ON MOMENTS OF ORDER STATISTICS AND QUASI-RANGES FROM NORMAL POPULATIONS

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1. Summary. The main purpose of the paper is to obtain the lower bound on the number of integrals to be evaluated in order to know the first, second and mixed (linear) moments of the normal order statistics (O.S.) in a sample of size N assuming that these moments are available for sample sizes less than N. Towards this, the recurrence relationships, identities, etc. among the moments of the normal order statistics, which have appeared in the literature have been collected with appropriate references. Also, these formulae are listed and stated in the most general form wherever possible. Simple and alternate proofs of some of these formulae are given. These results are also supplemented with new formulae or relationships. It is shown that it is sufficient to evaluate at most one single integral and $(N\text{-}4)/2$ double integrals when N is even and one single integral and $(N\text{-}3)/2$ double integrals when N is odd, in order to know the first, second and mixed (linear) moments of normal O.S. However, for these moments of O.S. in samples drawn from an arbitrary population symmetric about zero, one has to evaluate one more double integral in addition to the number of integrals required in the case of normal O.S. Also, a possible scheme of computing these moments which will be useful especially for small sample sizes, is presented in Section 5.

The lower moments of quasi-ranges in samples drawn from an arbitrary population symmetric about zero are expressed in terms of the moments of the corresponding O.S. Simple recurrence formulae among the expected values of quasi-ranges in samples drawn from an arbitrary continuous population are obtained. A modest list of references is provided at the end which is by no means exhaustive.

2. Introduction. Order statistics (O.S.) have been extensively used in problems on ranges, quasi-ranges, tolerance limits, estimation of location and scale-like parameters, censored samples, selection and ranking problems. Many contributions have been made to the problem of O.S. in normal populations. Tippett [50] gave the first, second, third and fourth moments of the extreme O.S. for a few sample sizes. Hastings et al [19] gave the means, variances, covariances and correlations of O.S. in samples of ten or less from normal populations. In his expository paper, Wilks [53] summarized the results on order statistics and listed all the references up to that time. Jones [24] obtained exact lower moments for small samples together with some relations among them. Godwin [16] recognized some recurrence relations among integrals leading to lower moments of

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O.S. and extended Jones' exact moments to samples of size six and tables of means, etc. of Hastings, et al, to samples of size ten, with more accuracy. Cole [9] obtained a simple recurrence formula among the moments of 'normalized' O.S. Rosser [40] computed the first eight moments of O.S. for samples of size 20 or less from normal populations; provided an asymptotic formula for samples of size more than 20 and furnished still more accurate asymptotic formulae for the means and standard deviations of the extreme observations in a sample. Ruben [41], considering a general class of integrals, showed that the moments of normal O.S. are expressible as linear functions of the contents of certain hyperspherical simplices and tabulated accurately to eight significant digits, the first ten moments of the extreme O.S. in normal samples of size 50 or less. Ruben [43] also showed that the product moments of the extreme O.S. in normal samples of even sizes are expressible as linear functions of the expectations of the extreme O.S. Renyi [36] with the use of a property of the exponential order statistics and the aid of probability integral transformation, studied the O.S. in a sample drawn from an arbitrary population with continuous cumulative distribution function (c.d.f.). He obtained a number of previously known results and test criteria analogous to the Kolmogorov-Smirnov tests. Teichroew [49] computed numerically the first, second and mixed (linear) moments of all O.S. for samples of size 20 or less, up to ten decimal places and gave some relations among integrals. Using Teichroew's tables, Sarhan and Greenberg [44] computed the variances and covariances of the O.S. in samples of sizes up to 20 accurate to ten decimal places. Harter [21] presented a five decimal-place table of the expected values of O.S. for samples of size 2(1) 100 and for sizes, none of whose prime factors exceeded seven, up to 400. Having realized the considerable amount of time Hojo [23] spent in evaluating numerically the integrals closely related to the lower moments of normal order statistics in samples of sizes up to 13, K. Pearson [31] and K. Pearson and M. V. Pearson [32], by expanding the abscissa of a frequency distribution in powers of its c.d.f., obtained approximate formulae for the lower moments of the normal O.S. By comparing his approximate values with those given by Hojo, he inferred that many terms of the series have to be included in order to have sufficient accuracy of the values and that for extreme order statistics in small samples, the approximations are liable to give significant deviations from the correct values. Modifying this idea, David and Johnson [11] expand an order statistic in powers of the deviation of the c.d.f. from the expected value of the order statistic in samples drawn from a uniform population on the unit interval. This approximating series for the moments of order statistics converges faster than the series due to K. Pearson and M. V. Pearson. Similarly, Chu and Hotelling [4] obtained series approximations for the moments of the sample median. In an analogous manner, Clark and Williams [7] derived series approximations for the first four moments of any O.S. and the mean and variance of the product of any two O.S. in a sample drawn from any population such that the inverse of its c.d.f. can be expressed as a Taylor series. Plackett [33], defining the logistic of any continuous distribution as $L = \ln[F/(1 - F)]$ where F denotes the c.d.f. of the population and expanding the ith order statistic in powers of

$L - EL$ (E stands for expectation), obtained approximating series for the expected value of an order statistic. He also gave bounds for the error involved in considering only a certain number of terms of this series. Saw [45], defining a general class of integrals of which the moments of normal O.S. are special cases, obtained approximations for the integrals using the results of David and Johnson. He also tabulated the necessary coefficients occurring in the series approximations for certain integrals. Saw [46] obtained bounds for the error in the series approximations of David and Johnson. He also found that Plackett's bounds are slightly sharper than his, although these differences would be compensated by the computational advantages of the David-Johnson technique over that of Plackett. However, it appears that the series approximation of Plackett unlike that of David and Johnson is applicable to all O.S. including the extreme order statistics. Ludwig [28] obtained a distribution-free upper limit for the expectation of the difference of order statistics in terms of the population standard deviation, the sample size and ranks. He further showed that the expected value of any order statistic in a sample of size N can be computed from the expected values of the smallest value in samples of sizes up to N. The present author [17] studied the O.S. in samples from the positive normal population and gave, besides other results, some recurrence formulae among the moments, especially the product moments of the O.S. in samples drawn from an arbitrary population having a continuous c.d.f. Harter [21] and Srikantan [48] studied the cumulative error propagated by using the recurrence formulae among the moments of O.S., repeatedly.

3. Notation. Let $X_{1,N} \leqq X_{2,N} \leqq \cdots \leqq X_{N,N}$ be the O.S. in a sample of size N drawn from an arbitrary population having a continuous cumulative distribution function (c.d.f.) $F(x)$. Set

$$h_{i,N}(x)\,dx = [N!/(i-1)!(N-i)!]F^{i-1}(x)[1-F(x)]^{N-i}\,dF(x), \qquad i = 1, 2, \cdots, N. \tag{3.1}$$

Also, let X denote the r.v. having $F(x)$ for its c.d.f. and $f(x)$ denotes its probability density function if it exists. Setting

$$h_{i,j,N}(x)\,dx\,dy = \frac{N!}{(i-1)!(j-i-1)!(N-j)!}F^{i-1}(x)[F(y)-F(x)]^{j-i-1} \cdot [1-F(y)]^{N-j}\,dF(x)\,dF(y) \qquad x < y \quad \text{and} \quad 1 \leqq i < j \leqq N, \tag{3.2}$$

we have the well known integrals

$$\mu_{i,N}^{(k)} = E(X_{i,N}^k) = \int_{-\infty}^{\infty} x^k h_{i,N}(x)\,dx, \quad 1 \leqq i \leqq N, \quad k = 1, 2, \cdots, \tag{3.3}$$

and

$$\mu_{i,j,N} = E(X_{i,N}X_{j,N}) = \iint_{-\infty<x<y<\infty} xy h_{i,j,N}(x,y)\,dx\,dy, \qquad 1 \leqq i < j \leqq N. \tag{3.4}$$

It will be convenient to define $\mu_{i,i,N}$ by $\mu_{i,i,N} = \mu_{i,N}^{(2)}$. Also define $\mu_{j,i,N}$ by $\mu_{j,i,N} = \mu_{i,j,N}$. Set

$$\sigma_{i,j,N} = E[(X_{i,N} - \mu_{i,N})(X_{j,N} - \mu_{j,N})]. \tag{3.5}$$

Let $((\mu_{i,j,N}))$ be the product-moment matrix of the vector of O.S. Hereafter, by an arbitrary distribution we mean any continuous distribution for which the corresponding quantities are meaningful. Results are true for all N unless otherwise specified.

4. Recurrence formulae, certain relationships and the minimum number of integrals. We are primarily interested in obtaining a lower bound on the number of integrals to be evaluated in order to know the first, second and the mixed (linear) moments of the normal O.S. Towards this, we list all the known recurrence formulae, relationships etc., among the moments of O.S. and give a few new results, although some of them are extremely trivial, since they will be pertinent for our discussion. Also, the previously known results will be suitably referenced and they will be stated in the most general form and sometimes simpler proofs of these results will be provided. The results of this section could be used for checking numerical values from existing tables of moments of O.S. and for computing some in terms of others. Towards the end of the section, we consider the linear constraints among the lower moments of the normal O.S. and obtain a lower bound on the number of integrals to be evaluated.

THEOREM 4.1. *For an arbitrary distribution, one has*

$$i\mu_{i+1,N}^{(k)} + (N - i)\mu_{i,N}^{(k)} = N\mu_{i,N-1}^{(k)}$$

$$i = 1, 2, \cdots, N - 1, k = 1, 2, \cdots.$$

PROOF. This is a variant of a result due to Cole [9] for the 'normalized' O.S.

COROLLARY 4.1.1. *If the arbitrary distributon is symmetric about the origin, with $i = N/2$, $k = 2$ and N even, Theorem 4.1 gives $\mu_{N/2,N}^{(2)} = \mu_{N/2,N-1}^{(2)}$.*

THEOREM 4.2. *For an arbitrary distribution and for $1 \leqq i \leqq j \leqq N$, one has $(i - 1)\mu_{i,j,N} + (j - i)\mu_{i-1,j,N} + (N - j + 1)\mu_{i-1,j-1,N} = N\mu_{i-1,j-1,N-1}$.*

PROOF. Multiply the integrand in the integral defining $\mu_{i-1,j-1,N-1}$ by unity, write $1 = F(x) + [F(y) - F(x)] + [1 - F(y)]$ and split up the integral as the sum of three integrals. If $i = j$, Theorem 4.2 gives Theorem 4.1 with $k = 2$. Teichroew's [49] Formula 6 is equivalent to Theorem 4.2 for the normal O.S.

THEOREM 4.3. *If the arbitrary distribution is symmetric about the origin, $\mu_{i,N} = -\mu_{N-i+1,N}$, $i = 1, 2, \cdots [N/2]$, and $\mu_{i,j,N} = \mu_{N-j+1,N-i+1,N}$, $1 \leqq i \leqq j \leqq N$.*

PROOF. This result for normal O.S. has been given by Jones [24]. The general result follows from the definition of $\mu_{i,j,N}$ and $F(-x) = 1 - F(x)$

THEOREM 4.4. *For an arbitrary distribution we have $\sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{i,j,N} = N \cdot$ variance X.*

PROOF. Consider the variance of $(X_{1,N} + X_{2,N} + \cdots + X_{N,N})$ which is equal to the variance of $(Y_1 + Y_2 + \cdots + Y_N)$ where the Y_i are the unordered $X_{i,N}$.

THEOREM 4.5. *For an arbitrary distribution and for $r, s \geqq 0$,*

(i) $$\sum_{i=1}^{N} \mu_{i,N}^{(k)} = NE(X^k),$$

and

(ii) $$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} E[X_{i,N}^r X_{j,N}^s] = \binom{N}{2} E[X_{1,2}^r X_{2,2}^s].$$

PROOF FOR (i). See Hoeffding [22] Result II.
PROOF FOR (ii).

$$\text{L.H.S.} = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{N!}{(i-1)!(j-i-1)!(N-j)!} \cdot \iint_{-\infty<x<y\infty} x^r y^s F^{i-1}(x)[F(y) - F(x)]^{j-i-1} \cdot [1 - F(y)]^{N-j}\, dF(x)\, dF(y)$$

$$= \sum_{i=1}^{N-1} \frac{N!}{(i-1)!(N-i-1)!} \cdot \iint_{x<y} x^r y^s F^{i-1}(x)[1 - F(x)]^{N-i-1}\, dF(x)\, dF(y)$$

$$= N(N-1) \iint_{x<y} x^r y^s\, dF(x)\, dF(y) = \binom{N}{2} E[X_{1,2}^r X_{2,2}^s].$$

COROLLARY 4.5.1. *For an arbitrary distribution and for $r \geqq 0$*

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} E[X_{i,N}^r X_{j,N}^r] = \binom{N}{2} [E(X^r)]^2.$$

PROOF. Follows from Theorem 4.5 and $E[X_{1,2}^r X_{2,2}^r] = [E(X^r)]^2$.

COROLLARY 4.5.2. *Corollary 4.5.1 with $r = 1$ gives*

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} E[X_{i,N} X_{j,N}] = \binom{N}{2} (EX)^2.$$

COROLLARY 4.5.3. *For an arbitrary distribution having mean zero, $\sum_{i=1}^{N} \mu_{i,N} = 0$ and if N is odd and the arbitrary distribution is symmetric about zero, then $\mu_{(N+1)/2,N} = 0$.*

REMARK 4.5.1. Jones [24] gives the result

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} E[X_{i,N}^r X_{j,N}^s] = \binom{N}{2} E(X^r)E(X^s) \quad \text{or}$$

$$\sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} E[X_{i,N}^r X_{j,N}^s] = N(N-1)E(X^r)E(X^s)$$

which is incorrect, because for $N = 2$ Jones' result implies that $E[X_{1,2}^r X_{2,2}^s] = E(X^r)E(X^s)$, which is, in general, not true unless $r = s$. Consequently, the relationships $\sum \mu_{i,N} = NEX$ and $\sum \mu_{i,N}^{(2)} = NE(X^2)$ do not follow as special cases ($r = 0$, $s = 1$) and ($r = 0$, $s = 2$) from Jones' result.

REMARK 4.5.2. Theorem 4.4, Theorem 4.5(i) and Corollary 4.5.3 are special cases of the general and very obvious theorem that if $E|\psi(Y_1, \cdots, Y_N)| < \infty$, then $E\psi(X_{1,N}, X_{2,N}, \cdots, X_{N,N}) = E\psi(Y_1, Y_2, \cdots, Y_N)$, where $\psi(\cdot)$ is invariant under all permutations of its arguments, and the Y_i are the unordered observations.

REMARK 4.5.3. Theorem 4.5 will not go through if one takes expectations about central values for the order statistics (that is, if one replaces $X_{i,N}$, $X_{j,N}$ and X by $X_{iN} - \mu_{i,N}$, $X_{j,N} - \mu_{j,N}$ and $X - EX$ respectively).

THEOREM 4.6. *If g is any differentiable function such that differentiation of $g(x)$ with respect to its argument and expectation of $g(X)$ with respect to an absolutely continuous distribution are interchangeable, then*

$$Eg'(X_{i,N}) = -\sum_{j=1}^{N} E[g(X_{i,N})f'(X_{j,N})/f(X_{j,N})], \qquad i = 1, 2, \cdots, N,$$

where f denotes the probability density function of the distribution.

PROOF. The method of proof adopted here is identical to the one used by Seal [47]. (I thank Professor Wassily Hoeffding for drawing my attention to the method of proof used by Seal.) For all real t one has

$$E[g(X_{i,N} + t)] = N! \int_{-\infty<x_1<\cdots<x_N<\infty} \cdots \int g(x_i + t) \prod_{j=1}^{N} f(x_j)\, dx_j$$

$$= N! \int_{-\infty<y_1<\cdots<y_N<\infty} \cdots \int g(y_i) \prod_{j=1}^{N} f(y_j - t)\, dy_j .$$

Changing the y's to x's, one gets

$$E[g(X_{i,N} + t)] = N! \int_{-\infty<x_1<\cdots<x_N<\infty} \cdots \int g(x_i) \prod_{j=1}^{N} f(x_j - t)\, dx_j .$$

Differentiating both sides of the above equation with respect to t and setting $t = 0$ one obtains

$$Eg'(X_{i,N}) = N! \int_{-\infty<x_1<\cdots<x_N<\infty} \cdots \int g(x_i) \left\{-\sum_{j=1}^{N} [f'(x_j)/f(x_j)]\right\} \prod_{k=1}^{N} f(x_k)\, dx_k$$

$$= -\sum_{j=1}^{N} E\{g(X_{i,N})[f'(X_{j,N})/f(X_{j,N})]\}.$$

This completes the proof of the theorem.

COROLLARY 4.6.1. *With $g(x) = x$, Theorem 4.6 gives*

$$\sum_{j=1}^{N} E[X_{i,N}f'(X_{j,N})/f(X_{j,N})] = -1, \qquad i = 1, 2, \cdots, N.$$

COROLLARY 4.6.2. *Corollary 4.6.1 with $f(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2)$ gives*

$\sum_{j=1}^{N} \mu_{i,j,N} = 1$, $i = 1, 2, \cdots, N$ (See also Seal). *Also, since* $\sum_1^N \mu_{i,N} = 0$, *it follows that* $\sum_{j=1}^{N} \sigma_{i,j,N} = 0$.

Corollary 4.6.2 follows directly from the well known property of the independence of the sample mean and the deviations of the normal O.S. from the sample mean. This independence property is given by McKay [29].

THEOREM 4.7. *If g is a twice differentiable function such that twice differentiation of* $g(x)$ *with respect to its argument and expectation of* $g(X)$ *with respect to an absolutely continuous distribution are interchangeable then*

$$Eg''(X_{i,N}) = \left[Eg(X_{i,N}) \sum_{j=1}^{N} f''(X_{j,N})/f(X_{j,N})\right] + E\left[g(X_{i,N}) \sum_{k\neq j}\sum f'(X_{j,N})f'(X_{k,N})/f(X_{j,N})f(X_{k,N})\right], \quad i = 1, 2, \cdots, N.$$

PROOF. Proceed in a similar manner to Theorem 4.6.

REMARK 4.7.1. If X is a standardized normal variable or generalized truncated normal variable (in other words, $f'(x) = -xf(x)$) and $g(x) = 1$, then Theorem 4.7 will take a much simpler form.

THEOREM 4.8. *If* $g(x)$ *and* $h(x)$ *are differentiable functions such that differentiation of* $g(x)h(x)$ *with respect to its argument and expectation of* $g(X)h(X)$ *with respect to an absolutely continuous distribution are interchangeable, then*

$$E[g'(X_{i,N})h(X_{j,N}) + g(X_{i,N})h'(X_{j,N})] = -\sum_{k=1}^{N} E[g(X_{i,N})h(X_{j,N})f'(X_{k,N})/f(X_{k,N})].$$

PROOF. Use a proof similar to Theorem 4.6.

REMARK 4.8.1. By specializing $g(x)$ and $h(x)$ in the preceding theorem, one can obtain identities among higher mixed moments of O.S.

THEOREM 4.9. *For an arbitrary distribution and even* N,

$$\mu_{1,N,N} = \sum_{i=1}^{(N-2)/2} (-1)^{i-1}\binom{N}{i}\mu_{i,i}\,\mu_{N-i,N-i} + \left(\frac{1}{2}\right)(-1)^{(N-2)/2}\binom{N}{N/2}\mu_{N/2,N/2}^2.$$

PROOF. The following method of proof is similar to the one used by Ruben [43] for the normal O.S. Consider

$$\mu_{1,N,N} = N(N-1)\iint_{-\infty<x<y<\infty} xy[F(y) - F(x)]^{N-2}\, dF(x)\, dF(y).$$

The integrand in the above integral is symmetric in x and y. Hence $\mu_{1,N,N} = (\frac{1}{2})N(N-1)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy[F(y) - F(x)]^{N-2}\, dF(x)\, dF(y)$. Now, expand $[F(y) - F(x)]^{N-2}$ in powers of $F(y)$ and $F(x)$, integrate on x and y and obtain

$$\mu_{1,N,N} = \left(\frac{1}{2}\right)\sum_{i=0}^{N-2}(-1)^i\binom{N}{i+1}\mu_{i+1,i+1}\,\mu_{N-i-1,N-i-1}$$
$$= \sum_{i=0}^{(N-4)/2}(-1)^i\binom{N}{i+1}\mu_{i+1,i+1}\,\mu_{N-i-1,N-i-1} + \left(\frac{1}{2}\right)(-1)^{(N-2)/2}\binom{N}{N/2}\mu_{N/2,N/2}^2,$$

after combining like terms. That is;

$$\mu_{1,N,N} = \sum_{i=1}^{(N-2)/2} (-1)^{i-1} \binom{N}{i} \mu_{i,i}\mu_{N-i,N-i} + \left(\frac{1}{2}\right)(-1)^{(N-2)/2} \binom{N}{N/2} \mu_{N/2,N/2}^2.$$

This completes the proof of the theorem.

For example, the formula for $N = 4$ and $N = 6$ respectively are $\mu_{1,4,4} = 4\mu_{1,1}\mu_{3,3} - 3\mu_{2,2}^2$ and $\mu_{1,6,6} = 6\mu_{1,1}\mu_{5,5} - 15\mu_{2,2}\mu_{4,4} + 10\mu_{3,3}^2$. If the population mean is zero, that is $\mu_{1,1} = 0$, then the above formulae reduce to $\mu_{1,4,4} = -3\mu_{2,2}^2$ and $\mu_{1,6,6} = 10\mu_{3,3}^2 - 15\mu_{2,2}\mu_{4,4}$.

REMARK 4.9.1. Theorem 4.9 for normal O.S. follows also from Teichroew's [49] Formula (5).

THEOREM 4.10. *For an arbitrary absolutely continuous distribution for which $f'(x) = -xf(x)$, (that is, for standard normal or generalized truncated normal densities) one has*

$$\mu_{i,N}^{(2)} = 1 + \frac{N!}{(i-1)!(N-i)!} \sum_{m=0}^{N-i} (-1)^m \frac{1}{i+m} \binom{N-i}{m} \mu_{i+m-1,i+m,i+m}, \qquad i = 1(1)N$$

PROOF. By definition

$$[(i-1)!(N-i)!/N!]\mu_{i,N}^{(2)} = \int_{-\infty}^{\infty} x^2 f(x)F^{i-1}(x)[1-F(x)]^{N-i}\,dx.$$

Writing $xf(x)\,dx = d[-f(x)]$ and integrating by parts once, one obtains

$$\begin{aligned}\frac{(i-1)!(N-i)!}{N!}\mu_{i,N}^{(2)} &= -xf(x)F^{i-1}(x)[1-F(x)]^{N-i}\Big|_{-\infty}^{\infty} \\ &\quad + \int_{-\infty}^{\infty} F^{i-1}(x)[1-F(x)]^{N-i}f(x)\,dx \\ &\quad + (i-1)\int_{-\infty}^{\infty} xf^2(x)F^{i-2}(x)[1-F(x)]^{N-i}\,dx \\ &\quad - (N-i)\int_{-\infty}^{\infty} xf^2(x)F^{i-1}(x)[1-F(x)]^{N-i-1}\,dx \\ &= \frac{(i-1)!(N-i)!}{N!} + (i-1)\sum_{m=0}^{N-i}(-1)^m\binom{N-i}{m}\int_{-\infty}^{\infty} xf^2(x)F^{i+m-2}(x)\,dx \\ &\quad + (N-i)\sum_{n=0}^{N-i-1}(-1)^{n+1}\binom{N-i-1}{n}\int_{-\infty}^{\infty} xf^2(x)F^{i+n-1}(x)\,dx.\end{aligned}$$

Since $yf(y)\,dy = d[-f(y)]$, $\int_x^\infty yf(y)\,dy = f(x)$.

Therefore

$$\begin{aligned}\int_{-\infty}^{\infty} xf^2(x)F^k(x)\,dx &= \int_{-\infty}^{\infty} xf(x)F^k(x)\left[\int_x^{\infty} yf(y)\,dy\right]dx \\ &= \iint_{-\infty<x<y<\infty} xyf(x)f(y)F^k(x)\,dx\,dy \\ &= [(k+1)(k+2)]^{-1}\mu_{k+1,k+2,k+2}, \qquad k = 1, 2, \cdots.\end{aligned}$$

Using the above result in the expression for $\mu_{i,N}^{(2)}$, one obtains

$$\frac{(i-1)!(N-i)!}{N!}\mu_{i,N}^{(2)} = \frac{(i-1)!(N-i)!}{N!} + (i-1)\sum_{m=0}^{N-i}(-1)^m\binom{N-i}{m}\frac{\mu_{i+m-1,i+m,i+m}}{(i+m)(i+m-1)} + (N-i)\sum_{m=0}^{N-i-1}(-1)^{n+1}\binom{N-i-1}{n}\frac{\mu_{i+n,i+n+1,i+n+1}}{(i+n+1)(i+n)}$$

$$= \frac{(i-1)!(N-i)!}{N!} + \frac{1}{i}\mu_{i-1,i,i} + \sum_{m=1}^{N-i}(-1)^m\frac{\mu_{i+m-1,i+m,i+m}}{(i+m)(i+m-1)}\binom{N-i}{m}(i-1+m)$$

$$= \frac{(i-1)!(N-i)!}{N!} + \sum_{m=0}^{N-i}(-1)^m\binom{N-i}{m}(i+m)^{-1}\mu_{i+m-1,i+m,i+m}.$$

Hence

$$\mu_{i,N}^{(2)} = 1 + \frac{N!}{(i-1)!(N-i)!}\sum_{m=0}^{N-i}(-1)^m(i+m)^{-1}\binom{N-i}{m}\mu_{i+m-1,i+m,i+m}.$$

This completes the proof of the theorem.

Setting $i = N$ gives

$$\mu_{N,N}^{(2)} = 1 + \mu_{N-1,N,N}. \tag{4.1}$$

Using Theorem 4.3 in Equation (4.1) one obtains

$$\mu_{1,N}^{(2)} = 1 + \mu_{1,2,N}. \tag{4.2}$$

Setting $i = N - 1$ gives

$$\mu_{N-1,N}^{(2)} = 1 + N\mu_{N-2,N-1,N-1} - (N-1)\mu_{N-1,N,N}. \tag{4.3}$$

Setting $i = 1$ gives

$$\mu_{1,N}^{(2)} = 1 + N\sum_{m=0}^{N-1}(-1)^m(m+1)^{-1}\binom{N-1}{m}\mu_{m,m+1,m+1}$$

$$= 1 + \sum_{m=0}^{N-1}(-1)^m\binom{N}{m+1}\mu_{m,m+1,m+1} \tag{4.4}$$

$$= 1 + \sum_{m=2}^{N}(-1)^{m-1}\binom{N}{m}\mu_{m-1,m,m},$$

since $\mu_{0,1,1} = 0$.

COROLLARY 4.10.1. *With the hypothesis of Theorem* 4.10 *one obtains*

$$\mu_{1,2,N} = \sum_{m=2}^{N}(-1)^{m-1}\binom{N}{m}\mu_{1,2,m}.$$

PROOF. It follows from Equations (4.2) and (4.4) that

$$\mu_{1,2,N} = \sum_{m=2}^{N} (-1)^{m-1} \binom{N}{m} \mu_{1,2,m}.$$

Corollary 4.10.1 is equivalent to

$$[1 + (-1)^N]\mu_{1,2,N} = \sum_{m=2}^{N-1} (-1)^{m-1} \binom{N}{m} \mu_{1,2,m}. \tag{4.5}$$

This completes the proof of the corollary.

If $N = 2$

$$\mu_{1,2,2} = 0. \tag{4.6}$$

This together with $\mu_{1,2}^{(2)} = 1 + \mu_{1,2,2}$ implies that $\mu_{1,2}^{(2)} = 1$. Put $N = 4$ in Equation (4.5), use (4.6) and obtain

$$\mu_{1,2,4} = 2\mu_{1,2,3}\ . \tag{4.7}$$

Put $N = 6$ in Equation (4.5), use (4.6) and (4.7) and get $\mu_{1,2,6} = 3\mu_{1,2,5} - 5\mu_{1,2,3}$, etc. For odd N the identity in Equation (4.5) leads to the same result as the one for the immediately preceding even number, namely $N - 1$.

THEOREM 4.11. *For any distribution symmetric about zero the matrix* $((\mu_{i,j,N}))$ *is doubly symmetric (i.e., symmetric with respect to the two major diagonals) and the distinct elements in* $((\mu_{i,j,N}))$ *are those lying in any wedge-shaped region (say*

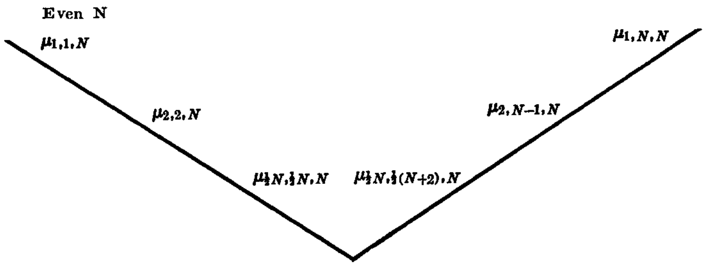

FIG. 4.11.1

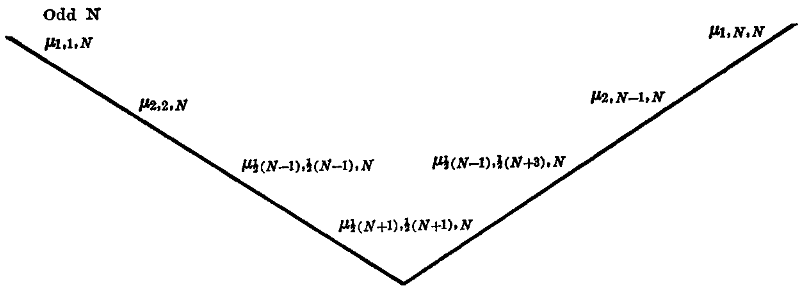

FIG. 4.11.2

the upper one) bounded by the two major diagonals. Hence, the number of distinct elements in $((\mu_{i,j,N}))$ *is* $N(N+2)/4$ *if* N *is even, and is* $(N+1)^2/4$ *if* N *is odd.*

PROOF. From Theorem 4.3, it is clear that the matrix $((\mu_{i,j,N}))$ is doubly symmetric. The distinct elements will be as shown in Figures 4.11.1 and 4.11.2. Therefore, the total number of distinct $\mu_{i,j,N}$ is $N + (N-2) + (N-4) + \cdots + 2 = N(N+2)/4$ when N is even, and the total number of distinct $\mu_{i,j,N}$ is $N + (N-2) + \cdots + 3 + 1 = (N+1)^2/4$ when N is odd.

COROLLARY 4.11.1. *From the consideration of the figures, one can obtain the Table* 4.11.1 *for the number of distinct elements in the matrix* $((\mu_{i,j,N}))$.

TABLE 4.11.1

	Even N	Odd N
$\mu_{i,N}^{(2)}$	$N/2$	$(N+1)/2$
$\mu_{i,j,N}$ $(i \neq j)$	$N^2/4$	$(N^2-1)/4$
Total	$N(N+2)/4$	$[(N+1)/2]^2$

THEOREM 4.12. *For an arbitrary distribution symmetric about zero, the number of distinct and independent constraints among the distinct* $\mu_{i,j,N}$ $(i \neq j)$ *imposed by the recurrence formula in Theorem* 4.2 *is* $[(N-1)^2 - 1]/4$ *if* N *is even and* $(N-1)^2/4$ *if* N *is odd.*

PROOF. The right-hand member of Theorem 4.2 is $N\mu_{i-1,j-1,N-1}$. Hence it is clear that for any N, the number of distinct and independent constraints among the $\mu_{i,j,N}$ imposed by the recurrence relation in Theorem 4.2 is exactly the number of distinct $\mu_{i,j,N-1}$ $(i \neq j)$ in the matrix $((\mu_{i,j,N-1}))$. Now the theorem follows after using the results presented in Table 4.11.1.

THEOREM 4.13. *For the normal distribution, if the* $\mu_{i,N-1}^{(2)}$ *are all known, the number of linear constraints among the distinct* $\mu_{i,N}^{(2)}$ *is* $N/2$ *if* N *is even and* $(N-1)/2$ *if* N *is odd. Also, if all* $\mu_{i,j,N-1}$ $(i \neq j)$, $\mu_{k,k}$ *for* $k = 1, 2, \cdots, N-1$ *and* $\mu_{1,N}^{(2)}$ *are known, the number of linear and independent constraints among the distinct* $\mu_{i,j,N}$ $(i \neq j)$ *and the number of linearly independent* $\mu_{i,j,N}$ $(i \neq j)$ *are as shown in Table* 4.13.1.

TABLE 4.13.1

	Theorem 4.2	Theorem 4.9	Theorem 4.10 (Equation 4.2)	Number of linearly independent $\mu_{i,j,N}$ $(i \neq j)$
Even N	$N(N-2)/4$	1	1	$(N-4)/2$
Odd N	$[(N-1)/2]^2$	0	1	$(N-3)/2$

PROOF.

N is even: $\mu_{1,N}^{(2)}$; $\mu_{2,N}^{(2)}$; $\cdots$ $\mu_{N/2,N}^{(2)}$ are the distinct $\mu_{i,N}^{(2)}$ (See Figure 4.11.1). The

recurrence formula of Theorem 4.1 with $k = 2$ and $i = 1, 2, \cdots, (N-2)/2$ together with $\mu^{(2)}_{N/2,N} = \mu^{(2)}_{N/2,N-1}$, will involve the distinct $\mu^{(2)}_{i,N}$. They constitute $N/2$ linear constraints among the distinct $\mu^{(2)}_{i,N}$. From Theorem 4.12, the number of independent constraints imposed by Theorem 4.2 on distinct $\mu_{i,j,N}$ $(i \neq j)$, is $N(N-2)/4$. Recall that Theorem 4.9 expresses $\mu_{1,N,N}$ in terms of $\mu_{k,k}$, $k = 1, 2, \cdots, N-1$. Also, Theorem 4.10 (Equation (4.2)) expresses $\mu_{1,2,N}$ in terms of $\mu^{(2)}_{1,N}$. Obviously, one cannot hope to obtain either Theorem 4.9 or (4.2) by a linear combination of the formulae in Theorem 4.2 with $i \neq j$. Now, since the relationship in Theorem 4.9 is linearly independent of (4.2), it follows that the constraints given by Theorem 4.9 and (4.2) are linearly independent of those constraints provided by Theorem 4.2 on distinct $\mu_{i,j,N}$ $(i \neq j)$. Thus, the number of linearly independent and distinct $\mu_{i,j,N}$ $(i \neq j)$ is given by $(\frac{1}{4})[N^2 - N(N-2)] - 2 = (N-4)/2$.

N is odd: $\mu^{(2)}_{1,N}, \mu^{(2)}_{2,N}, \cdots, \mu^{(2)}_{(N+1)/2,N}$ are the distinct $\mu^{(2)}_{i,N}$. (See Figure 4.11.2.) Theorem 4.1 with $k = 2$ for $i = 1, 2, \cdots, (N-1)/2$ involves the distinct $\mu^{(2)}_{i,N}$ and constitutes $(N-1)/2$ linear constraints among the distinct $\mu^{(2)}_{i,N}$. Also, from Theorem 4.10 we have that the number of independent constraints imposed by Theorem 4.2 on the distinct $\mu_{i,j,N}$ $(i \neq j)$ is $[(N-1)/2]^2$. Equation (4.2) gives one additional constraint. Hence the results of Table 4.13.1.

THEOREM 4.14. *In order to find the first, second and mixed (linear) moments of O.S. in a sample of size N drawn from an arbitrary population symmetric about zero, given these moments for all sample sizes less than N, one has to evaluate at most one single integral and $(N-2)/2$ double integrals if N is even; and one single integral and $(N-1)/2$ double integrals if N is odd.*

PROOF. Assume first, second and mixed (linear) moments are known for all sample sizes less than N. To compute $\mu_{i,N}$, consider

$$\mu_{i,N} = \frac{N!}{(i-1)!(N-i)!} \int xF^{i-1}(x)[1 - F(x)]^{N-i}\, dF(x),$$
$$= \sum_{k=0}^{N-i} (-1)^k \frac{N!}{(i-1)!(N-i)!} \binom{N-i}{k} \int xF^{i-1+k}(x)\, dF(x),$$

and all of these integrals for which the range of integration is $(-\infty, \infty)$, would have been computed previously except the integral $\int xF^{N-1}(x)\, dF(x)$. Hence, one has to evaluate at most one integral when N is even and 0 integrals when N is odd since $\mu_{(N+1)/2,N} = 0$. To get $\mu^{(2)}_{i,N}$, where

$$\mu^{(2)}_{i,N} = \frac{N!}{(i-1)!(N-i)!} \int x^2F^{i-1}(x)[1 - F(x)]^{N-i}\, dF(x)$$
$$= \sum_{k=0}^{N-i} (-1)^k \frac{N!}{(i-1)!(N-i)!} \binom{N-i}{k} \int x^2F^{i-1+k}(x)\, dF(x).$$

All of these integrals would be available except $\int x^2F^{N-1}(x)\, dF(x)$. Hence there would be at most one integral to be evaluated when N is odd and none when N is even, since $\mu^{(2)}_{N/2,N} = \mu^{(2)}_{N/2,N-1}$, when N is even (see Corollary 4.1.1).

To obtain

$$\mu_{i,j,N} = C \iint_{x<y} xyF^{i-1}(x)[F(y) - F(x)]^{j-i-1}[1 - F(x)]^{N-j}\, dF(x)\, dF(y)$$

where C is a constant depending on i and j, we write

$$\mu_{i,j,N} = \sum \sum C_{r,s} \iint_{x<y} xyF^{r}(x)F^{s}(y)\, dF(x)\, dF(y),$$

and all of these integrals would be available except where $r + s = N - 2$. Hence, at most, $(N - 1)$ integrals are needed and of these, $(N - 2)/2$ when N is even and $(N - 1)/2$ when N is odd, could be eliminated by symmetry. Also, $\mu_{1,N,N}$ is available (See Theorem 4.9) when N is even. Hence, at most $(N - 2)/2$ double integrals when N is even and $(N - 1)/2$ double integrals when N is odd, would be required. Thus, the proof of the theorem is complete.

5. A systematic procedure for evaluating the first, second and mixed (linear) moments of the normal O.S. In this section, we will indicate how one can go about evaluating the first, second and mixed (linear) moments of normal O.S. for any N given these moments of O.S. for all sample sizes up to and including $N - 1$. We will also demonstrate the procedure by considering a few values for N. *N is even:*

Evaluate $\mu_{1,N}$ and solve for the rest of $\mu_{i,N}$ using the $\mu_{i,N-1}$ and the recurrence formulae in Theorem 4.1 with $k = 1$. From Corollary 4.1.1 $\mu_{N/2,N}^{(2)}$ is known and the rest of $\mu_{i,N}^{(2)}$ will be known using Theorem 4.1 with $k = 2$. Also $\mu_{1,N,N}$ is available from Theorem 4.9. From Equation (4.2), namely $\mu_{1,N}^{(2)} = 1 + \mu_{1,2,N}$, $\mu_{1,2,N}$ is known. Now evaluate any $(N - 4)/2$ of the rest of the distinct $\mu_{i,j,N}$ $(i \neq j)$. For example evaluate $\mu_{1,3,N}, \mu_{1,4,N}, \cdots, \mu_{1,N/2,N}$. Use the recurrence relation of Theorem 4.2 with $i = 1$ and $j = 2, 3, \cdots, N - 1$; $i = 2$ and

$$j = 3, 4, \cdots, N - 2;$$

$i = 3$ and $j = 4, 5, \cdots, N - 3$; etc. until the total number of these relationships is $N(N - 2)/4$ and solve for the rest of the distinct $\mu_{i,j,N}$. Whenever one wishes to use the recurrence formula of Theorem 4.2 with $i = 1$ and $j \neq N$ write the formula with $i = N - j + 1$ and $j = N$. In the formula thus obtained, use the relationships $\mu_{i,j,N} = \mu_{j,i,N}$ and $\mu_{i,j,N} = \mu_{N-i+1,N-j+1,N}$, until the resultant formula involves only those $\mu_{i,j,N}$ that are in the upper wedge-shaped region of the matrix $((\mu_{i,j,N}))$.

N is odd: From Corollary 4.5.3, we have $\mu_{(N+1)/2,N} = 0$. Using Theorem 4.1 with $k = 1$ the rest of $\mu_{i,N}$ are known. Evaluate one $\mu_{i,N}^{(2)}$, for example, evaluate $\mu_{1,N}^{(2)}$. The rest of $\mu_{i,N}^{(2)}$ can be solved for by using Theorem 4.1 with $k = 2$. From Equation 4.2, $\mu_{1,2,N}$ is known. Now evaluate any $(N - 3)/2$ of the rest of the distinct $\mu_{i,j,N}$ $(i \neq j)$. For example evaluate $\mu_{1,3,N}, \mu_{1,4,N}, \cdots, \mu_{1,(N+1)/2,N}$, using the recurrence relation of Theorem 4.2 with $i = 1$ and $j = 2, 3, \cdots, N - 1$; $i = 2$ and $j = 3, 4, \cdots, N - 2$; $i = 3$ and $j = 4, 5, \cdots, N - 3$; etc. until the

total number of these is $(N-1)^2/4$. Then solve for the rest of the distinct $\mu_{i,j,N}$ $(i \neq j)$.

In the sequel, let us demonstrate the procedure by considering $N = 4$ and 5. Assume that these moments are available up to and including $N = 3$.

$N = 4$: Evaluate $\mu_{1,4}$ and obtain $\mu_{2,4}$ from the equation, $3\mu_{1,4} + \mu_{2,4} = 4\mu_{1,3}$. That is $\mu_{2,4} = 4\mu_{1,3} - 3\mu_{1,4}$. Also, from Corollary 4.1.2 one gets $\mu_{2,4}^{(2)} = \mu_{2,3}^{(2)}$. Also, from Theorem 4.1 and the preceding result, one gets $3\mu_{1,4}^{(2)} = 4\mu_{1,3}^{(2)} - \mu_{2,3}^{(2)}$.

Theorem 4.10 (Equation 4.2) gives $\mu_{1,2,4} = \mu_{1,4}^{(2)} - 1$ and Theorem 4.9 gives $\mu_{1,4,4} = -3\mu_{1,2}^{(2)}$. The distinct $\mu_{i,j,4}$ $(i \neq j)$ are $\mu_{1,2,4}$, $\mu_{1,3,4}$, $\mu_{1,4,4}$ and $\mu_{2,3,4}$. But, $\mu_{1,2,4}$ and $\mu_{1,4,4}$ are already known. Hence, use the recurrence formula of Theorem 4.2 $N(N-2)/4 = 2$ times, in order to involve $\mu_{1,3,4}$ and $\mu_{2,3,4}$. Using the recurrence formula with $i = 2, j = 4$ and $i = 2$ and $j = 3$ one obtains

$$\begin{pmatrix} \mu_{1,3,4} \\ \mu_{2,3,4} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 4\mu_{1,2,3} - 2\mu_{1,2,4} \\ 2\mu_{1,3,3} + 3\mu_{2,2}^{(2)} \end{pmatrix}.$$

That is $\mu_{1,3,4} = 2\mu_{1,3,3} + 3\mu_{1,2}^{(2)}$ and $\mu_{2,3,4} = 4\mu_{1,2,3} - 2\mu_{1,3,3} - 2\mu_{1,2,4} - 3\mu_{1,2}^{(2)}$. Now, one can write down the matrix $((\mu_{i,j,4}))$.

$N = 5$: $\mu_{3,5} = 0$, $\mu_{2,5} = (\frac{5}{3})\mu_{2,4}$ and $\mu_{1,5} = (\frac{5}{4})\mu_{1,4} - (\frac{1}{4})\mu_{2,5} = (\frac{5}{4})[\mu_{1,4} - (\frac{1}{3})\mu_{2,4}]$. Evaluate $\mu_{1,5}^{(2)}$ and obtain $\mu_{1,2,5}$ from the equation $\mu_{1,2,5} = \mu_{1,5}^{(2)} - 1$. Evaluate $\mu_{1,3,5}$ and solve for $\mu_{1,4,5}$, $\mu_{1,5,5}$, $\mu_{2,3,5}$ and $\mu_{2,4,5}$ from the following equations:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} \mu_{1,4,5} \\ \mu_{1,5,5} \\ \mu_{2,3,5} \\ \mu_{2,4,5} \end{pmatrix} = \begin{pmatrix} 5\mu_{1,2,4} - 3\mu_{1,2,5} - \mu_{1,3,5} \\ 5\mu_{1,3,4} - 2\mu_{1,3,5} \\ 5\mu_{1,4,4} \\ 5\mu_{2,3,4} \end{pmatrix}.$$

In order to find the first, second and mixed (linear) moments of O.S. in samples drawn from an arbitrary population symmetric about zero, one can use the above computational scheme keeping in mind that one has to evaluate one more double integral, perhaps $\mu_{1,2,N}$, since $\mu_{1,2,N}$ will not be available from Equation (4.2) which holds only for the family of normal populations. The computational scheme presented is useful especially if it is possible to evaluate exactly the lower moments of some O.S. for each N. However, for some distributions, especially the normal distribution, it will be difficult to evaluate these exactly except for small sample sizes. Also, as noted by Harter [21] and Srikantan [48], repeated application of the recurrence formulae of Theorems 4.1 and 4.2 for working "upwards" (that is, going from a smaller N to a larger N) causes loss of accuracy of 1 to 3 units in the last decimal place. Hence, the above scheme for numerical evaluation of these moments is feasible for small sample sizes due to the serious accumulation of round off errors. However, as Harter [21] and Srikantan point out, if the recurrence formulae are written as

$$\mu_{i,N-1} = (i/N)\mu_{i+1,N} + \{(N-i)/N\}\mu_{i,N}, \quad i = 1(1)N-1$$

and

$$\mu_{i-1,j-1,N-1} = \{(i-1)/N\}\mu_{i,j,N} + \{(j-i)/N\}\mu_{i-1,j,N} + \{(N-j+1)/N\}\mu_{i-1,j-1,N}, \qquad 1 < i \leqq j \leqq N-1,$$

then, they can be used for working "downwards" (that is, going from a larger N to a smaller N) with no serious accumulation of rounding errors. For this "downward" procedure one has to evaluate numerically all the first, second and mixed moments of O.S. in an arbitrary large sample size N, which may not be possible especially when one is using the series approximations for these moments suggested by David and Johnson [11], Clark and Williams [7] and Plackett [33].

6. Quasi-ranges. The ith quasi-range in samples of size N from an arbitrary population is defined as the range of $N - 2i$ sample observations omitting the i largest and the i smallest observations. Cadwell [2] developed a method of evaluating the probability density function of the ith quasi-range in a sample from a normal population. Mosteller [30] proposed a sample quasi-range as a useful "inefficient" estimate of the population standard deviation. Ruben [43] expressed the odd moments of the normal sample range when N is odd and its even moments when N is even, as linear functions of the expectations of the extreme O.S. Chu and Hotelling [4] and Chu [5] gave some uses of quasi-ranges. Carlson [3] obtained a recurrence formula for the mean range when N is odd in terms of the mean ranges for sample sizes up to and including $N - 1$. Harter [20] discussed estimates in terms of sample quasi-ranges, of the standard deviation in rectangular, exponential and normal populations. In order to obtain the best linear unbiased estimates of population standard deviation by means of sample quasi-ranges, one needs to know the expected values, variances and covariances of the quasi-ranges. Harter [20] has tabulated the expected values and variances of sample quasi-ranges for $i = 0(1)8$ and $N = (2i + 2)(1)100$ accurate to five or six decimal places. Rider [37] obtained formulae for the cumulants of these quasi-ranges from the exponential population. Leone et al [25] used sample quasi-ranges in setting up confidence intervals for the population standard deviation. In this section the expected values, variances and covariances of quasi-ranges in samples from any population symmetric about zero have been expressed in terms of expected values, variances and covariances of order statistics in the sample. Simple recurrence formulae among the expected values of sample quasi-ranges from an arbitrary population are obtained. For numerical evaluation, these formulae can be used for working "downwards" with no serious accumulation of rounding errors. We further need the following notation.

Let

$$W_{i,N} = X_{N-i,N} - X_{i+1,N} \quad (i = 0, 1, \cdots, [(N-2)/2]),$$

$$\omega_{i,N} = E(W_{i,N}) = \mu_{N-i,N} - \mu_{i+1,N} \quad (i = 0, 1, \cdots, [(N-2)/2]).$$

$W_{0,N}$ will be the sample range and $\omega_{0,N}$ will be its expected value. Also let

$$a_{i,j,N} = E(W_{i,N}W_{j,N}), \quad 0 \leqq i \leqq j \leqq [(N-2)/2].$$

$$a_{i,N}^{(2)} = a_{i,i,N}, \qquad (i = 0, 1, \cdots, [(N-2)/2]),$$

and let $\rho_{i,j,N}$ be the correlation between $X_{i,N}$ and $X_{j,N}$ $(1 \leqq i \leqq j \leqq N)$. Then we have the following results, some of which are extremely trivial. However, they are included for the sake of convenience.

THEOREM 6.1. *For an arbitrary distribution symmetric about zero,* $\omega_{i,N} = 2\mu_{N-i,N}$ $(i = 0, 1, \cdots, [(N-2)/2])$.

PROOF. Follows from the fact $\mu_{i+1,N} = -\mu_{N-i,N}$.

THEOREM 6.2. *For an arbitrary distribution one has* $(N-i)\omega_{i-1,N} + i\omega_{i,N} = N\omega_{i-1,N-1}$, $i = 0, 1, \cdots, [(N-2)/2]$.

PROOF. Theorem 4.1 with $k = 1$ gives $i\mu_{i+1,N} + (N-i)\mu_{i,N} = N\mu_{i,N-1}$ $(i = 1, 2, \cdots, N-1)$. Changing i to $(N-i)$ in the preceding equation one obtains $(N-i)\mu_{N-i+1,N} + i\mu_{N-i,N} = N\mu_{N-i,N-1}$ $(i = 1, 2, \cdots, N-1)$. From the above two equations one gets

$$(N-i)(\mu_{N-i+1,N} - \mu_{i,N}) + i(\mu_{N-i,N} - \mu_{i+1,N}) = N(\mu_{N-i,N-1} - \mu_{i,N-1}).$$

Using the definition of $\omega_{i,N}$, it now follows that $(N-i)\omega_{i-1,N} + i\omega_{i,N} = N\omega_{i-1,N-1}$, for $i = 1, 2, \cdots, [(N-2)/2]$. This completes the proof of the theorem.

THEOREM 6.3. *For any distribution symmetric about zero the distributions of* $X_{i+1,N}$ *and* $-X_{N-i,N}$ *are identical, and the distributions of* $X_{i+1,N}\cdot X_{j+1,N}$ *and* $X_{N-i,N}X_{N-j,N}$ *are identical.*

PROOF. Follows from the symmetry of the distribution.

COROLLARY 6.3.1. *For any distribution symmetric about zero and for*

$$0 \leqq i \leqq j \leqq [(N-2)/2]$$

(1) $a_{i,j,N} = 2[\mu_{i+1,j+1,N} - \mu_{i+1,N-j,N}]$
and
(2) $\text{var}(W_{i,N}) = 2\,\text{var}(X_{N-i,N})[1 - \rho_{i+1,N-i,N}]$.

PROOF. Write $W_{i,N}W_{j,N}$ in terms of $X_{i,N}$'s, take expectations and use Theorem 6.3. Result (2) follows from (1) with $i = j$ and Theorems 6.1 and 6.3.

THEOREM 6.4. *For distributions symmetric about zero,*

$$\text{Cov}(W_{i,N}, W_{j,N}) = 2[\text{Cov}(X_{i+1,N}, X_{j+1,N}) - \text{Cov}(X_{i+1,N}, X_{N-j,N})], \qquad 0 \leqq i \leqq j \leqq [(N-2)/2].$$

PROOF.

$$\begin{aligned}\text{Cov}(W_{i,N}, W_{j,N}) &= \text{Cov}(X_{N-i,N} - X_{i+1,N}, X_{N-j,N} - X_{j+1,N}) \\ &= \text{Cov}(X_{N-i,N}, X_{N-j,N}) - \text{Cov}(X_{N-i,N}, X_{j+1,N}) \\ &\quad - \text{Cov}(X_{i+1,N}, X_{N-j,N}) + \text{Cov}(X_{i+1,N}, X_{j+1,N}) \\ &= 2[\text{Cov}(X_{i+1,N}, X_{j+1,N}) - \text{Cov}(X_{i+1,N}, X_{N-j,N})],\end{aligned}$$

on using Theorem 6.3.

THEOREM 6.5. *For distributions symmetric about zero*

$$\rho(W_{i,N}, W_{j,N}) = \frac{\rho_{i+1,j+1,N} - \rho_{i+1,N-j,N}}{[(1 - \rho_{i+1,N-i,N})(1 - \rho_{j+1,N-j,N})]^{\frac{1}{2}}},$$

$$0 \leqq i \leqq j \leqq [(N-2)/2].$$

PROOF. Using Corollary 6.3.1 one obtains

$$\rho(W_{i,N}, W_{j,N}) = \frac{\operatorname{Cov}(X_{i+1,N}, X_{j+1,N}) - \operatorname{Cov}(X_{i+1,N}, X_{N-j,N})}{[\operatorname{Var}(X_{N-i,N}) \operatorname{Var}(X_{N-j,N})(1 - \rho_{i+1,N-j,N})(1 - \rho_{j+1,N-j,N})]^{\frac{1}{2}}}$$

$$= \frac{\rho_{i+1,j+1,N} - \rho_{i+1,N-j,N}}{[(1 - \rho_{i+1,N-i,N})(1 - \rho_{j+1,N-j,N})]^{\frac{1}{2}}}.$$

This completes the proof of the theorem.

The recurrence formula in Theorem 6.2 after dividing both sides by N, can be used for working "downwards" in numerical evaluation of the expected values of the sample quasi-ranges, without serious accumulation of rounding errors. The results of Theorems 6.1, 6.3 and 6.4 will enable one to prepare tables of the expected values, variances and covariances of quasi-ranges in samples drawn from populations symmetric about zero, provided tables of these for the corresponding O.S. are available.

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