PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 113, Number 2, October 1991

ON MONOTONE TRAJECTORIES

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(Communicated by Kenneth R. Meyer)

ABSTRACT. In this paper C^1 strongly monotone dynamical systems are investigated. It is proved that the set of points with precompact orbits which converge to a not unstable equilibrium but whose trajectories are not eventually strongly monotone is nowhere dense. This improves on and extends a recent result by P. Poláčik [13].

For a metric space X with metric d, by a *semiflow* on X we mean a continuous mapping $\phi\colon [0, \infty) \times X \to X$ satisfying the following (we denote $\phi_t(\cdot) = \phi(t, \cdot)$):

(S1) $\phi_0 = \mathrm{id}_X$

(S2) $\phi_t \circ \phi_s = \phi_{t+s}$, for $s, t \in [0, \infty)$.

The *trajectory* of $x \in X$ is the mapping $t \mapsto \phi_t x$, $t \in [0, \infty)$. The set $\{\phi_t x : t \in [0, \infty)\}$ is called the *orbit* of $x \in X$. An *equilibrium* is a point $e \in X$ such that $\phi_t e = e$ for all $t \geq 0$. We say a point $x \in X$ (or its trajectory) is *convergent* (to an equilibrium $e \in X$) if $d(\phi_t x, e) \to 0$ as $t \to \infty$. It is easy to check that $x \in X$ is convergent to $e \in X$ if and only if the orbit of x is precompact and $\omega(x) = \{e\}$, where the *ω-limit set* $\omega(x) := \{y \in X : \exists t_k \to \infty$ such that $\phi_{t_k} x \to y$ as $k \to \infty\}$.

A real Banach space V with norm $|\cdot|$ is called *strongly ordered* if V is endowed with a closed cone V_+ having nonempty interior $\mathring{V}_+$. For $v, w \in V$, we write $v \leq w$ if $w - v \in V_+$, $v < w$ if $w - v \in V_+ \setminus \{0\}$, and $v \ll w$ if $w - v \in \mathring{V}_+$. A semiflow ϕ on an open subset of a strongly ordered Banach space V is said to be *strongly monotone* if, for each $t > 0$, $x, y \in X$, the inequality $x < y$ implies $\phi_t x \ll \phi_t y$.

In a strongly monotone semiflow ϕ, the trajectory of $x \in X$ is called *eventually strongly decreasing* (resp. *eventually strongly increasing*) when there is a $T \geq 0$ such that if $T \leq t_1 < t_2$, then $\phi_{t_1} x \gg \phi_{t_2} x$ (resp. $\phi_{t_1} x \ll \phi_{t_2} x$). A trajectory that is eventually strongly decreasing or eventually strongly increasing is called *eventually strongly monotone.* As proved in [6, Theorem 6.4], if the tra-

Received by the editors January 16, 1990.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58D07, 47H07, 47H20, 58F10, 35B40, 35K10, 34D05.

jectory of $x \in X$ is eventually strongly decreasing (resp. eventually strongly increasing), it is convergent to some equilibrium e and $e \ll \phi_t x$ (resp. $e \gg \phi_t x$) for $t \geq T$, provided that x has precompact orbit.

The abstract theory of strongly monotone semiflows was initiated independently by Morris W. Hirsch in [6] (for an early survey see [5, Chapter III]) and by Hiroshi Matano in [9] and [10]. It is well known that a second order parabolic partial differential equation (PDE) satisfying some smoothness and growth conditions generates a semiflow on an appropriate subset of a Banach space V, where V is a (possibly proper) subspace (defined by some of the boundary conditions) of a fractional power space for L^p (see [4]), or of C^i, $i = 0, 1$ (see [11]), or else of a more abstract interpolation space for L^p (see [1, 2]).

If the PDE together with the boundary conditions admits the strong comparison principle and the space V is strongly ordered via the cone V_+ of nonnegative functions, then the resulting semiflow ϕ is strongly monotone (for details of the proof see [6, §4] or [13, §2]). Another source of strongly monotone (semi)flows is strongly cooperative systems of ordinary differential equations (for a survey of such systems the reader is referred to [14] and the papers mentioned therein).

To mention one result in the theory of strongly monotone semiflows, assume that the ambient Banach space V is separable and that each point $x \in X$ has precompact orbit. Then the set of points whose ω-limit sets consist of equilibria is residual in X ([6, Corollary 7.6]).

Recently Peter Poláčik in a series of papers [12, 13] has been investigating the asymptotic behavior of trajectories for strongly monotone semiflows generated by abstract semilinear parabolic differential equations:

$$u' + Gu = f(u), \tag{1}$$

with G being a sectorial operator on a Banach space V and $f: X \to V$ having derivative satisfying the Lipschitz condition, where X is an open subset of a fractional power space V^α for some $\alpha \in [0, 1)$ (for the theory of such equations see [3]). One of Poláčik's principal results (cf. [13, Theorem 1]) states that if all points have precompact orbits and $0 \in X$ is an equilibrium, with the principal eigenvalue of $G - f'(0)$ positive, then the set of points whose trajectories are convergent to 0 but that are not eventually strongly monotone is nowhere dense in the (open) set of points convergent to 0. In a sufficiently small neighborhood of 0, the former is equal to the one-dimensional "strongly stable" manifold at 0.

The main points of the present article are Theorems 1, $1'$, and 2. They are, in fact, reformulations of Poláčik's results in the more abstract setting of C^1 strongly monotone semiflows, the theory of which covers quasilinear parabolic PDE's (not necessarily covered by the existing theory of (1)). Moreover, our method of proof enables one to get rid of the restrictive (although generic) assumption contained in [13] that the equilibrium 0 be simple.

A semiflow ϕ on X is said to be of class C^1 if the mapping ϕ restricted to $(0, \infty) \times X$ has a continuous derivative. For $t > 0$, $x \in X$, $D\phi_t(x)$ denotes the derivative of ϕ with respect to the second variable. By the continuity of that derivative, the assignment $(t, x) \mapsto D\phi_t(x)$ is continuous as a mapping into the Banach space $\mathscr{L}(V)$ of bounded linear operators on V with the operator norm. From (S2) we deduce the following *cocycle property*:

$$D\phi_{t+s}(x) = D\phi_t(\phi_s x) D\phi_s(x) \quad \text{for } x \in X, \ s, t > 0. \tag{2}$$

By (2), the formula

$$D\phi(t, x, v) := (\phi_t x, D\phi_t(x)v) \quad \text{for } t \geq 0, \ x \in X, \ v \in V \tag{3}$$

defines on the (product) tangent bundle $X \times V$ an object having all properties of semiflow except perhaps continuity at $t = 0$. Slightly abusing the language, we call $D\phi$ the *derivative semiflow* of ϕ.

For $x \in X$, let

$$Fx := \left. \frac{\partial \phi_t x}{\partial t} \right|_{t=1}.$$

Differentiating (S2) in s, we obtain

$$D\phi_t(\phi_1 x) Fx = F\phi_t x \quad \text{for } t \geq 0, \ x \in X. \tag{4}$$

We say that ϕ is a C^1 *strongly monotone* semiflow if the following conditions are fulfilled:

(M1) ϕ is a semiflow of class C^1,

(M2) ϕ is strongly monotone, and

(M3) $D\phi_t(V_+ \setminus \{0\}) \subset \mathring{V}_+$, for any $t > 0$, $x \in X$.

Notice that if X is convex (in particular, if X is an order interval; see [6, p. 8]), the property (M2) follows from (M3).

For conditions guaranteeing that the semiflow generated by a PDE be of class C^1, the reader is referred to [3, §3.4] or to [1, 2]. In such cases the derivative semiflow is generated in a natural way by the linearized equation, so the property (M3) is simply the comparison principle for (nonautonomous) linear parabolic PDE's.

Now we state a result (due to D. Henry) which will be extensively used in the sequel:

Proposition 1. *Assume that A is a bounded linear operator on a Banach space U, such that there are positive reals $\lambda'' < \lambda'$ and closed invariant complementary subspaces U', U'' with $\sigma(A \mid U') \subset \{z \in \mathbf{C} : |z| \geq \lambda'\}$ and $\sigma(A \mid U'') \subset \{z \in \mathbf{C} : |z| \leq \lambda''\}$ where $\sigma(\cdot)$ denotes the spectrum. Denote the corresponding projections by P', P'', respectively.*

(a) *Let $u_n \in U \setminus \{0\}$, $n \in \mathbf{N}$, be a sequence such that*

$$|u_{n+1} - Au_n| = o(|u_n|) \quad as\ n \to \infty.$$

Then the following alternative holds: Either $|P'u_n|/|P''u_n| \to \infty$ *and* $|u_n|/\mu^n \to \infty$ *for each* $\mu \in (\lambda'', \lambda')$, *or* $|P'u_n|/|P''u_n| \to 0$ *and* $|u_n|/\mu^n \to 0$ *for each* $\mu \in (\lambda'', \lambda')$.

(b) *Let* $\{S(t) : t > 0\}$ *be a strongly continuous semigroup of bounded linear operators on* U, *such that* $A := S(1)$ *satisfies the assumption of part* (a). *Let* $u : [0, \infty) \to U\backslash\{0\}$ *be a continuous function such that*

$$|u(t+s) - S(s)u(t)| = o(|u(t)|) \quad \text{as } t \to \infty,$$

uniformly for $s \in [1, 2]$. *Then the following alternative holds:*

Either $|P'u(t)|/|P''u(t)| \to \infty$ *and* $|u(t)|/\mu^t \to \infty$ *for each* $\mu \in (\lambda'', \lambda')$,
or $|P'u(t)|/|P''u(t)| \to 0$ *and* $|u(t)|/\mu^t \to 0$ *for each* $\mu \in (\lambda'', \lambda')$.

Proof. See [4, Theorem 2 and the subsequent Corollary]. □

We recall a version of the Kreĭn-Rutman theorem:

Proposition 2. *Assume that* A *is a compact linear operator on a strongly ordered Banach space* V, *with the property* $A(V_+\backslash\{0\}) \subset \mathring{V}_+$. *Then* V *decomposes into a direct sum of two closed invariant subspaces* V_1, V_2 *such that* $\dim V_1 = 1$, $V_1\backslash\{0\} \subset \mathring{V}_+ \cup -\mathring{V}_+$, $V_2 \cap V_+ = \{0\}$, $\sigma(A \mid V_1) = \{\rho(A)\}$, *and* $\sigma(A \mid V_2) \subset \{z \in \mathbf{C} : |z| \le \nu\}$, *where* $\nu < \rho(A)$ (= *the spectral radius of* A).

In the applications of Proposition 2, w always stands for the (unique) unit vector contained in $V_1 \cap \mathring{V}_+$. Moreover, any time a Banach space V is represented as a direct sum $V_a \oplus V_b$ with the corresponding projections P_a, P_b, we assume that the norm satisfies $|v| = \max(|P_a v|, |P_b v|)$ for every $v \in V$.

Let $0 \in X$ be an equilibrium for a strongly monotone semiflow ϕ. $K(\varepsilon)$ denotes the closed ball of center 0 and radius $\varepsilon > 0$. Define

$B(\varepsilon) := \{x \in K(\varepsilon) : |\phi_t x| \to 0 \text{ as } t \to \infty\}$,

$B_+(\varepsilon) := \{x \in B(\varepsilon) :$ the trajectory of x is eventually strongly decreasing$\}$,

$B_-(\varepsilon) := \{x \in B(\varepsilon) :$ the trajectory of x is eventually strongly increasing$\}$,

$B_0(\varepsilon) := B(\varepsilon)\backslash(B_+(\varepsilon) \cup B_-(\varepsilon))$.

Let us formulate our Standing Hypothesis:

> 0 is an equilibrium for a C^1 strongly monotone semiflow ϕ such that $D\phi_1(0)$ is compact with spectral radius $\rho \le 1$.

Theorem 1. *There exists* $\varepsilon > 0$ *such that* $B_0(\varepsilon) \subset W_2(\varepsilon)$ *where* $W_2(\varepsilon)$ *is the local "strongly stable" invariant* C^1 *manifold tangent at* 0 *to the subspace* V_2 *in the Kreĭn–Rutman decomposition for* $D\phi_1(0)$. *In particular,* $B_0(\varepsilon)$ *is nowhere dense in* $K(\varepsilon)$.

Proof. Let ν be as in Proposition 2. By [8, Corollary 5.4], for some $\varepsilon > 0$ the local "strongly stable" invariant manifold $W_2(\varepsilon)$ for the time-one mapping $\phi_1|K(\varepsilon)$ is equal to $\{x \in K(\varepsilon) : |\phi_n x|/\mu^n \to 0$ for each $\mu \in (\nu, \rho)\}$. Fix $x \in B_0(\varepsilon)$.

If $F\phi_\tau x = 0$ for some $\tau \geq 0$, then (2) and (4) imply that $F\phi_t x = 0$ for $t \geq \tau$; hence $t \mapsto D\phi_t(x)$ is constant for $t \geq \tau + 1$. From this we conclude that $x \in W_2(\varepsilon)$.

Assume $F\phi_t x \neq 0$ for all $t \geq 0$. Put $U := V$, $U' := V_1$, $U'' := V_2$, $u(t) := F\phi_t x$, $t \in [0, \infty)$, and $s \in (0, \infty)$. We have

$$\begin{aligned} u(t+s) = F\phi_{t+s}x &= D\phi_{t+s}(\phi_1 x)Fx \\ &= D\phi_s(\phi_{t+1}x)\cdot D\phi_t(\phi_1 x)Fx = D\phi_s(\phi_{t+1}x)F\phi_t x, \end{aligned}$$

so

$$\begin{aligned} |u(t+s) - S(s)u(t)| &= |(D\phi_s(\phi_{t+1}x) - D\phi_s(0))F\phi_t x| \\ &\leq |D\phi_s(\phi_{t+1}x) - D\phi_s(0)| \cdot |F\phi_t x|. \end{aligned}$$

Due to the continuity of the mapping $(s, x) \mapsto D\phi_s(x)$ for $s > 0$, the assignment $x \mapsto (s \mapsto D\phi_s(x)) \in C([1, 2], \mathscr{L}(V))$ is continuous. We have $\phi_t x \to 0$, so $|D\phi_s(\phi_{t+1}x) - D\phi_s(0)| \to 0$ as $t \to \infty$ uniformly for $s \in [1, 2]$. By Proposition 1(b), we get either $|P_1F\phi_t x|/|P_2F\phi_t x| \to 0$ and $|F\phi_t x|/\mu^t \to 0$ as $t \to \infty$, for $\mu \in (\nu, \rho)$, or $|P_1F\phi_t x|/|P_2F\phi_t x| \to \infty$ and $|F\phi_t x|/\mu^t \to \infty$ as $t \to \infty$, for $\mu \in (\nu, \rho)$, where P_i, $i = 1, 2$ are projections onto V_i.

In the first case, integration yields $|\phi_t x|/\mu^t \to 0$ as $t \to \infty$; that is, $x \in W_2(\varepsilon)$. We claim the second case is impossible. In fact, for sufficiently large t we have

$$\left|\frac{F\phi_t x}{|F\phi_t x|} - \frac{P_1F\phi_t x}{|P_1F\phi_t x|}\right| = \left|\frac{P_2F\phi_t x}{|P_1F\phi_t x|}\right| = \frac{|P_2F\phi_t x|}{|P_1F\phi_t x|}, \tag{5}$$

which converges to 0 as $t \to \infty$. Since the range of P_1 is spanned by the vector $w = P_1F\phi_t x/|P_1F\phi_t x|$ belonging to the interior $\overset{\circ}{V}_+$ of V_+, for some $k > 0$ we have $E(k) := \{v \in V : |P_1 v| \geq k|P_2 v|\} \subset \overset{\circ}{V}_+ \cup - \overset{\circ}{V}_+$. The right-hand side of (5) tends to 0, so there is a $\tau \geq 0$ such that $F\phi_t x \in E(k)$ for $t \geq \tau$. By integration, we get that the trajectory of x is eventually strongly monotone.

We have thus proved that the set $B_0(\varepsilon)$ is contained in the one-dimensional manifold $W_2(\varepsilon)$. But the latter is C^1 diffeomorphic to $K(\varepsilon) \cap W_2$, so is nowhere dense. □

We also define the following sets:

$\Sigma_+(\varepsilon) := \{x \in B(\varepsilon) : \exists T \geq 0 \text{ such that } \phi_t x \gg 0 \text{ for all } t > T\}$,

$\Sigma_-(\varepsilon) := \{x \in B(\varepsilon : \exists T \geq 0 \text{ such that } \phi_t x \ll 0 \text{ for all } t > T\}$, and

$\Sigma_0(\varepsilon) := B(\varepsilon) \backslash \left(\Sigma_+(\varepsilon) \cup \Sigma_-(\varepsilon)\right)$.

By [6, Theorem 6.4] we have $B_+(\varepsilon) \subset \Sigma_+(\varepsilon)$ and $B_-(\varepsilon) \subset \Sigma_-(\varepsilon)$.

Theorem 1$'$. *There exists an $\varepsilon > 0$ such that $\Sigma_0(\varepsilon) \subset W_2(\varepsilon)$.*

Proof. The proof goes along the lines of the proof of Theorem 1, (almost) the only difference being that we put $u_n := \phi_n x$, $n \in \mathbf{N}$, and $A := D\phi_1(0)$ and make use of Proposition 1(a) instead of 1(b). The expression

$$|u_{n+1} - Au_n| = |\phi_1\phi_n x - D\phi_1(0)\phi_n x|$$

is $o(|\phi_n x|)$, since ϕ_1 is of class C^1. □

Theorem 2. *Assume that either of the following conditions holds:*

(a) *The set X_0 of points with precompact orbits is dense in $K(\varepsilon)$.*

(b) *For each $x \in B(\varepsilon)\setminus\{0\}$ we have $\xi := \inf\{\mu > 0 : \liminf_{t\to\infty} |\phi_t x|/\mu^t = 0\} > 0$.*

Then $\Sigma_0(\varepsilon) = W_2(\varepsilon)$.

Proof. In view of Theorem 1′, it suffices to show that $\Sigma_+(\varepsilon) \cap W_2(\varepsilon) = \Sigma_-(\varepsilon) \cap W_2(\varepsilon) = \varnothing$. Suppose to the contrary that some $x \in \Sigma_+(\varepsilon) \cap W_2(\varepsilon)$.

Assume that (a) is satisfied. Without loss of generality, we can write $x \gg 0$ (if not, substitute $\phi_T x$ for x where T is as in the definition of $\Sigma_+(\varepsilon)$). Since $x \in W_2(\varepsilon)$, for all $\mu \in (\nu, \rho)$ we have $|\phi_n x|/\mu^n \to 0$. Define $M := \{y \in K(\varepsilon) : 0 \ll y \ll x\}$. The set M is open and nonempty (for it contains the open segment joining 0 and x). The manifold $W_2(\varepsilon)$ is nowhere dense, so, because X_0 is dense, we can find $y \in M \cap X_0 \setminus W_2(\varepsilon)$. Since $y \gg 0$, we have $\omega(y) \geq \omega(0) = \{0\}$; on the other hand, $\omega(y) \leq \omega(x) = \{0\}$. Therefore $\phi_t y \to 0$ as $t \to \infty$. Furthermore $0 \ll \phi_t y \ll \phi_t x$ for $t \geq 0$. P_1 is the projection onto the subspace V_1 spanned by $w \in \mathring{V}_+$, so $0 \ll P_1\phi_t y \ll P_1\phi_t x$, and $0 < |P_1\phi_t y| < |P_1\phi_t x|$ for all $t \geq 0$. Hence it follows that $|P_1\phi_n y|/\mu^n \to 0$ for all $\mu \in (\nu, \rho)$. Because $y \notin W_2(\varepsilon)$, there exist a number $\kappa \in (\nu, \rho)$ and a subsequence $n_i \to \infty$ as $i \to \infty$ such that

$$(6) \qquad |\phi_{n_i} y|/\kappa^{n_i} \to \infty \quad \text{as } i \to \infty.$$

This is possible only if $|P_2\phi_{n_i} y|/\kappa^{n_i} \to \infty$, which implies that

$$(7) \qquad |P_1\phi_{n_i} y|/|P_2\phi_{n_i} y| \to 0.$$

Put $U := V$, $U' := V_1$, $U'' := V_2$, $A := D\phi_1(0)$, and $u_n := \phi_n y$. We have

$$|u_{n+1} - Au_n| = |\phi_1 u_n - D\phi_1(0)u_n|,$$

which is $o(|u_n|)$ (since ϕ_1 is C^1). In view of Proposition 1(a) we deduce from (7) that $|\phi_n y|/\kappa^n \to 0$, which contradicts (6). (The idea of this part of the proof is taken from the proof of [13, Lemma 3.2].)

Now assume that (b) is fulfilled. The finite-dimensional subspace V_3 corresponding to the eigenvalues with moduli in $(\xi/2, \rho]$ is nontrivial (notice that $\xi \leq \rho$, in particular $V_1 \subset V_3$). Denote by V_4 the subspace of V complementary to V_3, and by P_3, P_4 the respective projections. The case $V_4 = \{0\}$ (impossible if $\dim V = \infty$) is not excluded. Suppose $V_3 = V_1$. From this we obtain $\nu \leq \xi/2$. By [7, Corollary 5.4], for each $\mu \in (\xi/2, \xi) \subset (\nu, \rho)$ we have $|\phi_n x|/\mu^n \to 0$ (since $x \in W_2(\varepsilon)$). However, from the definition of ξ

it follows in a straightforward way that $|\phi_t x|/\mu^t \to \infty$ for any $\mu \in (0, \xi)$, a contradiction. Therefore V_1 is a proper subspace of V_3.

Putting $U := V$, $U' := V_3$, $U'' := V_4$, $A := D\phi_1(0)$ and $U(t) := \phi_t x$ and applying Proposition 1(b), we get that $|P_3\phi_t x|/|P_1\phi_t x| \to \infty$, from which it follows that $P_3\phi_t x \neq 0$ for sufficiently large t. This enables us to use Proposition 1(b) once more, this time with $U := V_3$, $U' := V_1$, $U'' := V_5 := V_3 \cap V_2$, $A := D\phi_1(0)$, and $u(t) := P_3\phi_t x$. We have either $|P_1\phi_t x|/|P_5\phi_t x| \to 0$ or $|P_1\phi_t x|/|P_5\phi_t x| \to \infty$ (notice that $P_1P_3 = P_1$ and $P_5P_3 = P_5$). For a subset Y of V_3, denote by ΠY the image of $Y\setminus\{0\}$ under projectivization. The above dichotomy can be formulated thus:

$$\text{Either} \quad \delta(\Pi P_3\phi_t x, \Pi V_5) \to 0 \quad \text{or} \quad \delta(\Pi P_3\phi_t x, \Pi V_1) \to 0,$$

with $\delta(N_1, N_2) := \inf\{\Delta(a_1, a_2) : a_1 \in N_1, a_2 \in N_2\}$, where Δ is a metric on the projective space ΠV_3. Since $x \in \Sigma_+(\varepsilon)$, we have $\Pi P_3\phi_t x \in \Pi(V_+ \cap V_3)$ for all t sufficiently large. Because the compact sets ΠV_5 and $\Pi(V_+ \cap V_3)$ are disjoint, there is $\beta > 0$ such that $\delta(\Pi V_5, \Pi(V_+ \cap V_3)) \geq \beta$. Therefore $\delta(\Pi P_3\phi_t x, \Pi V_1) \to 0$, and $|P_1\phi_t x|/|P_5\phi_t x| \to \infty$. Hence $|\phi_t x|/\mu^t \to \infty$ for all $\mu \in (\nu, \rho)$, which is in contradiction to the characterization of $W_2(\varepsilon)$. □

Corollary. *Under the assumptions of Theorem* 2, *the following equalities hold*:

$$\Sigma_0(\varepsilon) = B_0(\varepsilon) = W_2(\varepsilon), \qquad \Sigma_+(\varepsilon) = B_+(\varepsilon), \qquad \Sigma_-(\varepsilon) = B_-(\varepsilon).$$

Proof. By Theorem 2, $W_2(\varepsilon) = \Sigma_0(\varepsilon)$. From [6, Theorem 6.4] we deduce that $\Sigma_0(\varepsilon) \subset B_0(\varepsilon)$, while by our Theorem 1, $B_0(\varepsilon) \subset W_2(\varepsilon)$. From the first equality it follows that $\Sigma_+(\varepsilon) \cup \Sigma_-(\varepsilon) = B_+(\varepsilon) \cup B_-(\varepsilon)$. Since the sets on both sides are disjoint and, again by [6, Theorem 6.4], $B_+(\varepsilon) \subset \Sigma_+(\varepsilon)$ and $B_-(\varepsilon) \subset \Sigma_-(\varepsilon)$, the remaining equalities follow easily. □

Concluding remarks. 1. The additional assumptions imposed in Theorem 2 are fulfilled in many cases. For example, hypothesis (a) is satisfied when the semiflow ϕ is order compact (for the definition see [6, §2]). For conditions guaranteeing the fulfillment of hypothesis (b) see [8, Theorem 1′ in the Appendix].

2. One can define the global versions of the sets $B(\varepsilon)$, etc. by replacing everywhere the condition "$x \in K(\varepsilon)$" with "$x \in X$." All theorems proved in this article extend to the global case. We shall not go into details here; the interested reader is referred to [13, §4].

Acknowledgments

The author wishes to thank Peter Poláčik for sending a preprint of the first version of [13].

The present article was written during the author's stay at the Mathematics Research Centre, University of Warwick, Coventry, England. Support by the Stefan Batory Trust is gratefully acknowledged.

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