T_0)) \, .$$

From (18)

$$\int_0^\infty dx f(x) E_x(\phi(T_0)) = -\frac{1}{2} \widehat{E}_0 \left[\int_0^\infty du \, \frac{r'}{r}(R_u) \, \phi(u) f(R_u) \right] .$$

Duffie and Lando's result: For $\phi(t) = \mathbb{1}_{t<h}$ we obtain, as a consequence of property 6:

$$\lim_{h \to 0} \frac{1}{h} \int_0^\infty dx f(x) P_x(T_0 < h) = -\lim_{y \to 0} \frac{r'}{2r}(y) f(y)$$

as soon as the right-hand side limit exists. Let us remark that $-\frac{r'}{r}(x) = \frac{s'}{s}(x)$. Suppose that f is differentiable and that $f(0) = 0$. We know that $s'(x) = c \exp -2 \int_0^x dy \mu(y)$. If μ is locally integrable, we obtain $\lim_{x \to 0} \frac{s'}{s}(x) f(x) = f'(0)$.

An alternative proof of (18): It is also possible to give a proof of (18) based on a generalization of Pitman's representation of the $BES(3)$ process (Yor (1997)), which states that the decomposition of R in the enlarged filtration $\mathcal{R}_t^J = \mathcal{R}_t \vee \sigma(J_t)$ where $J_t \stackrel{def}{=} \inf_{s \geq t} R_s$ is

$$R_t = r + \widetilde{B}_t + \int_0^t du \left(\frac{r'}{r}(R_u) + c(R_u) \right) + 2 J_t \, . \tag{19}$$

From (19), we deduce that, for any $(\mathcal{R}_t)$-predictable process H,

$$\widehat{E}_0 \left[\int_0^\infty dJ_u H_u \right] = \frac{1}{2} \widehat{E}_0 \left[\int_0^\infty \left(dR_u - (\frac{r'}{r} + c)(R_u) du \right) H_u \right] . \tag{20}$$

From (17), the right-hand side of (20) equals $-\frac{1}{2} \widehat{E}_0 \left[\int_0^\infty du \frac{r'}{r}(R_u) \, H_u \right]$. By time change on the left-hand side of (20), we obtain

$$\widehat{E}_0 \left[\int_0^\infty dx H_{\gamma_x} \right] = -\frac{1}{2} \widehat{E}_0 \left[\int_0^\infty du \, \frac{r'}{r}(R_u) \, H_u \right] , \tag{21}$$

and the proof is completed.