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ON μ-RESOLVABLE AND AFFINE μ-RESOLVABLE BALANCED INCOMPLETE BLOCK DESIGNS

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The concept of resolvability and affine resolvability was generalized to μ-resolvability and affine μ-resolvability by Shrikhande and Raghavarao (1964). In this paper, a representation of parameters of an affine μ-resolvable BIB design is given and necessary conditions for the existence of this design are derived. Some methods of constructing (affine) μ-resolvable BIB designs are given and some inequalities for these designs are obtained. Finally, some information on the block structure of μ-resolvable BIB designs is provided.

0. Introduction and summary. In a Balanced Incomplete Block (BIB) design with parameters v, b, r, k and λ, we have the following relations:

$$vr = bk\,, \qquad \lambda(v-1) = r(k-1)\,, \qquad b \geqq v\,. \tag{0.1}$$

The concept of resolvability and affine resolvability, introduced by Bose [2], was generalized to μ-resolvability and affine μ-resolvability by Shrikhande and Raghavarao [11]. They gave necessary and sufficient conditions for a μ-resolvable BIB design to be affine μ-resolvable, a necessary condition for the existence of an affine μ-resolvable BIB design, and further in [10] gave a certain method of constructing these designs.

In this paper, a different approach is used. The parameters of an affine μ-resolvable BIB design expressed in terms of only three integral variables are given, and necessary conditions for the existence of an affine μ-resolvable BIB design are derived. Some methods of constructing (affine) μ-resolvable BIB designs are stated and further (affine) μ-resolvability of a BIB design based on a finite geometry over a Galois field is investigated. Some inequalities (including a generalization of Bose's one) for BIB designs with special parameters are given. Finally, we provide some information on the block structure of μ-resolvable BIB designs of a certain type.

1. Parameters and nonexistence of affine μ-resolvable BIB designs. A BIB design is called μ-resolvable if the blocks can be separated into t sets of m blocks each such that each set contains every treatment exactly μ times. For a μ-resolvable BIB design, we necessarily have

$$b = mt\,, \qquad r = \mu t\,, \qquad v\mu = mk\,, \qquad b\mu = mr\,. \tag{1.1}$$

A μ-resolvable BIB design is called affine μ-resolvable if any pair of blocks belonging to the same set contain q_1 treatments in common, whereas any pair of

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blocks belonging to different sets contain q_2 treatments in common. An (affine) 1-resolvable design may be simply called (affine) resolvable.

THEOREM A (cf. [11]). *The necessary and sufficient condition for a μ-resolvable BIB design to be affine μ-resolvable is*

$$b - v = t - 1 . \tag{1.2}$$

From (0.1), (1.1) and the definition of affine μ-resolvability, we have $q_1 = (\mu - 1)k/(m - 1)$ and $q_2 = \mu k/m = k^2/v$. Moreover, since from (1.1) and (1.2) $q_1 = (\mu - 1)k/(m - 1) = k + \lambda - r$ can be obviously shown, we have

THEOREM 1.1. *A necessary condition for the existence of an affine μ-resolvable BIB design with parameters v, $b = mt$, $r = \mu t$, k and λ, where $b - v = t - 1$, is that $k^2/v = \mu k/m$ is an integer.*

This is essentially an alternative derivation of Theorem 3 of Shrikhande and Raghavarao [11]. We use Theorems A and 1.1 to obtain a representation of parameters of an affine μ-resolvable BIB design. The result is given in Theorem 1.2.

THEOREM 1.2. *The parameters v, b, r, k and λ of an affine μ-resolvable BIB design can be essentially expressed in terms of only three integral variables μ ($\geqq 1$), m ($\geqq 2$) and j ($\geqq (1 - \mu)/m_1$) as follows:*

$$\begin{aligned} &v = \frac{m}{\mu}\{m_1(m-1)j + m\mu\}, \qquad b = \frac{m}{\mu}\{mm_1 j + (m+1)\mu\}, \\ &r = mm_1 j + (m+1)\mu, \qquad k = m_1(m-1)j + m\mu, \\ &\lambda = m_1\mu j + \mu^2 + \mu(\mu-1)/(m-1), \end{aligned} \tag{1.3}$$

where $j \geqq 0$ when $(\mu, m) = 1$, and m_1 is an integer satisfying $(\mu, m) = g$; $\mu = \mu_1 g$, $m = m_1 g$ and $(\mu_1, m_1) = 1$.

PROOF. Let v, b, r, k and λ be the parameters of an affine μ-resolvable BIB design. Then substituting (1.1) into (1.2), we have

$$r = k + (k - \mu)/(m - 1) . \tag{1.4}$$

Since r and k are integers, $k - \mu$ must be divisible by $m - 1$. Hence

$$k = (m - 1)p + \mu , \tag{1.5}$$

where p is a positive integer. Substituting (1.5) into (1.4), we have

$$r = mp + \mu . \tag{1.6}$$

Moreover from (0.1), (1.1) and (1.5) we have

$$\lambda = \mu p + \mu(\mu - 1)/(m - 1) . \tag{1.7}$$

The $(\mu, m) = g$ leads to an expression $\mu = \mu_1 g$, $m = m_1 g$, $(\mu_1, m_1) = 1$. Further, from Theorem 1.1 $\mu_1\{(m - 1)p + \mu\}/m_1$ must be an integer. Hence from $(\mu_1, m_1) = 1$, $(m - 1)p + \mu$ must be a multiple of m_1, i.e., there exists a positive integer α such that $(m-1)p + \mu = \alpha m_1$ or $p = \mu + (gp - \alpha)m_1$. Let $j = gp - \alpha$, then from $p \geqq 1$ we have $j \geqq 0$ when $(\mu, m) = 1$. Thus using (1.1), (1.5), (1.6), (1.7) and $p = \mu + m_1 j$ we obtain (1.3).

Bose [2] has obtained the particular case (i.e., $\mu = 1$) of Theorem 1.2. From Theorem 1.2 we have the following theorems for the nonexistence of an affine μ-resolvable BIB design:

THEOREM 1.3. *When r divides b, an affine μ-resolvable* BIB *design with parameters v, $b = mt$, $r = \mu t$, k and λ does not exist for an integer μ satisfying $\mu \geqq 2$.*

PROOF. From the assumption and (1.1) m is a multiple of μ. Hence $(\mu, m-1) = 1$. Since in (1.3) $\lambda - m_1\mu j - \mu^2 = \mu(\mu-1)/(m-1)$ is an integer, $(\mu, m-1) = 1$ implies that $(\mu-1)/(m-1)$ is an integer. This contradicts $\mu < m$ derived from $k < v$ and (1.1). Note that when $\mu = 1$ this approach is essentially meaningless.

THEOREM 1.4. *When r does not divide b, a necessary condition for the existence of an affine μ-resolvable* BIB *design with parameters v, $b = mt$, $r = \mu t$, k and λ for an integer μ satisfying $\mu \geqq 2$ is that there exists a positive integer p_2 satisfying the following conditions*:

(i) $p_2 \leqq \mu - 1$, (ii) $\mu(\mu-1)$ *is divisible by* p_2.

In this case

(1.8) $$m = \mu(\mu-1)/p_2 + 1\,,$$

(1.9) $$j = \mu_1 p_1\,,$$

(1.10) $$\lambda = m_1\mu_1\mu p_1 + \mu^2 + p_2\,,$$

where p_1 is an integer, but in particular p_1 is a nonnegative integer when $(\mu, m) = 1$, and μ_1 is an integer satisfying $(\mu, m) = g$; $\mu = \mu_1 g$, $m = m_1 g$ and $(\mu_1, m_1) = 1$.

PROOF. From the assumption and (1.1) m is not a multiple of μ. The $(\mu, m) = g$ leads to an expression $\mu = \mu_1 g$, $m = m_1 g$, $(\mu_1, m_1) = 1$. Since in (1.3) $v = m_1 k/\mu_1$ and $b = m_1 r/\mu_1$ are integers, from $(\mu_1, m_1) = 1$ both r and k must be the multiples of μ_1. From $r = mm_1 j + (m+1)\mu$, $k = m_1(m-1)j + m\mu$ in (1.3) and $(\mu_1, m_1) = 1$, there exist both integers p_1^* and p_2^* satisfying $mj = p_1^*\mu_1$ and $(m-1)j = p_2^*\mu_1$. Setting $p_1 = p_1^* - p_2^*$ leads to (1.9). When $(\mu, m) = 1$, from $j \geqq 0$ and $m > \mu \geqq 2$ we have $p_1^* \geqq p_2^*$, i.e., $p_1 \geqq 0$. Since in (1.3) λ is an integer, $m-1$ must divide $\mu(\mu-1)$, i.e., there exists a positive integer p_2 satisfying $\mu(\mu-1) = p_2(m-1)$. This implies (ii), (1.8) and moreover (i) by $m > \mu$. Substituting (1.9) and $\mu(\mu-1) = p_2(m-1)$ into λ of (1.3), we obtain (1.10). Thus the proof is completed. Note that when $\mu = 1$, this approach is essentially meaningless and further, in this case an affine resolvable BIB design does not exist from the definition of resolvability.

Note from (1.1) that when r is a prime, a μ-resolvable BIB design with parameters v, b, r, k and λ does not exist for an integer μ satisfying $\mu \geqq 2$. It follows from Theorems 1.3 and 1.4 that an affine μ-resolvable BIB design with parameters v, b, r, k and $\lambda \leqq 4$ does not exist for an integer μ satisfying $\mu \geqq 2$. Further note that $\mu_1 = \mu$ in Theorem 1.4 provided that μ is a prime or a prime power, because we can show $(\mu, m) = 1$ from (1.8). For example when $\mu = 2$, from Theorem 1.4 we have $\lambda = 5 + 12p_1$ ($p_1 \geqq 0$). Hence when $\lambda \not\equiv 5 \pmod{12}$ affine 2-resolvable BIB designs do not exist. Thus only one affine 2-resolvable BIB design with parameters $v = 9$, $b = 12$, $r = 8$, $k = 6$ and $\lambda = 5$, which is constructed by a method described in the next section, exists for $r \leqq 15$.

2. Method of construction. From a BIB design with parameters v, b, r, k and λ we can construct its complementary BIB design with parameters $v^* = v$, $b^* = b$, $r^* = b - r$, $k^* = v - k$ and $\lambda^* = b - 2r + \lambda$, and vice versa. Then we have

THEOREM 2.1. *The existence of an (affine) μ-resolvable* BIB *design with parameters v, $b = mt$, $r = \mu t$, k and λ implies the existence of an (affine) $(m - \mu)$-resolvable* BIB *design with parameters $v^* = v$, $b^* = b$, $r^* = (m - \mu)t$, $k^* = v - k$ and $\lambda^* = (m - 2\mu)t + \lambda$ by the complementary method, and vice versa.*

PROOF. It is sufficient to show (affine) $(m - \mu)$-resolvability of a BIB design constructed by the complementary method. Now since each treatment occurs μ times among the m blocks in each of t sets of a μ-resolvable BIB design with parameters v, b, r, k and λ, each treatment obviously occurs exactly $m - \mu$ times in each of t sets of its complementary BIB design with parameters $v^* = v$, $b^* = b$, $r^* = b - r = (m - \mu)t$, $k^* = v - k$ and $\lambda^* = (m - 2\mu)t + \lambda$. This implies $(m - \mu)$-resolvability. Moreover, when the original design is affine μ-resolvable, we have $v^* + t - 1 = v + t - 1 = b = b^*$, $q_1^* = (m - \mu - 1)(v - k)/(m - 1) = v - 2k + (\mu - 1)k/(m - 1)$, and $q_2^* = (m - \mu)(v - k)/m = v - 2k + \mu k/m$. Hence affine $(m - \mu)$-resolvability is shown from Theorems A and 1.1.

Thus we can construct many μ-resolvable or affine μ-resolvable BIB designs by using known solutions of resolvable or affine resolvable BIB designs from Theorem 2.1. For example an affine resolvable BIB design with parameters $v = 9$, $b = 12$, $r = 4$, $k = 3$ and $\lambda = 1$ having a solution [6], i.e., PC(4)[(1, 6, 7), (2, 3, 5), (0, 4, ∞)] mod 8, gives an affine 2-resolvable BIB design with parameters $v = 9$, $b = 12$, $r = 8$, $k = 6$ and $\lambda = 5$ having a solution, i.e., PC(4)[(0, 2, 3, 4, 5, ∞), (0, 1, 4, 6, 7, ∞), (1, 2, 3, 5, 6, 7)] mod 8, where PC(4) means a partial cycle of order 4, i.e., only 0, 1, 2, 3 are to be added to the initial blocks when developed mod 8. It follows from the preceding section that this design is an affine μ-resolvable BIB design with the least set of parameters for $\mu \geqq 2$. Similarly, from an affine resolvable BIB design with parameters $v = 16$, $b = 20$, $r = 5$, $k = 4$ and $\lambda = 1$ ([6], [7]), we can construct an affine 3-resolvable BIB design with parameters $v = 16$, $b = 20$, $r = 15$, $k = 12$ and $\lambda = 11$. Noting that an affine μ-resolvable BIB design with $r \leqq 15$ does not exist provided $\mu \geqq 4$, it is clear that existent affine μ-resolvable BIB designs with $\mu \geqq 2$ and $r \leqq 15$ are only two designs described above.

In the rest of this paper unless otherwise specified, both a design and its incidence matrix may be denoted by the same symbol. By using the idea of Rao [8], we clearly obtain

THEOREM 2.2. *Let N_1 be a* BIB *design with parameters v_1, b_1, r_1, k_1, λ_1, and $N_1' = (\mathbf{n}_1', \mathbf{n}_2', \cdots, \mathbf{n}_{v_1}')$, where $\mathbf{n}_i\mathbf{n}_j' = r_1$ $(i = j)$ or λ_1 $(i \neq j)$, N_1' is the transpose of a matrix N_1. Let N_2 be a μ_2-resolvable* BIB *design with parameters v_2, $b_2 = m_2t_2$, $r_2 = \mu_2t_2$, $k_2 = v_1$, λ_2. Substitute v_1 distinct row vectors $\mathbf{n}_i$ $(1 \times b_1)$ in place of v_1 distinct units and* $\mathbf{0}$ $(1 \times b_1)$ *in place of $v_2 - v_1$ distinct* 0 *(zero) in every block of N_2. Then the resulting matrix is an α-resolvable* BIB *design with parameters $v = v_2$, $b = b_1b_2$, $r = r_1r_2$, $k = k_1$, $\lambda = \lambda_1\lambda_2$, $m = m_2b_1$, $t = t_2$ and $\alpha = r_1\mu_2$.*

For example a BIB design with parameters $v_1 = b_1 = 4$, $r_1 = k_1 = 3$, $\lambda_1 = 2$ and a resolvable BIB design with parameters $v_2 = 8$, $b_2 = 14$, $r_2 = 7$, $k_2 = 4$, $\lambda_2 = 3$ lead to a 3-resolvable BIB design with parameters $v = 8$, $b = 56$, $r = 21$, $k = 3$, $\lambda = 6$ by Theorem 2.2.

THEOREM 2.3. *If N_1 is a μ-resolvable* BIB *design with parameters v_1, $b_1 = mt$, $r_1 = \mu t$, k_1, λ_1 satisfying $b_1 = 4(r_1 - \lambda_1)$, and N_2 is a* BIB *design with parameters v_2, b_2, r_2, k_2, λ_2 satisfying $b_2 = 4(r_2 - \lambda_2)$, then $N = N_1 \otimes N_2 + N_1^* \otimes N_2^*$ is an α-resolvable* BIB *design with parameters $v = v_1v_2$, $b = b_1b_2$, $r = r_1r_2 + (b_1 - r_1)(b_2 - r_2)$, $k = k_1k_2 + (v_1 - k_1)(v_2 - k_2)$, $\lambda = r - b/4$, $\alpha = \mu r_2 + (m - \mu)(b_2 - r_2)$, where N_i^* is the complement of a* BIB *design N_i $(i = 1, 2)$ and $A \otimes B = ||a_{ij}B||$ denotes the Kronecker product of matrices $A = ||a_{ij}||$ and B.*

Since it is proved by Shrikhande [9] and Sillitto [12] that $N = N_1 \otimes N_2 + N_1^* \otimes N_2^*$ is a BIB design with the above parameters, α-resolvability is easily shown.

From the definition of an affine μ-resolvable BIB design, it follows that the existence of an affine μ-resolvable BIB design with parameters v, $b = mt$, $r = \mu t$, k and λ implies the existence of a BIB design with parameters $v' = m$, $b' = v$, $r' = k$, $k' = \mu$ and $\lambda' = k + \lambda - r$.

It is interesting to note that if r is a multiple of an integer α, then grouping of α complete sets each of blocks in a resolvable BIB design leads to an α-resolvable BIB design with the same set of parameters. Moreover, from two μ_i-resolvable BIB designs $(i = 1, 2)$ with common parameters v, k and $t_1 = t_2$, a $(\mu_1 + \mu_2)$-resolvable BIB design can be constructed. Finally, by using Bose's first Module Theorem [1] it follows that the BIB designs with parameters v, b, r, k and λ in some series of Bose ([1], [3]) and of Sprott ([13], [14]) are k-resolvable.

3. *d*-flats in PG(t, q) and EG(t, q). A finite projective t-dimensional geometry over a Galois field GF(q), where q is a prime or a prime power, is denoted by PG(t, q) and the corresponding Euclidean geometry by EG(t, q). It is known that the BIB designs with parameters $v = \phi(t, 0, q)$, $b = \phi(t, d, q)$, $r = \phi(t - 1, d - 1, q)$, $k = \phi(d, 0, q)$, $\lambda = \phi(t - 2, d - 2, q)$ and $v = q^t$, $b = q^{t-d}\phi(t - 1, d - 1, q)$, $r = \phi(t - 1, d - 1, q)$, $k = q^d$, $\lambda = \phi(t - 2, d - 2, q)$ are obtained by choosing the points as treatments and all d-dimensional linear subspaces (d-flats) as blocks from PG(t, q) and EG(t, q), respectively, where $\phi(t, d, q) = (q^{t+1} - 1)(q^t - 1) \cdots (q^{t-d+1} - 1)/(q^{d+1} - 1)(q^d - 1) \cdots (q - 1)$ is the number of d-flats in PG(t, q) [1]. Designs so obtained are denoted by PG$(t, q): d$ and EG$(t, q): d$, respectively.

It is not difficult to verify that when $(t+1, d+1) = 1$, a BIB design PG(t, q): d is k-resolvable, where $k = \phi(d, 0, q)$, and that there does not exist an affine resolvable BIB design PG$(t, q): d$.

Since Rao [6] consequentially showed that a BIB design EG$(t, q): d$ is resolvable, it follows from Theorem A and its direct calculation that a necessary and sufficient condition for a resolvable BIB design EG$(t, q): d$ to be affine resolvable is $d = t - 1$. Further, since the parameters of a BIB design EG$(t, q): d$ satisfy the condition of Theorem 1.3, it follows that an affine μ-resolvable BIB design EG$(t, q): d$ does not exist for $\mu \geqq 2$. The construction of an affine resolvable BIB design EG$(t, q): t-1$ is given by Rao [6], [7].

It should be noted that a BIB design, which is constructed by Theorem 2.1 from a resolvable BIB design EG$(t, q): d$, is $(q^{t-d} - 1)$-resolvable, and that in particular a BIB design, which is constructed from an affine resolvable BIB design EG$(t, q): t-1$, is affine $(q-1)$-resolvable. Finally, as a complement we point out an apparent error of Rao with respect to EG design. As stated in this section, an EG$(t, q): d$ is a resolvable BIB design. Nevertheless, Rao [7] carelessly listed an EG(3.3): 1 as a non-resolvable BIB design. A non-cyclical resolvable geometrical solution of this BIB design EG(3, 3): 1 with parameters $v = 27$, $b = 117$, $r = 13$, $k = 3$ and $\lambda = 1$, however, is easily given by a method of constructing parallel pencils in EG(3, 3) and omitted here.

4. Some inequalities among the parameters. We consider inequalities for BIB designs with parameters $b = mt$ and $r = \mu t$.

THEOREM 4.1. *For a μ-resolvable* BIB *design with parameters v, $b = mt$, $r = \mu t$, k and λ, then*

$$b \geqq v + t - 1 . \tag{4.1}$$

PROOF. Let N be the incidence matrix of a μ-resolvable BIB design. In each of t sets of m blocks (or columns) each in N, where a set of the m columns is such that each treatment occurs exactly μ times, adding the 1st, 2nd, $\cdots$, $(m-1)$th columns to the mth column of a set, we obtain a column consisting of μ only. As there are such t sets evidently $v = \text{Rank}\, N \leqq b - (t-1)$. Therefore we have $b \geqq v + t - 1$.

Theorem 4.1 shows that the concept of μ-resolvability cannot be introduced in a symmetrical BIB design. The particular case $b \geqq v + r - 1$ of the above theorem when $\mu = 1$ was derived by Bose [2]. Since $b \geqq v + r - 1$ holds for any BIB design with the assumption that v is a multiple of k [4], if m is a multiple of μ, then (4.1) can be improved to $b \geqq v + r - 1$. As another improvement of (4.1), we have

THEOREM 4.2. *For a* BIB *design with parameters v, $b = mt$, $r = \mu t$, k and λ, then*

$$\text{(i)} \qquad b \geqq \frac{v-1}{\mu} + r . \tag{4.2}$$

If, in addition, $v \leqq r$, *then*

$$(4.3) \qquad \text{(ii)} \qquad b \geqq \frac{2(v-1)}{\mu} + r\,.$$

PROOF. (i) Multiplying (0.1) by μ, from (1.1) we have $\mu(r-\lambda) = (\mu r - m\lambda)k$. Since $r - \lambda > 0$ and hence $\mu(r-\lambda)$ is a positive integer, we have $\mu r - m\lambda \geqq 1$. Now from (0.1) and $\mu(r-\lambda) = (\mu r - m\lambda)k$, we get $b = v\{(\mu r - m\lambda)k + \mu\lambda\}/\mu k$ or

$$(4.4) \qquad b = \frac{1}{\mu}(v-1)(\mu r - m\lambda) + r\,.$$

Hence from $\mu r - m\lambda \geqq 1$ and (4.4) we obtain (4.2). (ii) Since $\mu r - m\lambda \geqq 1$, assume on the contrary that $\mu r - m\lambda = 1$. Then from (4.4) $v - 1 + \mu r = b\mu = mr$, i.e., $m - \mu = (v-1)/r$. Since $m - \mu$ is an integer, $(v-1)/r$ is an integer, which is a contradiction since $v \leqq r$. Hence we have $\mu r - m\lambda \geqq 2$. Thus from $\mu r - m\lambda \geqq 2$ and (4.4) we obtain (4.3).

Since $2(v-1)/\mu + r > (v-1)/\mu + r \geqq v + t - 1$ for $v \leqq r$, if (4.1) is compared with (4.2) and (4.3), then (4.3) is more stringent than (4.1) provided $v \leqq r$. As an example which attains the bound of (4.3), we have a 2-resolvable BIB design with parameters $v = 6$, $b = 15$, $r = 10$, $k = 4$ and $\lambda = 6$ which is not affine 2-resolvable. The particular case $b \geqq 2v + r - 2$ of (4.3) when $\mu = 1$ was derived without a condition $v \leqq r$ by Kageyama [4]. Finally, one should be referred to Kageyama [5] as a further improvement of inequalities in this section.

5. Block structure of a certain type. We give a certain aspect of the block structure of a special class of μ-resolvable BIB designs (derived in Theorem 2.3).

THEOREM 5.1. *If an α-resolvable* BIB *design N is the Kronecker product $N = N_1 \otimes N_2 + N_1^* \otimes N_2^*$ of an affine μ-resolvable* BIB *design N_1 with parameters v_1, $b_1 = mt$, $r_1 = \mu t$, k_1, λ_1, $q_1 = k_1 + \lambda_1 - r_1$, $q_2 = k_1^2/v_1$ and a symmetrical* BIB *design N_2 with parameters $v_2 = b_2$, $r_2 = k_2$, λ_2, and $b_i = 4(r_i - \lambda_i)$, $i = 1, 2$, then with respect to any block B in N, the other blocks fall into five groups such that the group* (1) *contains $b_2 - 1$ blocks each having $\lambda_2 k_1 + (b_2 - 2r_2 + \lambda_2)(v_1 - k_1)$ treatments in common with B, the group* (2) *contains $m - 1$ blocks each having $k_2 q_1 + (v_2 - k_2)(v_1 + q_1 - 2k_1)$ treatments in common with B, the group* (3) *contains $(m-1)(b_2-1)$ blocks each having $\lambda_2 q_1 + 2(k_2 - \lambda_2)(k_1 - q_1) + (b_2 - 2r_2 + \lambda_2)(v_1 + q_1 - 2k_1)$ treatments in common with B, the group* (4) *contains $m(t-1)$ blocks each having $k_2 q_2 + (v_2 - k_2)(v_1 + q_2 - 2k_1)$ treatments in common with B, and the group* (5) *contains $m(t-1)(b_2-1)$ blocks each having $\lambda_2 q_2 + 2(k_2 - \lambda_2)(k_1 - q_2) + (b_2 - 2r_2 + \lambda_2)(v_1 + q_2 - 2k_1)$ treatments in common with B.*

PROOF. Under the assumption, the blocks of a BIB design N are separated into t sets of mb_2 blocks each such that each set contains every treatment exactly $\mu r_2 + (m-\mu)(b_2 - r_2)$ times. Since in each set of an affine μ-resolvable BIB design N_1, any two blocks contain (1.1), (1.0), (0.1) and (0.0) exactly q_1, $k_1 - q_1$,

$k_1 - q_1$ and $v_1 + q_1 - 2k_1$ times respectively, (N_2, N_2), (N_2, N_2^*), (N_2^*, N_2) and (N_2^*, N_2^*) in each set of N occur exactly q_1, $k_1 - q_1$, $k_1 - q_1$ and $v_1 + q_1 - 2k_1$ times, respectively. Since any two blocks belonging to different sets of N_1 contain (1.1), (1.0), (0.1) and (0.0) exactly q_2, $k_1 - q_2$, $k_1 - q_2$ and $v_1 + q_2 - 2k_1$ times respectively, (N_2, N_2), (N_2, N_2^*), (N_2^*, N_2) and (N_2^*, N_2^*) in a different set of N occur exactly q_2, $k_1 - q_2$, $k_1 - q_2$ and $v_1 + q_2 - 2k_1$ times, respectively. On the other hand, from symmetry of N_2, the scalar product of any two columns of (N_2, N_2) (or (N_2^*, N_2^*)) is k_2 and λ_2 (or $v_2 - k_2$ and $b_2 - 2r_2 + \lambda_2$). The scalar product of any two columns of (N_2, N_2^*) is 0, λ_2 and $k_2 - \lambda_2$. Hence fix a block B arbitrarily in such a set and consider the block structure between B and the remaining blocks, i.e., consider $N'N$. With respect to any block B in N, the required results are obtained.

For example an affine resolvable BIB design with parameters $v_1 = 9$, $b_1 = 12$, $r_1 = 4$, $k_1 = 3$, $\lambda_1 = 1$, $q_2 = 1$ and a symmetrical BIB design with parameters $v_2 = b_2 = 4$, $r_2 = k_2 = 3$, $\lambda_2 = 2$ lead to a 5-resolvable BIB design with parameters $v = 36$, $b = 48$, $r = 20$, $k = 15$, $\lambda = 8$ by Theorem 2.3. This example would also illustrate Theorem 5.1. Similarly, we have the following

THEOREM 5.2. *If an α-resolvable* BIB *design N is the Kronecker product $N = N_1 \otimes N_2 + N_1^* \otimes N_2^*$ of two affine μ_i-resolvable* BIB *designs N_i with parameters v_i, $b_i = m_i t_i$, $r_i = \mu_i t_i$, k_i, λ_i, $q_{i1} = k_i + \lambda_i - r_i$, $q_{i2} = k_i^2/v_i$ and $b_i = 4(r_i - \lambda_i)$, $i = 1, 2$, then with respect to any block B in N, the other blocks fall into eight groups such that the group* (1) *contains $m_2 - 1$ blocks each having $q_{21}k_1 + q_{21}^*(v_1 - k_1)$ treatments, the group* (2) *contains $m_2(t_2 - 1)$ blocks each having $q_{22}k_1 + q_{22}^*(v_1 - k_1)$ treatments, the group* (3) *contains $m_1 - 1$ blocks each having $k_2 q_{11} + (v_2 - k_2)(v_1 + q_{11} - 2k_1)$ treatments, the group* (4) *contains $(m_1 - 1)(m_2 - 1)$ blocks each having $q_{21}q_{11} + 2(k_2 - q_{21})(k_1 - q_{11}) + q_{21}^*(v_1 + q_{11} - 2k_1)$ treatments, the group* (5) *contains $m_2(m_1 - 1)(t_2 - 1)$ blocks each having $q_{22}q_{11} + 2(k_2 - q_{22})(k_1 - q_{11}) + q_{22}^*(v_1 + q_{11} - 2k_1)$ treatments, the group* (6) *contains $m_1(t_1 - 1)$ blocks each having $k_2 q_{12} + (v_2 - k_2)(v_1 + q_{12} - 2k_1)$ treatments, the group* (7) *contains $m_1(m_2 - 1)(t_1 - 1)$ blocks each having $q_{21}q_{12} + 2(k_2 - q_{21})(k_1 - q_{12}) + q_{21}^*(v_1 + q_{12} - 2k_1)$ treatments, and the group* (8) *contains $m_1 m_2(t_1 - 1)(t_2 - 1)$ blocks each having $q_{22}q_{12} + 2(k_2 - q_{22})(k_1 - q_{12}) + q_{22}^*(v_1 + q_{12} - 2k_1)$ treatments, in common with B, where $q_{21}^* = v_2 + \lambda_2 - k_2 - r_2$ and $q_{22}^* = (v_2 - k_2)^2/v_2$.*

Note that if the complementary designs are not considered, i.e., we consider $N = N_1 \otimes N_2$ only, then Theorems 5.1 and 5.2 contain Corollaries 3.2.1 and 3.2.2 of Vartak [15] for $\mu = \mu_1 = \mu_2 = 1$, respectively as a special case.

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