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On morphisms from projective space $\boldsymbol{P}^n$ to the Grassmann variety $\boldsymbol{Gr(n, d)}$

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§ 1. Introduction and notation.

In [6] we proved that there exist no morphisms, expect constant ones, from the m-dimensional projective space $\boldsymbol{P}^m$ to the Grassmann variety $Gr(n, d)$ if $m>n$.

In the present article, we study such a case that $m=n$ *and* $n-1>d>0$, and we obtain the following:

Theorem. *There exist no morphisms, except constant ones, from* $\boldsymbol{P}^n$ *to* $Gr(n, d)$ *if one of the following conditions holds*:

i) n *is even and* $n-1>d>0$.

ii) d *is even*, $n-1>d>0$ *and* $(n, d)\neq(5, 2)$.

In § 3, we give an example of a non-constant morphism from $\boldsymbol{P}^3$ to $Gr(3, 1)$. Furtheremore, in the case when the defining field is of characteristic 2, we give an example of a non-constant morphism from $\boldsymbol{P}^5$ to $Gr(5, 2)$. Using this example, we give an example of indecomposable vector bundle of rank 2 on $\boldsymbol{P}^5$.

We use the same notation as in [6], i. e.; $\boldsymbol{P}^n$ is the n dimensional projective space defined over an algebraically closed field k of an arbitrary characteristic p: H is a hyperplane of $\boldsymbol{P}^n$; $Gr(n, d)$ is the Grassmann variety which parametrizes d dimensional linear subspaces of $\boldsymbol{P}^n$; $E(n, d)$ is the universal subbundle of rank $d+1$ on $Gr(n, d)$ and $Q(n, d)$ is the universal quotient bundle of rank $n-d$ on $Gr(n, d)$;

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$\omega_{a_0,a_1,\cdots,a_d}(=\Omega_{n-d-a_0,n-d+1-a_1,\cdots,n-a_d})$ is the Schubert cycle of codimension $\sum_{i=0}^{d} a_i$ of $Gr(n,d)$ defined for an arbitary $(d+1)$-tuple of integers $(a_0, a_1, \cdots, a_d)$ such that $n-d\geqq a_0\geqq a_1\geqq\cdots\geqq a_d\geqq 0$ (cf. [6]); these Schubert cycles satisfy the following Pieri's formula

$$\omega_{a_0,a_1,\cdots,a_d}\omega_{h,0,\cdots,0}=\sum \omega_{b_0,b_1,\cdots,b_d} \tag{1}$$

where the summation runs over all the $(d+1)$-tuples of integers $(b_0, b_1, \cdots, b_d)$ which satisfy the relation,

$$n-d\geqq b_0\geqq a_0\geqq b_1\geqq a_1\geqq b_2\geqq\cdots\geqq a_{d-1}\geqq b_d\geqq a_d\geqq 0$$

and

$$\sum_{i=0}^{d} b_i=\sum_{i=0}^{d} a_i+h;$$

$\mathrm{Dr}(n, d, 0)=\{(x, P)\in Gr(n, d)\times \boldsymbol{P}^n | L_x\ni P\}$ is the flag variety, where L_x is the d-dimensional linear subspace of $\boldsymbol{P}^n$ which is represented by x; $c_i(E)$ is the i-th Chern class of a vector bundle E and $c(E)=1+c_1(E)+c_2(E)+\cdots$ is the total Chern class of E; $\check{E}$ is the dual vector bundle of E.

§ 2. Proof of the theorem.

Let f be a morphism from $\boldsymbol{P}^n$ to $Gr(n, d)$. Let c_i and d_j be the integers such that

$$\begin{cases} c_i(f^*\check{E}(n,d))=c_iH^i & 1\leqq i\leqq d+1 \\ c_j(f^*Q(n,d))=d_jH^j & 1\leqq j\leqq n+d \end{cases} \tag{2}$$

Then, we have

$$\begin{aligned}(1-c_1t+c_2t^2+\cdots+(-1)^{d+1}c_{d+1}t^{d+1})&(1+d_1t+d_2t^2+\cdots+d_{n-d}t^{n-d}) \\ &=1+(-1)^{d+1}c_{d+1}d_{n-d}t^{n+1},\end{aligned} \tag{3}$$

where t is an indeterminate, cf. [6].
When $c_{d+1}d_{n-d}=0$, we see easily that $c_1=0$. This implies that $f(\boldsymbol{P}^n)$ is one point, cf. the proof of [6, Corollary 4.2]. Thus we may assume that $c_{d+1}d_{n-d}\neq 0$. The theorem is proved if we show that this assumption leads to a contradiction. Since nd is even, we have that $c_1, c_2, \cdots, c_{d+1}, d_1, d_2, \cdots, d_{n-d}$ are positive integers and $a=\sqrt[n+1]{c_{d+1}d_{n-d}}$ is a po-

sitive integer, by virtue of [6, Lemma 3.3]. Set

$$G(t)=1-C_1t+C_2t^2+\cdots+(-1)^{d+1}C_{d+1}t^{d+1}$$

and

$$H(t)=1+D_1t+D_2t^2+\cdots+D_{n-d}t^{n-d},$$

where

$$C_i=c_i/a^i \text{ and } D_j=d_j/a^j \quad (1\leqq i\leqq d+1,\ 1\leqq j\leqq n-d).$$

Then, we have

$$G(t)H(t)=1+(-1)^{d+1}t^{n+1}, \tag{4}$$

and hence C_i, D_j are positive integers.

Case (*i*). Suppose that n is even and $n-1>d>0$. Since $Gr(n, d)$ and $Gr(n, n-d-1)$ are isomorphic to each other and since $d+1$ or $n-d$ is odd, we may assume, furtheremore, that $d+1$ is odd. By virtue of the equality (4), we have

$$G(-1)H(-1)=2.$$

Hence, we have

$$1+d+1\leqq 1+C_1+C_2+\cdots+C_{d+1}=G(-1)\leqq 2.$$

This contradicts the assumption that $d>0$.
In order to prove the theorem in case (ii), we need the following lemma.

Lemma 1. *Let n be odd, d even, and $n-1>d>0$. Suppose that there exists a non-constant morphism f from $\boldsymbol{P}^n$ to $Gr(n, d)$. Let C_i and D_j* be as above, *then* $n+1=2(d+1)=2(n-d)$, $C_1=C_2=\cdots=C_d=D_1=D_2=\cdots=D_{n-d-1}=2$ *and* $C_{d+1}=D_{n-d}=1$.

Proof. Set $2s=n+1$. We may assume that $n-d\geqq s$. $1-t$ divides $G(t)$ since $G(1)H(1)=0$ and $H(1)\neq 0$. Hence, we have

$$\frac{G(t)}{1-t}\cdot H(t)=1+t+t^2+\cdots+t^n.$$

Therefore, $H(1)$ divides $n+1(=2s)$. This and the fact that $H(1)=1+D_1+D_2+\cdots D_{n-d}\geqq 1+n-d\geqq 1+s$, show that $H(1)=2s$. Now we show that $D_1\geqq 2$. Assume that $D_1=1$. Then, by (4), we have

$$C_2=D_1^2-D_2=1-D_2\leqq 0.$$

This contradicts the assumption that $C_2>0$. Next we claim that $D_1, D_2, \cdots, D_{n-d-1}\geqq 2$. In order to see it let us suppose to the contary, that there exists a positive interger $i\leqq n-d-1$ such that $D_i=1$. The assertion is proved if we show that this assumption leads a contradiction Set $j=\min\{i|D_i=1\}$. Since $D_1\geqq 2$, $D_{j-1}\geqq 2$. It is easy to see that $H(t)=t^{n-d}H(t^{-1})$ cf. [6, § 3]. Hence, we have

$$1=D_{n-d},\ D_1=D_{n-d-1},\ D_2=D_{n-d-2},\cdots$$

Therefore, we have $2j\leqq n-d$. Since $\omega_{j,j,0,\dots,0}$ is numerically non-negative, so is $f^*\omega_{j,j,0,\dots,0}$ (cf. [6, Corollary 1.2]). On the other hand

$$\begin{aligned}
f^*\omega_{j,j,0,\dots,0}&=f^*(\omega^2_{j,0,\dots,0}-\omega_{j+1,0,\dots,0}\omega_{j-1,0,\dots,0}) \quad \text{(cf. (1) and (2))}\\
&=f^*(c_j(Q(n,d)))^2-f^*(c_{j+1}(Q(n,d)))f^*(c_{j-1}(Q(n,d)))\\
&\qquad\qquad\qquad\qquad \text{(cf. [6]]}\\
&=(d_j^2-d_{j+1}d_{j-1})H^{2j}\\
&=(D_j^2-D_{j+1}D_{j-1})a^{2j}H^{2j}.
\end{aligned}$$

Since $D_j^2-D_{j+1}D_{j-1}\leqq 1-2<0$, this contradicts the fact that $f^*\omega_{j,j,0,\dots,0}$ is numerically non-negative. Hence, we have

$$\begin{aligned}
n+1=H(1)&=1+D_1+D_2+\cdots D_{n-d}\leqq 1+2(n-d-1)+1\\
&=2(n-d)\leqq 2s.
\end{aligned}$$

This shows that $n-d=s$ and $D_1=D_2=\cdots=D_{n-d-1}=2$. It is easy to see that $C_1=C_2=\cdots=C_d=2$ by virtue of (4). q.e.d.

Let us continue the proof of the theorem.

Case (ii). Let n be odd, d even, $n-1>d>0$ and $(n,d)\neq(5,2)$. Assuming that there exists a non-constant morphism from $\boldsymbol{P}^n$ to $Gr(n,d)$, we show that this assumption leads to a contradiction. Let a and D_i be as above. Since $(n,d)\neq(5,2)$, we have $d\geqq 4$ and $D_1=D_2=D_3=2$. Therefore, we have

$$\begin{aligned}
f^*\omega_{2,2,0,\dots,0}&=f^*(\omega^2_{2,0,\dots,0}-\omega_{3,0,\dots,0}\omega_{1,0,\dots,0})\\
&=(D_2^2-D_3D_1)a^4H^4=0.
\end{aligned}$$

This shows that

$$\begin{aligned} 0 &= f^*(\omega_{2,2,0,\dots,0}\omega_{n-d-2,0,\dots,0}) \\ &= f^*\omega_{n-d,2,0,\dots,0} + f^*\omega_{n-d-1,2,1,0,\dots,0} \\ &\quad + f^*\omega_{n-d-2,2,2,0,\dots,0} \quad \text{(cf. (1))}. \end{aligned}$$

Since $f^*\omega_{n-d-1,2,1,0,\dots,0}$ and $f^*\omega_{n-d-2,2,2,0\dots,0}$ are numerically non-negative, we have that $f^*\omega_{n-d,2,0,\dots,0}=0$. On the other hand we have,

$$\begin{aligned} f^*\omega_{n-d,2,0,\dots0} &= f^*(\omega_{n-d,0,\dots,0}\omega_{2,0,\dots,0}) \\ &= f^*(c_{n-d}(Q(n, d)))f^*(c_2(Q(n, d))) \quad \text{(cf. (6))} \\ &= D_{n-d}D_2 a^{n-d+2}H^{n-d+2} \neq 0\,. \end{aligned}$$

This is a contradiction. q.e.d.

§ 3. Examples.

Example 1. Let S_4 be the quadric hypersurface of $\boldsymbol{P}^5$ defined by the homogeneous equation

$$X_0X_1 + X_2X_3 + X_4X_5 = 0\,.$$

Then, it is well-known that $Gr(3, 1)$ is isomorphic to S_4 (cf. [2] Chapter XIV). Let f be a morphism from $\boldsymbol{P}^3$ to S_4 defined by

$$f(x_0, x_1, x_2, x_3) = (x_0^2, -x_1^2, x_2^2, x_3^2, x_0x_1 + x_2x_3, x_0x_1 - x_2x_3)\,.$$

It is easy to see that f is not a constant morphism.

Example 2. Let S_6 be the quadric hypersurface of $\boldsymbol{P}^7$ defined by the homogeneous equation

$$X_0X_1 + X_2X_3 + X_4X_5 + X_6X_7 = 0\,.$$

For a generic point $P = (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ of S_6, let $h(P)$ be the point of $Gr(5, 2)$ which represents the 2-dimensional linear subspace of $\boldsymbol{P}^5$ spanned by the three points

$$\begin{aligned} &(x_0^2, 0, 0, x_2^2, x_2x_4 + x_0x_7, x_2x_6 - x_0x_5) \\ &(0, x_0^2, 0, x_2x_4 - x_0x_7, x_4^2, x_4x_6 + x_0x_3) \\ &(0, 0, x_0^2, x_2x_6 + x_0x_5, x_4x_6 - x_0x_3, x_6^2)\,. \end{aligned} \tag{5}$$

Lemma 2. *h is a morphism from S_6 to $Gr(5, 2)$.*

Proof. The plücker coordinate of $h(P)$ is

$(x_0^6, x_0^4(x_2x_6+x_0x_5), x_0^4(x_4x_6-x_0x_3), x_0^4x_6^2, x_0^4(x_0x_7-x_2x_4),$

$-x_0^4x_4^2, -x_0^4(x_4x_6+x_0x_3), x_0^4(x_1x_4+x_3x_7), x_0^4(x_1x_6-x_3x_5),$

$x_0^4x_3^2, x_0^4x_2^2, x_0^4(x_2x_4+x_0x_7), x_0^4(x_2x_6-x_0x_5), x_0^4(x_5x_7-x_1x_2),$

$-x_0^4x_5^2, x_0^4(x_1x_6+x_3x_5), x_0^4x_7^2, -x_0^4(x_1x_2+x_5x_7), x_0^4(x_3x_7$

$-x_1x_4), x_0^4x_1^2)$.

Hence, it is easy to see that h is a morphism. q.e.d.

Let S_5 be the hyperplane section of S_6 by $x_0-x_1=0$ and g be the morphism from S_5 to $\boldsymbol{P}^5$ defined by

(6) $$g(Q)=(y_3, y_5, y_7, -y_2, -y_4, -y_6)$$

where $Q=(y_0, y_0, y_2, y_3, y_4, y_5, y_6, y_7)$ is a generic point of S_5. Since

$y_3(y_0^2, 0, 0, y_2^2, y_2y_4+y_0y_7, y_2y_6-y_0y_5)+y_5(0, y_0^2, 0, y_2y_4-y_0y_7, y_4^2,$

$y_4y_6+y_0y_3)+y_7(0, 0, y_2y_6+y_0y_5, y_4y_6-y_0y_3, y_6^2)$

$=y_0^2(y_3, y_5, y_7, -y_2, -y_4, -y_6)$,

the point $g(Q)$ lies on the plane represented by $h(Q)$. Hence, we have

***Lemma* 3.** *(h, g) is a morphism from S_5 to the flag variety $Dr(5, 2, 0)$.*

From now on we assume that the defining field is of charactristic 2. Let f be the morphism from $\boldsymbol{P}^5$ to S_5 defined by

$f(x_0, x_1, x_2, x_3, x_4, x_5)$

$=(x_0x_1+x_2x_3+x_4x_5, x_0x_1+x_2x_3+x_4x_5, x_0^2, x_1^2, x_2^2, x_3^2, x_4^2, x_5^2)$

It is easy to see that $h\cdot f$ is not a constant morphism.

Set $h'=h\cdot f$ and $g'=g\cdot f$. Then, Lemma 3 shows that the vector bundle $h'^*(\check{E}(5, 2))$ contains $g'^*(\mathcal{O}_{\boldsymbol{P}^5}(-1))$ as a sublinebundle. We show that the quotient bundle $h'^*(\check{E}(5, 2))/g'^*(\mathcal{O}_{\boldsymbol{P}^5}(-1))$ is an indecomposable bundle of rank 2. Let c_1, c_2 and c_3 be integrs such that

$$c_i(h'^*(\check{E}(5, 2))) = c_i H^i \quad i=1, 2, 3.$$

Then, we have $c_1 = 2a$, $c_2 = 2a^2$, $c_3 = a^3$ with a positive integer a, by virtue of Lemma 1. Hence, the total Chern class of $h'^*(\check{E}(5, 2))$ is

$$1-2aH+2a^2H^2-a^3H^3=(1-aH)(1-aH+a^2H^2).$$

This shows that the total Chern class of

$$h'^*(\check{E}(5, 2))/g'^*(\mathcal{O}_{\boldsymbol{P}^5}(-1)) \text{ is}$$

$$1-aH+a^2H^2$$

and the hundle is indecomposable.

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