On monomial characters and central idempotents of rational group algebras

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February 20, 2004

Abstract

We give a method to obtain the primitive central idempotent of the rational group algebra $\mathbb{Q}G$ over a finite group G associated to a monomial irreducible character which does not involve computations with the character field nor its Galois group. We also show that for abelian-by-supersolvable groups this method takes a particularly easy form that can be used to compute the Wedderburn decomposition of $\mathbb{Q}G$.

Let G be a finite group and $\mathbb{Q}G$ the rational group algebra over G. A good understanding of the Wedderburn decomposition of $\mathbb{Q}G$, that is the decomposition of $\mathbb{Q}G$ as a direct sum of simple algebras, is a good tool to deal with several problems. For example, it is useful to study the group of automorphisms of $\mathbb{Q}G$ [5] or the group of units of the integral group ring $\mathbb{Z}G$ [6, 9, 15, 16, 18]. In theory the Wedderburn decomposition of $\mathbb{Q}G$ can be computed using powerful but rather complicated methods (see the introduction of [5] for a complete description of these methods).

The problem of computing the Wedderburn decomposition of $\mathbb{Q}G$ leads naturally to the problem of describing the primitive central idempotents of $\mathbb{Q}G$. The classical method to do that is first computing the primitive central idempotents $e(\chi)$ of $\mathbb{C}G$ associated to the irreducible characters of G and then adding the primitive central idempotents of the form $e(\sigma \circ \chi)$, with $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$. (See [20] or Section 2).

Recently Jespers, Leal and Paques [7] have discovered that if G is nilpotent then every primitive central idempotent of $\mathbb{Q}G$ is determined by a pair (H, K) of subgroups of G satisfying suitable conditions and that the primitive central idempotent $e(G, K, H)$ of G associated to (H, K) can be easily computed. In this paper we show that the results in [7] can be generalized to monomial groups and that the description of the pairs of subgroups leading to primitive central idempotent can be simplified. Namely we show that the primitive central idempotent of $\mathbb{Q}G$ associated to a monomial complex character of G is of the form $\alpha e(G, K, H)$ for $\alpha \in \mathbb{Q}$ and H and K subgroups of G satisfying some easy to check conditions. The advantage of this approach with respect to the classical method is avoiding computations on extensions of the rationals.

Moreover, we prove that if G is abelian-by-supersolvable then the pairs of subgroups (H, K) that realize the primitive central idempotents can be taken so that $e(G, K, H)$ is a primitive central idempotent and one can give a description of the simple algebra $\mathbb{Q}Ge(G, K, H)$.

*Partially supported by the D.G.I. of Spain and Fundación Séneca of Murcia

1 Preliminaries

We start fixing some notation. Throughout G is a finite group. The order of G is denoted by $|G|$, its derived subgroup by G' and the rational group algebra over G by $\mathbb{Q}G$. By $H \leq G$ we mean that H is a subgroup of G and by $H \trianglelefteq G$ that H is a normal subgroup of G. If X is a subset of G then $\langle X \rangle$ denotes the subgroup generated by G and we simplify the notation to $\langle Y, Z \rangle$ if either $X = Y \cup Z$ or $X = Y \cup \{Z\}$ or $X = \{Y, Z\}$. If $H \leq G$ then $N_G(H)$ denotes the normalizer of H in G. For $g \in G$ and $x \in \mathbb{Q}G$ set $x^g = g^{-1}xg$ and let us denote by $\mathrm{Cen}_G(x)$ the centralizer of x in G. If $H \leq G$ then we also set $H^g = g^{-1}Hg$. We sometimes also use exponential notation for the action of an automorphism of a group or an algebra.

For every positive integer n, ξ_n denotes a complex primitive n-th root of unity.

All the characters of a group are assumed to be complex characters. If χ is an irreducible character of G then the primitive central idempotent of $\mathbb{C}G$ associated to χ is denoted by $e(\chi)$ and the primitive central idempotent of $\mathbb{Q}G$ associated to χ by $e_{\mathbb{Q}}(\chi)$; that is $e_{\mathbb{Q}}(\chi)$ is the only primitive central idempotent e of $\mathbb{Q}G$ such that $\chi(e) \neq 0$.

The group $\mathcal{A} = \mathrm{Aut}(\mathbb{C})$ of automorphisms of the complex numbers acts on $\mathbb{C}G$ by acting on the coefficients, that is

$$\sigma \cdot \sum_{g \in G} a_g g = \sum_{g \in G} \sigma(a_g) g, \quad (\sigma \in \mathcal{A}, a_g \in \mathbb{C})$$

Furthermore $\mathcal{A}$ acts on the set of character of G by composition:

$$\sigma \cdot \chi = \sigma \circ \chi, \quad (\sigma \in \mathcal{A}, \chi \text{ a character of G})$$

If χ is an irreducible character of G then $e(\sigma \cdot \chi) = \sigma \cdot e(\chi)$ and the stabilizers of $e(\chi)$ and χ coincide with the group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q}(\chi))$ of automorphisms of $\mathbb{C}$ that fixes the character field $\mathbb{Q}(\chi)$ of χ. Therefore both the orbit of χ and $e(\chi)$ have $[\mathbb{Q}(\chi) : \mathbb{Q}]$ elements and can be computed applying the elements of $\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$ to χ and $e(\chi)$ respectively. Furthermore by [20, Proposition 1.1] one has

$$e_{\mathbb{Q}}(\chi) = \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} \sigma \cdot e(\chi) \tag{1.1}$$

If $H \leq G$ then set $\widehat{H} = \frac{1}{|H|} \sum_{x \in H} x \in \mathbb{Q}G$. If $g \in G$, then let $\hat{g} = \widehat{\langle g \rangle}$. Note that $\widehat{H}$ is an idempotent of $\mathbb{Q}G$ which is central in $\mathbb{Q}G$ if and only if $H \trianglelefteq G$. Following the notation of [7] let $\mathcal{M}(G)$ denote the set of all minimal normal non trivial subgroups of G and

$$\varepsilon(G) = \prod_{M \in \mathcal{M}(G)} (1 - \widehat{M}).$$

By convention $\varepsilon(1) = 1$.

If $N \trianglelefteq G$ then the kernel of the augmentation map $\omega_N : \mathbb{Q}G \to \mathbb{Q}(G/N)$ is $\mathbb{Q}G(1 - \widehat{N})$ and so ω_N induces an isomorphism $\mathbb{Q}G\widehat{N} \simeq \mathbb{Q}(G/N)$ (see [18]). Let $\varepsilon(G, N)$ denote the preimage of $\varepsilon(G/N)$ under this isomorphism, that is

$$\varepsilon(G, N) = \begin{cases} \widehat{N} & \text{if } N = G \\ \prod_{M/N \in \mathcal{M}(G/N)} (\widehat{N} - \widehat{M}) & \text{if } N \neq G. \end{cases}$$

The primitive (central) idempotents of $\mathbb{Q}A$ for A abelian are well known (see [1], [4] or [10]). A description in terms of the idempotents of the form $\varepsilon(A, H)$ has been given recently in [7].

Proposition 1.1 *[7] If A is an abelian group, then the primitive central idempotents of $\mathbb{Q}A$ are the elements of the form $\varepsilon(A, H)$ where H is a subgroup of A such that A/H is cyclic.*

If χ is a linear character of G and $H = \ker\chi$ then G/H is cyclic. By Proposition 1.1 $\omega_H(\varepsilon(G, H)) = \varepsilon(G/H)$ is a primitive central idempotent of $\mathbb{Q}(G/H)$ and therefore $\varepsilon(G, H)$ is a primitive central idempotent of $\mathbb{Q}G$. In fact one has the following:

Lemma 1.2 *If χ is a linear character of G and $H = \ker\chi$ then $e_{\mathbb{Q}}(\chi) = \varepsilon(G, H)$.*

Proof. If $G/H = \langle gH \rangle$ and $[G : H] = n$ then $\chi(g) = \xi$, a primitive n-th root of 1. The minimal subgroups of G/H are of the form $L_p = \langle H, g^{n/p} \rangle$ where p is a prime divisor of n. For every prime divisor p of n, one has

$$\chi(\widehat{H} - \widehat{L_p}) = 1 - \frac{1}{p} \sum_{i=0}^{p-1} \xi^{in/p} = 1.$$

Then $\chi(\varepsilon(G, H)) = \prod_{p|n} \chi(\widehat{H} - \widehat{L_p}) = 1$ and the lemma follows. ∎

We recall an old Theorem of Shoda [19] that can be deduced from Mackey's Theorem (see [2]).

Theorem 1.3 *[19] Let χ be a linear character defined on a subgroup K of G. Then the induced character χ^G is irreducible if and only if for every $g \in G \setminus K$ there is $k \in K \cap K^g$ such that $\chi(gkg^{-1}) \neq \chi(k)$.*

Definition 1.4 *A pair (H, K) of subgroups of G is called a* Shoda pair *if it satisfies the following conditions:*

(S1) $H \trianglelefteq K$,

(S2) K/H is cyclic and

(S3) If $g \in G$ and $[K, g] \cap K \subseteq H$ then $g \in K$.

Note that giving an $\mathcal{A}$-orbit of a linear character of a subgroup of K of G is equivalent to give a pair a subgroups (H, K) of G satisfying (S1) and (S2). The $\mathcal{A}$-orbit associated to such a pair (H, K) is formed by the characters χ of K with kernel H. Then Shoda's Theorem can be rephrased as follows: if χ is a linear character of a subgroup of K of G with kernel H then the induced character χ^G is irreducible if and only if (H, K) is a Shoda pair.

The group G acts on the right on $\mathbb{C}G$ by conjugation

$$x \cdot g = x^g, \quad (x \in \mathbb{C}G, g \in G).$$

Moreover G acts on the right on the set of characters of subgroups of G by defining $\chi \cdot g$, for χ a character of a subgroup K of G and $g \in G$, as the character of K^g given by

$$(\chi \cdot g)(k) = gkg^{-1}.$$

This action restricts to an action on the set $\mathcal{C}$ of irreducible characters of subgroups of G and this restriction is related with the action of G on $\mathbb{C}G$ by conjugation by the formula

$$e(\chi \cdot g) = e(\chi) \cdot g, \quad (\chi \in \mathcal{C}, g \in G).$$

Thus, the G-stabilizers of a character χ of a subgroup K of G and the stabilizer of the corresponding idempotent $e(\chi)$ coincide and are exactly

$$G_\chi = G_{e(\chi)} = \{g \in N_G(K) : \chi(k^g) = \chi(k), \text{ for every } k \in K\}.$$

Given two subgroups H and K of G such that $H \trianglelefteq K$, let $e(G,K,H)$ denote the sum of all G-conjugates of $\varepsilon(K,H)$, that is if T is a right transversal of $\mathrm{Cen}_G(\varepsilon(K,H))$ in G then

$$e(G,K,H) = \sum_{t \in T} \varepsilon(K,H)^t.$$

Clearly $e(G,K,H)$ is a central element of $\mathbb{Q}G$ and if the G-conjugates of $\varepsilon(K,H)$ are orthogonal, then $e(G,K,H)$ is a central idempotent of $\mathbb{Q}G$.

2 The central idempotent associated to a monomial irreducible character

In this section we show that the primitive central idempotent of $\mathbb{Q}G$ associated to a monomial irreducible complex character can be computed using the elements of the form $e(G,K,H)$. Namely we show the following.

Theorem 2.1 *Let G be a finite group, K a subgroup of G, χ a linear character of K and χ^G the induced character of χ on G. If χ^G is irreducible then the primitive central idempotent of $\mathbb{Q}G$ associated to χ^G is*

$$e_{\mathbb{Q}}(\chi^G) = \frac{[\mathrm{Cen}_G(\varepsilon(K,H)) : K]}{[\mathbb{Q}(\chi) : \mathbb{Q}(\chi^G)]} e(G,K,H)$$

where H is the kernel of χ.

Proof. The actions of $\mathcal{A}$ and G on $\mathbb{C}G$ are compatible in the sense that

$$\sigma \cdot (x \cdot g) = (\sigma \cdot x) \cdot g, \quad (\sigma \in \mathcal{A}, g \in G, x \in \mathbb{C}G).$$

Thus the notation $\sigma \cdot x \cdot g$ is unambiguous and one can consider $\mathcal{A} \times G$ acting on the left on $\mathbb{C}G$ and $\mathcal{C}$ by $(\sigma, g) \cdot x = \sigma \cdot x \cdot g^{-1}$.

Let $e = e(\chi)$. The elements of the $\mathcal{A} \times G$-orbit of an element e can be collected in a table

$$\begin{array}{cccc} \sigma_1 \cdot e \cdot g_1 & \sigma_1 \cdot e \cdot g_2 & \cdots & \sigma_1 \cdot e \cdot g_m \\ \sigma_2 \cdot e \cdot g_1 & \sigma_2 \cdot e \cdot g_2 & \cdots & \sigma_2 \cdot e \cdot g_m \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_n \cdot e \cdot g_1 & \sigma_n \cdot e \cdot g_2 & \cdots & \sigma_n \cdot e \cdot g_m \end{array}.$$

where $T_{\mathcal{A}} = \{\sigma_1, \dots, \sigma_n\}$ is a left transversal of the $\mathcal{A}$-stabilizer of e and $T_G = \{g_1, \dots, g_m\}$ is a right transversal of the G-stabilizer of e. One may take $T_{\mathcal{A}} = \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$. But one should be careful, although each row and each column does not have repeated elements there may exist repeated elements in the table. When two rows (resp. two columns) have an element in common then they have the same entries in a different order. By Theorem 1.3, the G-stabilizer of e is $G_\chi = K$ and hence T_G is a right transversal of G modulo K. Thus

$$e(\chi^G) = \sum_{i=1}^{m} e \cdot g_i \tag{2.2}$$

We complete the table by adding a column (respectively a row), the entries of which are the sum of the corresponding row (respectively column).

$$\begin{array}{cccc|c}
\sigma_1 \cdot e \cdot g_1 & \sigma_1 \cdot e \cdot g_2 & \cdots & \sigma_1 \cdot e \cdot g_m & \sigma_1 \cdot e(\chi^G) \\
\sigma_2 \cdot e \cdot g_1 & \sigma_2 \cdot e \cdot g_2 & \cdots & \sigma_2 \cdot e \cdot g_m & \sigma_2 \cdot e(\chi^G) \\
\cdots & \cdots & \cdots & \cdots & \\
\sigma_n \cdot e \cdot g_1 & \sigma_n \cdot e \cdot g_2 & \cdots & \sigma_n \cdot e \cdot g_m & \sigma_n \cdot e(\chi^G) \\
\hline
\varepsilon(K,H) \cdot g_1 & \varepsilon(K,H) \cdot g_2 & \cdots & \varepsilon(K,H) \cdot g_n & *
\end{array}.$$

Note that by (2.2) the sum of the elements in the i-th row is $\sigma_i \cdot e(\chi^G)$ an by (1.1) and Lemma 1.2 the sum of the elements of the j-th column is $e_{\mathbb{Q}}(\chi) \cdot g_j = \varepsilon(K,H) \cdot g_j$. We can compute the total sum $*$ by adding the elements of the last column or the elements of the last row:

$$* = \sum_{i=1}^{n} \sigma_i \cdot e_{\mathbb{C}}(\chi^G) = \sum_{j=1}^{m} \varepsilon(K,H) \cdot g_j. \tag{2.3}$$

In the first sum of (2.3) the elements to add are the elements of the $\mathcal{A}$-orbit of $e(\chi^G)$, each of them repeated $[\mathbb{Q}(\chi) : \mathbb{Q}(\chi^G)]$ times. Using (1.1) once more one has

$$* = [\mathbb{Q}(\chi) : \mathbb{Q}(\chi^G)] \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi^G)/\mathbb{Q})} \sigma \cdot e(\chi^G) = [\mathbb{Q}(\chi) : \mathbb{Q}(\chi^G)] e_{\mathbb{Q}}(\chi^G) \tag{2.4}$$

Similarly the second sum of (2.3) adds up the elements of the G-orbit of $\varepsilon(K,H)$, each of them repeated $[\mathrm{Cen}_G(\varepsilon(K,H)) : K]$ times. Therefore

$$* = [\mathrm{Cen}_G(\varepsilon(K,H)) : K] e(G,K,H) \tag{2.5}$$

and the Theorem follows comparing (2.4) and (2.5). ∎

The following two corollaries of Theorem 2.1 follow immediately.

Corollary 2.2 *If (H,K) is a Shoda pair of G then there is an $\alpha \in \mathbb{Q}$, necessarily unique, such that $\alpha e(G,K,H)$ is a primitive central idempotent of $\mathbb{Q}G$.*

Recall that a finite group is said to be monomial (or M-group) if every irreducible complex character of the group is monomial, that is induced by a linear character of a subgroup.

Corollary 2.3 *A finite group G is monomial if and only if every primitive central idempotents of $\mathbb{Q}G$ is of the form $\alpha e(G,K,H)$ for $\alpha \in \mathbb{Q}$ and (H,K) a Shoda pair of G.*

3 Sufficient conditions for $\alpha = 1$ and an approach to the structure of the simple algebra

In the previous section we have seen that the primitive central idempotent of $\mathbb{Q}G$ associated to a monomial irreducible character χ^G is of the form $\alpha e(G,K,H)$ for $\alpha \in \mathbb{Q}$ and a Shoda pair (H,K) of G. Clearly $\alpha = 1$ if and only if $e(G,K,H)$ is an idempotent (this happens, for example, if the G-conjugates of $\varepsilon(K,H)$ are orthogonal).

We say that a subgroup K of G is a maximal abelian subgroup of G if it is maximal in $\{A \leq G : A \text{ is abelian}\}$.

Definition 3.1 *A* strongly Shoda pair *of G is a pair (H,K) of subgroups of G satisfying the following conditions:*

(SS1) $H \leq K \trianglelefteq N_G(H)$;

(SS2) K/H is cyclic and a maximal abelian subgroup of $N_G(H)/H$ and

(SS3) for every $g \in G \setminus N_G(H)$, $\varepsilon(K,H)\varepsilon(K,H)^g = 0$.

In this section we show that if (H,K) is a strongly Shoda pair of G then it is a Shoda pair of G and then $e(G,K,H)$ is a primitive central idempotent of $\mathbb{Q}G$. Strongly Shoda pairs have one small and one big advantage with respect to Shoda pairs. The small advantage is that to compute the primitive central idempotent $\alpha e(G,K,H)$ associated to the strongly Shoda pair it is not necessary to compute $\alpha = \frac{[\mathrm{Cen}_G(\varepsilon(K,H)):K]}{[\mathbb{Q}(\chi):\mathbb{Q}(\chi^G)]}$. Instead α can be computed by forcing $\alpha e(G,K,H)$ to be an idempotent, that is, α is the quotient of the coefficients of 1 in $e(G,K,H)$ and $e(G,K,H)^2$. (Recall that the coefficient of 1 of a non zero idempotent of $\mathbb{Q}G$ is non zero [18, Proposition 1.8].) The big advantage is that if (H,K) is a strongly Shoda pair then one can give an approach to determine a description of the structure of the simple component $\mathbb{Q}Ge(G,K,H)$ as we will see in Proposition 3.4.

We use several times the following obvious facts for every $N \leq H \trianglelefteq K \leq G$ such that $N \trianglelefteq G$ and $g \in G$:

$$\varepsilon(K,H)^g = \varepsilon(K^g,H^g), \tag{3.6}$$

$$\omega_N(\varepsilon(K,H)) = \varepsilon(K/N,H/N). \tag{3.7}$$

Lemma 3.2 *Let $H \trianglelefteq K \leq G$.*

1. *If $K \trianglelefteq N_G(H)$, then $N_G(H) \leq \mathrm{Cen}_G(\varepsilon(K,H))$.*
2. *If K/H is cyclic then $\mathrm{Cen}_G(\varepsilon(K,H)) \leq N_G(H)$ and the following conditions are equivalent for $g \in G$:*
   - *(i) $g \in H$,*
   - *(ii) $g\varepsilon(K,H) = \varepsilon(K,H)$ and*
   - *(iii) $\hat{g}\varepsilon(K,H) = \varepsilon(K,H)$.*

Proof. 1 is a consequence of (3.6).

2. Assume that K/H is cyclic. (i) implies (ii) and the equivalence between (ii) and (iii) are obvious. The inclusion $\mathrm{Cen}_G(\varepsilon(K,H)) \leq N_G(H)$ is a direct consequence of (3.6) and the equivalence between (i)-(iii). So we only have to prove that (ii) implies (i).

By (3.7) and [7, Lemma 2] one has that $\varepsilon(K,H) \neq 0$. Assume that $g\varepsilon(K,H) = \varepsilon(K,H)$. Then $g \in K$ because $0 \neq \varepsilon(K,H) = g\varepsilon(K,H) \in \mathbb{Q}K$. Assume that $g \notin H$ and let $M/H \in \mathcal{M}(\langle H,g\rangle/H)$. Thus $M/H \in \mathcal{M}(K/H)$. Moreover $\langle g,H\rangle = \langle g,M\rangle$ and hence $\hat{g}\widehat{H} = \widehat{\langle g,H\rangle} = \widehat{\langle g,M\rangle} = \hat{g}\widehat{M}$. Therefore $\varepsilon(K,H) = \hat{g}\varepsilon(K,H) = 0$, a contradiction. ∎

Proposition 3.3 *The following conditions are equivalent for a pair (H,K) of subgroups of G:*

1. *(H,K) is a strongly Shoda pair of G;*
2. *(H,K) is a Shoda pair of G, $K \trianglelefteq N_G(H)$ and the G-conjugates of $\varepsilon(K,H)$ are orthogonal.*

Moreover if the previous conditions holds then $\mathrm{Cen}_G(K,H) = N_G(H)$ *and* $e(G,K,H)$ *is a primitive central idempotent.*

Proof.

2 implies 1 is a consequence of Lemma 3.2.

Now we prove 1 implies 2. Assume that (H,K) is a strongly Shoda pair. Clearly (H,K) satisfies (S1) and (S2). By Lemma 3.2, $\mathrm{Cen}_G(\varepsilon(K,H)) = N_G(H)$ and hence condition (SS3) implies that the G-conjugates of $\varepsilon(K,H)$ are orthogonal and so $e(G,K,H)$ is a central idempotent. It only remains to show that (H,K) satisfies condition (S3). Let $g \in G$ be such that $[K,g] \cap K \subseteq H$ and χ a linear character of K with kernel H. If $k \in K$ and $g^{-1}kg \in K$ then $[k,g] \in [K,g] \cap K$ and thus $\chi([k,g]) = 1$. Then

$$e(\chi)e(\chi \cdot g^{-1}) = \frac{1}{|K|^2} \sum_{k_1,k_2 \in K} \chi(k_1^{-1})\chi(k_2^{-1})k_1 g k_2 g^{-1}.$$

and hence the coefficient of 1 in $e(\chi)e(\chi \cdot g^{-1})$ is

$$\frac{1}{|K|^2} \sum_{k, g^{-1}kg \in K} \chi(k^{-1})\chi(g^{-1}kg) = \frac{1}{|K|^2} \sum_{k, g^{-1}kg \in K} \chi[k,g] = \frac{|K \cap K^g|}{|K|^2} \neq 0.$$

Since $\varepsilon(K,H) = \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} e(\sigma \cdot \chi)$ (Lemma 1.2) and the different $e(\sigma \cdot \chi)$'s are orthogonal

$$0 \neq e(\chi)e(\chi \cdot g^{-1}) = e(\chi)\varepsilon(K,H)\varepsilon(K,H)^g e(\chi \cdot g^{-1}),$$

and hence $\varepsilon(K,H)\varepsilon(K,H)^g \neq 0$. By condition (SS3), $g \in N_G(H)$ and then $[K,g] \subseteq K$, by condition (SS1). Thus $[K,g] \subseteq K \cap [K,g] \subseteq H$ and hence $\langle K,g \rangle / H$ is abelian. We deduce that $g \in K$ from condition (SS2). This proves (S3). ∎

By $S * G$ we denote the crossed product of a group G over a coefficient ring S [13]. We write $S *_\tau^\sigma G$ to emphasize the action σ and twisting τ of the crossed product $S * G$.

Proposition 3.4 *Let* (H,K) *be a strongly Shoda pair and let* $k = [K:H]$, $N = N_G(H)$, $n = [G:N]$, x *a generator of* K/H *and* $\phi : N/K \to N$ *a left inverse of the projection* $N \to N/K$. *Then* $\mathbb{Q}Ge(G,K,H)$ *is isomorphic to* $M_n(\mathbb{Q}(\xi_k) *_\tau^\sigma N/K)$ *and the action and twisting are given by*

$$\begin{aligned} \xi_k^{\sigma(a)} &= \xi_k^i, \quad \text{if } x^a = x^i; \\ \tau(a,b) &= \xi_k^j, \quad \text{if } \phi(ab)^{-1}\phi(a)\phi(b)H = x^j, \end{aligned}$$

for $a, b \in N/K$ *and integers* i *and* j.

Proof. Set $f = \varepsilon(K,H)$, $e = e(G,K,H)$ and T a right transversal for N in G. By Proposition 3.3, e is a primitive central idempotent of $\mathbb{Q}G$ and $e = \sum_{g \in T} f^g$. By Lemma 1.2, f is a primitive central idempotent of $\mathbb{Q}K$ and $\mathbb{Q}Kf$ is isomorphic to $\mathbb{Q}(\xi_k)$ via the map given by $H \mapsto 1$, $x \mapsto \xi_k$. Furthermore $\mathbb{Q}Nf = \mathbb{Q}Kf *_{\tau_1}^{\sigma_1} N/K$ is a crossed product of N/K over the field $\mathbb{Q}Kf$. Clearly the isomorphism $\mathbb{Q}Kf \simeq \mathbb{Q}(\xi_k)$ extends naturally to an N/K-graded isomorphism $\mathbb{Q}Nf = \mathbb{Q}Kf *_{\tau_1}^{\sigma_1} N/K \simeq \mathbb{Q}(\xi_k) *_\tau^\sigma N/K$

If $g \in G$ then the map $x \mapsto xg$ is an isomorphism between the $\mathbb{Q}G$-modules $\mathbb{Q}Gf$ and $\mathbb{Q}Gf^g$. Therefore ${}_{\mathbb{Q}G}\mathbb{Q}Ge = \oplus_{g \in T} \mathbb{Q}Gf^g \simeq (\mathbb{Q}Gf)^n$. Moreover $f\mathbb{Q}Gf = \mathbb{Q}Nf$, because f is central in $\mathbb{Q}N$. Thus

$$\mathbb{Q}Ge \simeq \mathrm{End}_{\mathbb{Q}G}(\mathbb{Q}Ge) \simeq M_n(\mathrm{End}_{\mathbb{Q}G}(\mathbb{Q}Gf)) \simeq M_n(f\mathbb{Q}Gf) = M_n(\mathbb{Q}Nf) \simeq M_n(\mathbb{Q}(\xi_k) *_\tau^\sigma N/K).$$

∎

Remark 3.5 Note that one can obtain an alternative proof of the fact that if (H, K) is a strongly Shoda pair then $e(G, K, H)$ is a primitive central idempotent, which does not use Theorem 2.1 by extending the arguments of the previous proof as follows:

If aK belongs to the kernel of σ_1 and $B = \langle a, K\rangle$ then $\mathbb{Q}Bf = \mathbb{Q}Kf * B/K$ is commutative. Thus $bf = f$ for every $b \in B'$. By Lemma 3.2, $B' \subseteq H$ and so B/H is abelian. Thus $a \in K$ as a consequence of (SS2). This proves that σ_1 is faithful. By [14, Theorem 29.6], $\mathbb{Q}Nf$ is simple and hence so is $\mathbb{Q}Ge \simeq M_n(\mathbb{Q}Nf)$.

Although condition (SS3) is very easy to check, it is conceptually disappointing because it has to be checked in $\mathbb{Q}G$ rather than in the lattice of subgroups of G. The following corollary shows some sufficient conditions to be checked only in the lattice of subgroups of G for $e(G, K, H)$ to be a primitive central idempotent.

Corollary 3.6 *Let (H, K) be a pair of subgroups of a finite group G satisfying the following conditions:*

1. *$H \trianglelefteq K \trianglelefteq G$;*
2. *K/H is cyclic and maximal abelian subgroup of $N_G(H)/H$.*

Then (H, K) is a strongly Shoda pair and hence $e(G, K, H)$ is a primitive central idempotent of $\mathbb{Q}G$.

Proof. Clearly (H, K) satisfies (SS1) and (SS2). By condition 1 and equation (3.6) the G-conjugates of $\varepsilon(K, H)$ are of the form $\varepsilon(K, H^g)$, with $g \in G$. Since K/H^g is cyclic, Proposition 1.1 yields that $\varepsilon(K, H^g)$ is a primitive central idempotent of $\mathbb{Q}K\widehat{H^g} \simeq \mathbb{Q}(K/H)$, and thus also of $\mathbb{Q}K$. So the G-conjugates of $\varepsilon(K, H)$ are primitive central idempotents of $\mathbb{Q}K$. Then the G-conjugates of $\varepsilon(K, H)$ are mutually orthogonal and hence (SS3) follows from the equality $N_G(H) = \mathrm{Cen}_G(\varepsilon(K, H))$ which is a consequence of Lemma 3.2. ∎

4 Abelian-by-Supersolvable groups

In this section we show that if G is an abelian-by-supersolvable finite group then every primitive central idempotent of $\mathbb{Q}G$ is of the form $e(G, K, H)$ for a strongly Shoda pair (H, K).

Recall that a group G is supersolvable if there is a series of normal subgroups of G with cyclic factors. A group G is said abelian-by-supersolvable if it has an abelian normal subgroup A such that G/A is supersolvable. Notice that the class of abelian-by-supersolvable groups is closed under subgroups and epimorphic images.

We need three lemmas. The first one elementary.

Lemma 4.1 *Let $\{H_1, H_2, \dots, H_k\}$ be an ordered list of non trivial subgroups of a finite group G satisfying the following condition: If $h_1h_2\cdots h_k = h'_1h'_2\cdots h'_k$ with $h_i, h'_i \in H_i$ for every $i = 1, \dots, k$, then $h_i = h'_i$ for all i. Then*

$$(1 - \widehat{H_1})(1 - \widehat{H_2})\cdots(1 - \widehat{H_k}) \neq 0.$$

Lemma 4.2 *Let H and K be subgroups of G such that $H \trianglelefteq K$, K/H is cyclic and the G-conjugates of $\varepsilon(K, H)$ are mutually orthogonal, then* $\mathrm{Cen}_G(\varepsilon(K, H)) = N_G(H)$.

Proof. By Lemma 3.2, $\mathrm{Cen}_G(\varepsilon(K,H)) \le N_G(H)$. To prove the converse inclusion we show that if $g \in N_G(H)$ then $\varepsilon(K,H)\varepsilon(K,H)^g \ne 0$. Let $g \in N_G(H)$. Then $\varepsilon(K,H)^g = \varepsilon(K^g,H)$, so that $\varepsilon(K,H)$ and $\varepsilon(K,H)^g$ belong to $\mathbb{Q}N_G(H)$. Therefore we may assume, without loss of generality, that $N_G(H) = G$, that is $H \trianglelefteq G$. Factoring out by H and using the isomorphism $\omega_H : \mathbb{Q}G\widehat{H} \simeq \mathbb{Q}(G/H)$ and the equality (3.7) one may assume that $H = 1$, so that K is cyclic and we have to prove that $\varepsilon(K)\varepsilon(K)^g \ne 0$. Let $M_1, \dots, M_k$ be the minimal subgroups of K, ordered so that there is an integer $h \le k$ such that $M_i^g = M_i$ if and only if $h < i \le k$. Since $M_1 M_2 \cdots M_k \cap M_1^g M_2^g \cdots M_h^g = \{1\}$ and $M_1 M_2 \cdots M_k = M_1 \oplus M_2 \oplus \cdots \oplus M_k$, the list of groups $\{M_1, \dots, M_k, M_1^g, \dots, M_h^g\}$ satisfies the conditions of Lemma 4.1. Then

$$\varepsilon(K)\varepsilon(K)^g = (1 - \widehat{M_1}) \cdots (1 - \widehat{M_k})(1 - \widehat{M_1^g}) \cdots (1 - \widehat{M_h^g}) \ne 0.$$

∎

We say that N is a maximal abelian normal subgroup of G if N is maximal in $\{A \trianglelefteq G : A \text{ is abelian}\}$. In general not every maximal abelian normal subgroup of G is a maximal abelian subgroup of G.

Lemma 4.3 *If G is a supersolvable group and N is a maximal abelian normal subgroup of G then N is a maximal abelian subgroup of G.*

Proof. Let $K = \mathrm{Cen}_G(N)$. We have to show that $K = N$, so assume the opposite. Since G/N is supersolvable and K/N is a normal subgroup of G/N then G/N has a series of normal subgroups of G/N with cyclic factors containing K/N [17, page 230]. Thus there exists $x \in K \setminus N$ such that $\langle N, x\rangle/N$ is normal in G/N. Since $x \in K$, $\langle N, x\rangle$ is an abelian normal subgroup of G containing N properly, a contradiction. ∎

Theorem 4.4 *Let G be a finite abelian-by-supersolvable group and $e \in \mathbb{Q}G$. Then the following conditions are equivalent.*

1. *e is a primitive central idempotent of $\mathbb{Q}G$.*
2. *$e = e(G,K,H)$ for a strongly Shoda pair (H,K) of G.*
3. *$e = e(G,K,H)$ for a pair (H,K) of subgroups of G satisfying the following conditions:*
   - *(A) $H \trianglelefteq K \trianglelefteq \mathrm{Cen}_G(\varepsilon(K,H))$;*
   - *(B) K/H is cyclic and maximal abelian subgroup of $\mathrm{Cen}_G(\varepsilon(K,H))/H$ and*
   - *(C) the G-conjugates of $\varepsilon(K,H)$ are orthogonal.*

Proof. To show that 2 and 3 are equivalent it is enough to prove that if a pair (H,K) of subgroups of G satisfies either condition 2 or condition 3 then $\mathrm{Cen}_G(\varepsilon(K,H)) = N_G(H)$. That the latter holds in the respective cases follows from Lemma 3.2 and Lemma 4.2.

2 implies 1 is a consequence of Proposition 3.3.

It remains to show that 1 implies 3. Let us call a good idempotent to be one of the form $e(G,K,H)$ with (H,K) a pair of subgroups of G satisfying conditions (A)-(C). Since 3 implies 1, every good idempotent is a primitive central idempotent and we want to show that every primitive central idempotent of $\mathbb{Q}G$ is good or equivalently that 1 is a sum of good idempotents. We argue by induction on $|G|$ with the case $|G| = 1$ being trivial.

Let A be a maximal element in the set of abelian normal subgroups of G such that G/A is supersolvable. Let $\mathcal{N}$ be the set of non trivial normal subgroups of G contained in A. Set $E = \prod_{N\in\mathcal{N}}(1-\widehat{N})$. We are going to obtain our aim by showing that E and $1-E$ are both sums of good idempotents.

Let e be a primitive central idempotent of $\mathbb{Q}G(1-E)$. Then there is $N \in \mathcal{N}$ such that $e\widehat{N} \neq 0$, for otherwise $0 = eE = e$. Thus $e = e\widehat{N} \in \mathbb{Q}G\widehat{N}$ and $\omega_N(e)$ is a primitive central idempotent of $\mathbb{Q}(G/N)$. By the induction hypothesis $\omega_N(e) = e(G/N, K/N, H/N)$ where $(H/N, K/N)$ satisfies conditions (A)-(C) as a pair of subgroups of G/N. By equation (3.7), $e = e(G, K, H)$ and (H, K) satisfies conditions (A)-(C) as a pair of subgroups of G. This proves that $1 - E$ is a sum of good idempotents.

Let $\mathcal{B}$ be the set of subgroups H of A such that A/H is cyclic. By Proposition 1.1, $\{\varepsilon(A,H) : H \in \mathcal{B}\}$ is the set of primitive central idempotents of $\mathbb{Q}A$. Let $\mathcal{B}_1 = \{H \in \mathcal{B} : E\varepsilon(A,H) = \varepsilon(A,H)\}$ and $\mathcal{B}_2$ the complement of $\mathcal{B}_1$ in $\mathcal{B}$. Then $E = \sum_{H\in\mathcal{B}_1}\varepsilon(A,H)$ and $1-E = \sum_{H\in\mathcal{B}_2}\varepsilon(A,H)$.

We claim that $\mathcal{B}_1 = \{H \in \mathcal{B} : N \not\subseteq H \text{ for every } N \in \mathcal{N}\}$. Let $H \in \mathcal{B}$. If $N \subseteq H$ for some $N \in \mathcal{N}$ then $\widehat{N}\varepsilon(A,H) = \varepsilon(A,H)$, by Lemma 3.2. Therefore $E\varepsilon(A,H) = 0$ and hence $H \in \mathcal{B}_2$. Conversely, assume that $N \not\subseteq H$ for every $N \in \mathcal{N}$. By Lemma 3.2, for every $N \in \mathcal{N}$ there exists $n \in N$ such that $n\varepsilon(A,H) \neq \varepsilon(A,H)$ and hence $\widehat{N}\varepsilon(A,H) \neq \varepsilon(A,H)$. Then $\widehat{N}\varepsilon(A,H) = 0$ and so $E\varepsilon(A,H) = \varepsilon(A,H)$. Thus $H \in \mathcal{B}_1$. This proves the claim.

Since $A \trianglelefteq G$, $\mathcal{B}_1$ is closed under conjugation by elements of G and hence E is the sum of the idempotents of the form $e(G, A, B)$ with B running through a set of representatives of the G-conjugates of elements in $\mathcal{B}_1$. Therefore we only have to show that $e(G, A, B)$ is a sum of good idempotents for every $B \in \mathcal{B}_1$. In the remainder of the proof B is an element of $\mathcal{B}_1$.

If $B = 1$ then A is cyclic and hence G is supersolvable. By Lemma 4.3, A is a maximal abelian subgroup of G and hence the pair $(1, A)$ satisfies conditions (A)-(C), so that $e(G, A, 1)$ is a good idempotent.

Assume now that $B \neq 1$. Thus B is a non trivial subgroup of G which does not contain any non trivial normal subgroup of G and in particular B is not normal in G, that is $N_G(B) \neq G$. Let $S = \mathrm{Cen}_G(\varepsilon(A,B))$. By Lemma 3.2, $S = N_G(B)$. Since $\varepsilon(A,B)$ is a central idempotent of $\mathbb{Q}S$ and S is a proper subgroup of G, by induction hypothesis

$$\varepsilon(A,B) = \sum_{i=1}^{k} e(S, K_i, H_i),$$

where each (H_i, K_i) is a pair of subgroups of S satisfying conditions (A)-(C) as subgroups of S.

Claim: If $g \in G \setminus S$ then $\varepsilon(K_i, H_i)\varepsilon(K_i, H_i)^g = 0$.

This is because $\varepsilon(A,B)$ and $\varepsilon(A,B)^g$ are two different primitive central idempotents of $\mathbb{Q}A$ and hence

$$\varepsilon(K_i, H_i)\varepsilon(K_i, H_i)^g = \varepsilon(K_i, H_i)\varepsilon(A,B)\varepsilon(A,B)^g\varepsilon(K_i, H_i)^g = 0.$$

From the Claim it follows that $\mathrm{Cen}_G(\varepsilon(K_i, H_i)) \subseteq S$ and therefore

$$\mathrm{Cen}_G(\varepsilon(K_i, H_i)) = \mathrm{Cen}_S(\varepsilon(K_i, H_i)). \tag{4.8}$$

Let T be a right transversal of S in G and, for each $i = 1, \ldots, k$, let R_i be a right transversal of $\mathrm{Cen}_S(\varepsilon(K_i, H_i))$ in S. By (4.8), R_iT is a right transversal for $\mathrm{Cen}_G(\varepsilon(K_i, H_i))$ in G. So

$$e(G, A, B) = \sum_{t\in T}\varepsilon(A,B)^t = \sum_{t\in T}\sum_{i=1}^{k} e(S, K_i, H_i)^t = \sum_{t\in T}\sum_{i=1}^{k}\sum_{r\in R_i}\varepsilon(K_i, H_i)^{rt} = \sum_{i=1}^{k} e(G, K_i, H_i).$$

Recall that the (H_i, K_i)'s satisfy conditions (A)-(C) as subgroups of S. Now we show that they also satisfy these conditions as subgroups of G. (A) and (B) follow from (4.8) and (C) is a consequence of the Claim. Thus $e(G, A, B)$ is a sum of good idempotents and this finishes the proof. ∎

As a direct consequence of Lemma 4.2 and Theorem 4.4 one has:

Corollary 4.5 *Let G be a finite supersolvable group and $e \in \mathbb{Q}G$. Then the following conditions are equivalent.*

1. *e is a primitive central idempotent of $\mathbb{Q}G$.*
2. *$e = e(G, K, H)$ for a pair (H, K) of subgroups of G satisfying the following conditions:*
   - *(a) $H \leq K \leq N_G(H)$;*
   - *(b) K/H is cyclic and maximal abelian normal subgroup of $N_G(H)/H$ and*
   - *(c) if $g \in G \setminus N_G(H)$, then $\varepsilon(K, H)\varepsilon(K, H)^g = 0$.*
3. *$e = e(G, K, H)$ for a pair (H, K) of subgroups of G satisfying the following conditions:*
   - *(A) $H \trianglelefteq K \leq \mathrm{Cen}_G(\varepsilon(K, H))$;*
   - *(B) K/H is cyclic and maximal abelian normal subgroup of $\mathrm{Cen}_G(\varepsilon(K, H))/H$ and*
   - *(C) the G-conjugates of $\varepsilon(K, H)$ are orthogonal.*

It is well known that every abelian-by-supersolvable group is monomial. Note that we have not used this fact in the proof of Theorem 4.4. In fact one can deduce an stronger result from Theorem 2.1 and Theorem 4.4:

Corollary 4.6 *If G is a abelian-by-supersolvable finite group then every irreducible character of G is a monomial character induced by a linear character χ of a subgroup K of G such that K is normal in $N_G(\mathrm{Ker}\ \chi)$ and the elements of the G-orbit of $e(\chi)$ are mutually orthogonal.*

Now we show how to modify the proof of Theorem 4.4 to show that the primitive central idempotents provided by Corollary 3.6 are enough to describe the primitive central idempotents of $\mathbb{Q}G$ for a metabelian group G. Recall that a group G is metabelian if it contains an abelian normal subgroup A so that G/A is abelian too or equivalently if G' is abelian.

Theorem 4.7 *Let G be a metabelian finite group and let A be a maximal abelian subgroup of G containing G'. The primitive central idempotents of $\mathbb{Q}G$ are the elements of the form $e(G, K, H)$ where (H, K) is a pair of subgroups of G satisfying the following conditions:*

1. *K is a maximal element in the set $\{B \leq G : A \leq B \text{ and } B' \leq H \leq B\}$ and*
2. *K/H is cyclic.*

Proof. Let G and A be as in the statement of the theorem. Note that every subgroup K of G containing A is normal in G. Moreover, if $H \leq B \leq G$ then $B' \leq H$ if and only if $B \subseteq N_G(H)$ and B/H is abelian. Thus if the pair (H, K) satisfies conditions 1 and 2 then it also satisfies conditions 1 and 2 of Corollary 3.6 and so $e = e(G, K, H)$ is a primitive central idempotent of $\mathbb{Q}G$.

Now we want to prove that every primitive central idempotent is of this form. Note that this is equivalent to prove that 1 is a sum of primitive central idempotents of the desired form. We argue by induction on the order of G/A, the case $|G/A| = 1$ follows from Proposition 1.1.

Let $\mathcal{B}$ be the set of subgroups of G containing A properly. Note that every element B in $\mathcal{B}$ is normal in G and hence so is B'. Let $E = \prod_{B\in\mathcal{B}}(1-\widehat{B'})$, which is a central idempotent of $\mathbb{Q}G$. We will show that both E and $1-E$ are a sum of $e(G,K,H)$'s with K and H satisfying conditions 1 and 2.

Let e be a primitive central idempotent of $\mathbb{Q}G(1-E)$. Then there is $B \in \mathcal{B}$ such that $e\widehat{B'} \neq 0$, for otherwise $0 = eE = e$. Thus $e\widehat{B'} = e \in \mathbb{Q}G\widehat{B'}$ and $\omega_{B'}(e)$ is a primitive central idempotent of $\mathbb{Q}(G/B')$. By the induction hypothesis $\omega_{B'}(e) = e(G/B', K/B', H/B')$ where $(H/B', K/B')$ satisfies conditions 1 and 2. Then $e = e(G,K,H)$ and (H,K) satisfies conditions 1 and 2. This proves that $1-E$ is a sum of primitive central idempotents of the desired form.

Let $\mathcal{H}$ be the set of subgroups H of A such that A/H is cyclic. By Proposition 1.1, $1 = \sum_{H\in\mathcal{H}} \varepsilon(A,H)$. Let $\mathcal{H}_1 = \{H \in \mathcal{H} : \varepsilon(A,H)E = \varepsilon(A,H)\}$ and $\mathcal{H}_2$ the complement of $\mathcal{H}_1$ in $\mathcal{H}$. Then $E = \sum_{H\in\mathcal{H}_1} \varepsilon(A,H)$ and $1-E = \sum_{H\in\mathcal{H}_2}\varepsilon(A,H)$.

We claim that $\mathcal{H}_1 = \{H \in \mathcal{H} : B' \not\subseteq H \text{ for every } B \in \mathcal{B}\}$. Let $H \in \mathcal{H}$. If $B' \subseteq H$ for some $B \in \mathcal{B}$ then $\widehat{B'}\varepsilon(A,H) = \varepsilon(A,H)$, by Lemma 3.2. Therefore $\varepsilon(A,H)E = 0$ and hence $H \in \mathcal{H}_2$. Conversely, assume that $B' \not\subseteq H$ for every $B \in \mathcal{B}$. By Lemma 3.2 for every $B \in \mathcal{B}$ there exists $b \in B'$ such that $b\varepsilon(A,H) \neq \varepsilon(A,H)$ and hence $\widehat{B'}\varepsilon(A,H) \neq \varepsilon(A,H)$. Then $\widehat{B'}\varepsilon(A,H) = 0$ and thus $\varepsilon(A,H)E = \varepsilon(A,H)$. Thus $H \in \mathcal{H}_1$. This proves the claim.

By the previous paragraph, if $H \in \mathcal{H}_1$ then A is maximal in the set of subgroups B of G such that $B' \subseteq H$, that is (H,A) satisfies conditions 1 and 2. Consider G acting on $\mathcal{H}_1$ by conjugation and let $\mathcal{R}$ be a set of representatives of this action. Then $E = \sum_{H\in\mathcal{H}_1}\varepsilon(A,H) = \sum_{H\in\mathcal{R}} e(G,A,H)$. This finishes the proof. ∎

Often when the primitive central idempotents of $\mathbb{Q}G$ are sought one recalls that the primitive central idempotents of $\mathbb{Q}G\widehat{G'} \simeq \mathbb{Q}(G/G')$ are easy to compute by using the description of the primitive central idempotents of rational group algebras over abelian groups and one concentrates on computing the primitive central idempotents of $\mathbb{Q}G(1-\widehat{G'})$ [5, 8]. Now we can go further. Indeed, for every group G, the quotient group G/G'' is metabelian and using the isomorphism $\mathbb{Q}G\widehat{G''} \simeq \mathbb{Q}(G/G'')$ one deduces the following from Theorem 4.7.

Corollary 4.8 *Let G be a finite group and A a maximal element in $\{H \leq G : G' \leq H$ and H/G'' is abelian$\}$. Then every primitive central idempotent of $\mathbb{Q}G\widehat{G''}$ is of the form $e(G,K,H)$ for a pair of subgroups (H,K) of G satisfying the following conditions:*

1. *$G'' \leq H$,*
2. *K is a maximal element in the set $\{B \leq G : A \leq B$ and $B' \leq H \leq B\}$ and*
3. *K/H is cyclic.*

5 Examples

In this section we first show a straightforward method to compute the primitive central idempotents of $\mathbb{Q}G$ for G a finite metacyclic group. Then we compare the different methods to compute primitive central idempotents of $\mathbb{Q}G$ we have introduce in the previous sections.

Example 5.1 Recall that a metacyclic finite group is a finite group G having a normal cyclic subgroup $\langle a\rangle$ such that $G/\langle a\rangle$ is cyclic; that is G has a presentation of the form

$$G = \langle a, b \mid a^m = 1, b^n = a^t, bab^{-1} = a^r\rangle.$$

Let u be the multiplicative order of r module m. For every $d|u$ let $G_d = \langle a, b^d \rangle$. Note that $G'_d = \langle a^{r^d-1} \rangle$ and hence G_u is a maximal abelian normal subgroup of G containing G'. By Theorem 4.7 the primitive central idempotents of $\mathbb{Q}G$ are the elements of the form $e(G, G_d, H)$ where d is a divisor of u and H is a subgroup of G_d satisfying the following conditions:

1. $d = \min\{x|u : a^{r^x-1} \in H\}$ and
2. G_d/H is cyclic.

∎

If G is as in Example 5.1 with n a prime number then there are only two kinds of idempotents: those of the form $e(G, G, H)$ with G/H cyclic and those of the form $\varepsilon(\langle a \rangle, \langle a^d \rangle)$ where d is a divisor of m which does not divide $r - 1$. This was the starting point of the investigation of Herman [5] of the group of automorphisms $\mathrm{Aut}(\mathbb{Q}G)$ of $\mathbb{Q}G$ for this case. We quote the following from [5]: "In order to generalize the results of the above metacyclic groups (that is for n prime) to the class of general metacyclic groups, we need an algorithm means to determining the entire collection of non-abelian simple components that would appear. This appears to be a complicated process to work out for a general m and n." This process has three tasks: first determining the primitive central idempotents of $\mathbb{Q}G$; second studying for each primitive central idempotent e of $\mathbb{Q}G$ the exact form of the $\mathbb{Q}Ge$ as a matrix ring over a division ring and third deciding which pairs of primitive central idempotents give rise to isomorphic simple rings.

Another problem that often reduces to determining the structure of the simple components of $\mathbb{Q}G$ is the study of the group of units $\mathbb{Z}G^*$ of the integral group ring $\mathbb{Z}G$. By results of Jespers and Leal [6], and Ritter and Sehgal [16], the Bass cyclic and bicyclic units of $\mathbb{Z}G$ (see [18]) generate a subgroup of finite index of $\mathbb{Z}G^*$ if the simple components of $\mathbb{Q}G$ do not belong to a small list of simple algebras. Jespers, Leal and Polcino [8] have studied when this is the case for G a metacyclic group as above with $n = 2$ and $t = 0$. In this case the problem reduces to the first two tasks of the previous paragraph.

We have seen how to accomplish the first task. Proposition 3.4 provides a tool for the second and third tasks but a careful study of the number theoretic information of the crossed product of Proposition 3.4 is required to complete this process. This study is beyond the scope of this paper.

In the remainder we use the notation $C_n = \langle a \rangle$ to denote a cyclic group of order n and S_n for the symmetric group on n letters.

Let G be a finite group and let E be the set of primitive central idempotents of $\mathbb{Q}G$ and set

$$\begin{aligned}
E_1 &= \{\alpha e(G, K, H) \in E : (H, K) \text{ is a Shoda pair of } G\};\\
E_2 &= \{e(G, K, H) \in E : (H, K) \text{ is a Shoda pair of } G\};\\
E_3 &= \{e(G, K, H) : (H, K) \text{ is a strongly Shoda pair of } G\} \text{ and}\\
E_4 &= \{e(G, K, H) : (H, K) \text{ satisfies conditions 1 and 2 of Corollary 3.6}\}.
\end{aligned}$$

Then $E_4 \subseteq E_3 \subseteq E_2 \subseteq E_1 \subseteq E$. Furthermore $E = E_1$ if and only if G is monomial (Corollary 2.3), $E = E_3$ if G is abelian-by-supersolvable (Theorem 4.4) and $E = E_4$ if G is metabelian. Now we compare the different E_i's.

Example 5.2 *($E_3 \neq E_4$ for G minimal monomial and non metabelian.)* Every subgroup of order at most 23 is metabelian. There are two non isomorphic non metabelian groups of order 24: S_4, which is abelian-by-supersolvable, and the special linear group $\mathrm{SL}(2, 3)$, which is not monomial.

$\mathbb{Q}S_4$ has exactly five primitive central idempotents but there are only 3 pairs (H,K) of subgroups of S_4 satisfying conditions 1 and 2 of Corollary 3.6, namely (S_4,S_4), (A_4,S_4) and (V,A_4), where $V=\langle(1\,2)(3\,4),(1\,3)(2\,4)\rangle$. So E_4 has 3 elements. The other two primitive central idempotents of $\mathbb{Q}S_4$ are $e(S_4,K,\langle(1\,2\,3\,4)\rangle)$ and $e(S_4,K,\langle(1\,3),(2\,4)\rangle)$ where $K=\langle(1\,2\,3\,4),(1\,3)\rangle$. ∎

Example 5.3 *($E_3\neq E_4$ for supersolvable groups and non metabelian.)* Consider the supersolvable group

$$G=\langle a_1,a_2,b,c|a_1^3=a_2^3=b^3=c^2=[a_1,a_2]=[a_1,b]=[a_1,c]=1,a_2^b=a_1a_2,a_2^c=a_2^{-1},b^c=b^{-1}\rangle$$

and let $H=\langle a_2\rangle$. Then $N_G(H)=\langle a_1,a_2,c\rangle$ and $(H,N_G(H))$ is a strongly Shoda pair, so that $e(G,N_G(H),H)$ is a primitive central idempotent of $\mathbb{Q}G$. However a computer search has shown that $e(G,N_G(H),H)\notin E_4$. ∎

Example 5.4 *($E_2\neq E_3$.)* Let $G=(\langle x,y\rangle\times\langle b\rangle)\rtimes\langle a\rangle$ where $\langle x,y\rangle\simeq Q_8$ is the quaternion group of order 8, $\langle b\rangle\simeq C_7$, $\langle a\rangle\simeq C_3$ and the action of $\langle a\rangle$ on $\langle x,y\rangle\times\langle b\rangle$ is given by $x^a=y$, $y^a=xy$ and $b^a=b^2$. Then $(H=1,K=\langle x,b\rangle)$ is a Shoda pair of F and the G-conjugates of $\varepsilon(K,H)$ are orthogonal so that $e(G,K,H)\in E_2$. However (H,K) is not strongly Shoda pair because K is not normal in $G=N_G(H)$ and in fact a computer search shows that $e(G,K,H)\notin E_3$.

Example 5.5 *($E_1\neq E_2$.)* Let $G=(D\times C_7)\rtimes C_3$ where $C_7=\langle b\rangle$ and $C_3=\langle a\rangle$ are cyclic groups of order 7 and 3 respectively and D is the group given by the following presentation:

$$D=\langle x,y,z|z^4=1,x^2=y^2=z^2,[x,z]=[y,z]=1,x^y=x^{-1}\rangle.$$

A computer search shows that for this group E_1 has 11 elements but both E_3 and E_2 has only 10 elements. The element of $E_1\setminus E_2$ is $e=\frac{1}{3}e(G,K,H)$ where $H=\langle xz\rangle$ and $K=N_G(H)=\langle x,z,b\rangle$.

Note that the groups of Examples 5.4 and 5.5 are not monomial. A computer search has shown that $E=E_3$ for all the monomial groups of order at most 500. This yields the question of whether Theorem 4.4 can be extended to monomial groups.

By Theorem 4.4 if G is abelian-by-supersolvable then $E=E_3$ and therefore for every primitive central idempotent e of $\mathbb{Q}G$ there is a linear character χ of a subgroup K of G with kernel H such that $e=e_{\mathbb{Q}}(\chi^G)=e(G,K,H)$, that is $[\mathrm{Cen}(\varepsilon(K,H)):K]=[\mathbb{Q}(\chi):\mathbb{Q}(\chi^G)]$. Since abelian-by-supersolvable groups are monomial, if the equality $[\mathrm{Cen}(\varepsilon(K,H)):K]=[\mathbb{Q}(\chi):\mathbb{Q}(\chi^G)]$ would hold for every Shoda pair (H,K) and every character χ of K with kernel H then Theorem 4.4 would follow immediately. The next example shows that this is not the case.

Example 5.6 Let $C_3=\langle a\rangle$, $C_2=\langle b\rangle$, $Q_8=\langle x,y\rangle$ and consider $G=C_3\rtimes(Q_8\rtimes C_2)$ where the actions are given by

$$a^b=a^{-1},x^b=x^{-1},y^b=yx,a^x=a,a^y=a^{-1}.$$

If $H=\langle b\rangle$ and $K=\langle a,b,x^2\rangle$ then (H,K) is a Shoda pair and therefore the linear character of K, $\chi:K\to\mathbb{C}$ given by $\chi(ax^2)=\xi_6$ and $\chi(b)=1$ induces an irreducible monomial character χ^G of G. Notice that $K=N_G(H)=\mathrm{Cen}_G(\varepsilon(K,H))$ but the G-conjugates of $\varepsilon(K,H)$ are not orthogonal. In fact $e(G,K,H)$ is not idempotent but $e_{\mathbb{Q}}(\chi^G)=e(G,K,H)/2$. Notice that G is supersolvable and so, according to Theorem 4.4, $e_{\mathbb{Q}}(\chi^G)=e(G,K_1,H_1)$ for some strongly Shoda pair (H_1,K_1) of G. Specifically this can be obtained with $H_1=1$ and $K_1=\langle a,x\rangle$. ∎

We finish with computing the primitive central idempotents of the smallest non abelian-by-supersolvable group (which coincides with the smallest non monomial group) in terms of elements of the form $e(G,K,H)$.

Example 5.7 The minimal non monomial group is $\mathrm{SL}(2,3)$ which is isomorphic to $G = Q_8 \rtimes C_3$ where $Q_8 = \langle x, y\rangle$, $C_3 = \langle a\rangle$ and $x^a = y$ and $y^a = xy$. Notice that $G/\langle x^2\rangle \simeq A_4 = \langle \bar{x}, \bar{y}\rangle \rtimes \langle a\rangle$, which is metabelian and the primitive central idempotents of $\mathbb{Q}A_4$ are $\varepsilon(A_4, A_4)$, $\varepsilon(A_4, \langle \bar{x}, \bar{y}\rangle)$ and $e(A_4, \langle \bar{x}, \bar{y}\rangle, \langle \bar{x}\rangle)$. Thus, for G, E_4 has at least 3 elements: $f_1 = \varepsilon(G, G) = \widehat{G}$, $f_2 = \varepsilon(G, Q_8) = \widehat{Q^8} - \widehat{G}$ and $f_3 = e(G, Q_8, \langle x\rangle)$. Let $e = f_1 + f_2 + f_3 = 1 - \widehat{\langle x^2\rangle} = \varepsilon(\langle x^2\rangle, 1)$. Note that the pair $(1, \langle x^2\rangle)$ satisfies conditions (a)-(c) and (A)-(C) of Corollary 4.5. However e is not a primitive central idempotent, namely e is the sum of the following two primitive central idempotents of $\mathbb{Q}G$

$$\begin{aligned} e_1 &= \frac{e(G,B,A)}{2} &= \frac{(1-x^2)(2-(1+(x+y+xy))a-(1-(x+y+xy))a^2)}{12}, \\ e_2 &= \frac{e(G,B,1)-e(G,B,A)}{4} &= \frac{(1-x^2)(4+(1+(x+y+xy))a+(1-(x+y+xy))a^2)}{12}, \end{aligned}$$

where $A = \langle a\rangle$ and $B = \langle x^2 a\rangle$. Since G is not monomial $E_1 \neq E$ and therefore either e_1 or e_2 do not belong to E_1. This is a consequence of the well known fact that the irreducible characters of G associated to e_1 and e_2 are not monomial. We give an alternative proof of this fact by showing directly that e_1 and e_2 do not belong to E_1.

Assume that $e_i = \alpha e(G, K, H)$ for $i = 1$ or 2, $\alpha \in \mathbb{Q}$ and some Shoda pair (H, K) of G. Then $\alpha = \frac{[\mathrm{Cen}_G(\varepsilon(K,H)):K]}{[\mathbb{Q}(\chi):\mathbb{Q}(\chi^G)]}$ for some linear character of K with kernel H and hence $|K|[\mathbb{Q}(\chi) : \mathbb{Q}(\chi^G)]e_i \in \mathbb{Z}G$. Thus $|K|[\mathbb{Q}(\chi) : \mathbb{Q}(\chi^G)]$ is a multiple of 12. Let $k = |K|$. Then $[\mathbb{Q}(\chi) : \mathbb{Q}(\chi^G)]$ divides $\phi([K : H])$ and $\phi([K : H])$ divides $\phi(k)$. Therefore $k\phi(k)$ is a multiple of 12. We deduce that $k = 6, 12$ or 24.

If $k \neq 6$ then K is a normal subgroup of G. Since the normal subgroups of G are $1, \langle x^2\rangle, \langle x, y\rangle$ and G, none of them of order 12, we have that $K = G$ and so $H \trianglelefteq G$. This implies that (G, H) is a strongly Shoda pair and hence $\alpha = 1$. However $e_i \neq e(G, G, H)$ for any normal subgroup H of G.

Thus $k = 6$. Conjugating by an appropriate element of G one may assume without loss of generality that K contains A. The only subgroup of G of order 6 that contains A is B, so that $K = B$. However (B, H) is not a Shoda pair for any subgroup H of B because $[B, x] \cap B = 1$ and $x \notin B$.

Note that e_1 is of the form $\alpha e(G, K, H)$ for $\alpha \in \mathbb{Q}$ and a pair of subgroups (H, K) of G such that $H \trianglelefteq K$. However an exhaustive search of the idempotents of $\mathbb{Q}G$ of this form shows that e_2 cannot be written in this form. ∎

Example 5.7 also shows that conditions (a)-(c) and (A)-(C) of Corollary 4.5 are not enough to ensure that $e(G, K, H)$ is a primitive central idempotent even if G is solvable.

Final Remark. Some of the computations necessary to compute the different E_i's of the groups of the previous examples have been done using a package [11] for System Gap [3] that we have developed and is explained in [12].

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