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On multi-field flows in gravity and holography

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Abstract: We perform a systematic analysis of flow-like solutions in theories of Einstein gravity coupled to multiple scalar fields, which arise as holographic RG flows as well as in the context of cosmological solutions driven by scalars. We use the first order formalism and the superpotential formulation to classify solutions close to generic extrema of the scalar potential, and close to "bounces," where the flow is inverted in some or all directions and the superpotential becomes multi-valued. Although the superpotential formulation contains a large redundancy, we show how this can be completely lift by suitable regularity conditions. We place the first order formalism in the context of Hamilton-Jacobi theory, where we discuss the possibility of non-gradient flows and their connection to non-separable solutions of the Hamilton-Jacobi equation. We argue that non-gradient flows may be useful in the presence of global symmetries in the scalar sector.

Keywords: AdS-CFT Correspondence, Gauge-gravity correspondence

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1 Introduction and summary

Theories of gravity coupled to scalar fields are an important ingredient in many subjects including cosmology (inflation), large distance gravitational physics (models of dark energy, modified gravity theories), high energy theory and phenomenology (string theory, supergravity, brane world models), as well as in the context of the gauge/gravity duality. The presence of multiple scalars often makes the physics qualitatively different from the single scalar case. The latter is relatively manageable when looking for simple solutions in which the fields depend on only one coordinate (as in the case of cosmology or simple holographic renormalisation group flows), since one can use the scalar field as the coordinate which describes the evolution. That said, in many cases, having multiple scalars with non-trivial evolution is inevitable, and the reduction to a single-field may be too simplistic. This is the case for example in cosmology and in gauge/gravity duality, the latter of which will be the main focus of this work.

The gauge/gravity duality is the conjectured equivalence between a large N gauge theory in d-dimensional flat space-time (*boundary theory*), and a higher dimensional theory with dynamical gravity in higher dimensional curved space-time (*bulk theory*), [1–3]. In this context bulk scalar fields correspond to couplings of single-trace scalar operators in the dual, boundary field theory. Evolution in the bulk geometry corresponds to evolution under the renormalisation group (RG) in the boundary theory. Solutions of the bulk equations in which the scalars run along the holographic coordinate (which parametrise a non-compact direction among those which are *extra* with respect to the boundary coordinates) are called *holographic RG flow* solutions [4–11].

Holographic RG flows have been widely studied, and often (especially in phenomenological models) it is assumed for simplicity that only one of the scalars runs. However, from the field theory point of view, it is clear that this is an oversimplification: any QFT has an infinite number of operators, which will generically mix under the RG flow. Many operators will start running even when the corresponding couplings are not turned on in the far UV. Therefore, even though in some cases one can, to a first approximation, hope to neglect this mixing, in general this will not be possible and one should consider solutions with multiple scalars evolving.

The need to consider multifield scenarios also arises in a cosmological setting, for instance inflation, see e.g. [12–14]. This is because in many cases, the truncation to a single field fails to capture some important aspects of the full dynamics. Similar issues arise in the context of supergravity truncations, see e.g. [15].

In this work, we consider $d+1$-dimensional Einstein gravity coupled to N scalar fields ϕ^r with a generic scalar potential, and we focus on *flow* solutions, which depend on a single coordinate u (space-like in holography and time-like in cosmology) and which can be brought to the general form

$$ds^2 = du^2 + e^{A(u)}\eta_{\mu\nu}dx^\mu dx^\nu, \qquad \phi^r = \phi^r(u), \qquad r = 1, \ldots, N. \tag{1.1}$$

In holography these solutions describe Poincaré-invariant vacuum (or false vacuum) states of the dual d-dimensional field theory; in cosmology they represent flat FRW space-times. Our analysis will be mostly framed in the language of holographic RG flows, but many of our results carry over unchanged to more general contexts.

When studying solutions of the form (1.1) it has often been very useful to rewrite Einstein's equations as first order flow equations, governed by a *superpotential* $W(\{\phi^r\})$ [8, 9, 16], i.e. a function $W(\phi)$ on the scalar field manifold such that the ansatz (1.1) solves the equations

$$\frac{d\phi^r}{du} = \mathcal{G}^{rs}\frac{\partial W}{\partial \phi^s}, \qquad \frac{dA}{du} = -2(d-1)W, \tag{1.2}$$

where $\mathcal{G}^{rs}$ is the (inverse) metric on field space. The superpotential is determined by a coordinate-invariant differential equation given schematically by

$$\mathcal{G}^{rs}\partial_r W \partial_s W - W^2 = V, \tag{1.3}$$

where V is the scalar field potential. The superpotential formulation is a way of grouping together solutions into classes which share the same geometric features. Solutions in the same class differ by the initial condition of the flow equations (1.1). This way of organising the space of solutions also has applications in cosmology, as argued in [17].

In holography, the superpotential equation in the *single field* case has been widely studied and the qualitative features of the solutions are known for general potentials [18–22]. In particular, the first order formalism is a very convenient way of classifying solutions close to an extremum of the potential (IR or UV fixed point) and when the scalar runs to infinity [23]; it determines the holographic β-function for the coupling dual to the bulk scalar, by $\beta(\phi) = -\partial_\phi \log W$ [8, 20]; it provides a c-function which interpolates monotonically between the UV and IR central charges [5, 7, 16]; it gives a simple way to write counterterms for holographic renormalisation [9, 24, 25] and the gravitational on-shell action [20]. The first order formalism also allows one to uncover and classify "exotic" features which cannot occur in perturbative field theory, such as inversion of the direction of a holographic RG flow occuring at points where the superpotential becomes multi-valued, as was observed early on in [26, 27] and discussed in detail in [22]. A generalization of this formalism was recently used in [28] to describe holographic RG-flows for field theories in curved space-times.

Another important aspect of the superpotential formalism is its connection to the properties of gravitational stability of the system. In [29, 30] it was shown that, if the potential is given by (1.3), then the AdS critical point can be proved to be stable perturbatively and, with some extra assumptions, also non-perturbatively. These results have been generalized to domain walls in [16, 31].

The multi-field case is much more involved, and a systematic analysis has so far been missing. The main reason is that the multi-field superpotential equation is a non-linear partial differential equation (as opposed to an ordinary differential equation in the single-field case). As a consequence, several properties which are automatic in the simple-field case (e.g. the very existence of a superpotential, the gradient property of all flows which we discuss below), are less obvious when many scalars are involved.

Another major difference is that, in the single-field case, the number of integration constants in the first order formulation is *exactly* the same as in the standard second order form of Einstein's equation. This implies that there is a one-to-one correspondence between solutions of Einstein equations and solutions of the flow equations, and given a solution of the form (1.1) there always exists a unique superpotential which can be reconstructed following an algorithmic procedure. Instead, with multiple scalars, the correspondence between first order and second order formulation is many to one: the same solution of the Einstein-scalar system can arise from *many* different superpotentials, as we will see in explicit examples.

This ambiguity goes beyond the usual dependence of the field theory β-functions on field redefinitions. In holography, once the boundary conditions in the UV are fixed, different superpotentials (different β-functions) correspond to different vacua of the same theory. For a single field, there is a one-to-one correspondence between vacua and β-functions, whereas in the multi-field case there seem to be many more β-functions to describe the same set of physical solutions (flows). One of our results will be to show that this degeneracy is lifted by appropriate regularity conditions in the IR, which can eliminate all but one (or a discrete set) of beta-functions.

In this work we pursue two main goals. On the one hand, we perform a general analysis of the space of solutions in terms of first order flows (1.1). On the other hand, we will analyse the system from the point of view of Hamilton-Jacobi theory and investigate if and how the existence of a superpotential is justified, and more generally when one can describe the space of solutions using only gradient flows.

In the first part of the paper we analyse the space of solutions of the superpotential equation and the corresponding flows, extending the systematic analysis that was performed in [22] to the multi-field case. In particular we give a general classification of the behavior at singular points, where some or all partial derivatives of W vanish. These may be of two types:

1. Extrema of the potential: these may be reached in the far UV or the far IR, and they are the endpoints of the flows.

2. Points which are not extrema of the potential: here, one or more directions of the flow are inverted, and W becomes multi-branched. These are not endpoints, as the solution can be continued smoothly past these points. They were referred to as *bounces* in [22]

Unlike in the case of a single field, a generic extremum of the potential can play the role both of an IR and a UV fixed point, depending on which directions in field space are running.[1] Close to the extremum, the behaviour of the superpotential is a generalisation of the single field case: an analytic part which is universal (up to discrete choices), plus a subleading part which contains the integration constants. The new feature in the multi-field case is that we are now in the presence of integration *functions*, therefore the description is highly redundant. We will identify a restricted class of solutions which contain just enough integration constants to provide all possible solutions of Einstein's equations of the form (1.1).

Furthermore, we find that imposing an appropriate regularity condition around *minima* of the scalar potential lifts all continuous deformations, leaves only one physical vacuum (or at most a discrete set), and eliminates the redundancy in the first order description *while still allowing a continuous choice of the UV source parameters.* This last requirement is crucial because we do not want to restrict the values of the UV couplings, which enter as initial conditions of the flow around a maximum of the potential. Indeed, in a QFT it

[1]With the exception of a local maximum, which can only be a UV fixed point.

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should be possible to change the couplings continuously, at least in a certain range, without making the theory inconsistent.[2]

As in the single-field case, away from extrema of the potential a solution to the superpotential equation can *bounce* and become multi-branched at certain special points. When this occurs, the flow of one or more scalars inverts its direction, causing a breakdown of the first-order formalism. In the multi-field case the structure of bounces is much richer than for a single field. First, bounces can now occur on a hyper-surface of any dimension up to $N-1$ of the scalar manifold. Second, there are two qualitatively different kinds of bounces:

- Complete bounces, where *all* the scalars change direction at the same time
- Partial bounces, where only some of the scalars invert their flow.

Complete bounces occur on sub-manifold lying on equipotential hyper-surfaces whereas partial bounces can occur anywhere in field space except at extrema of $V(\phi)$. Close to a bounce the superpotential has several branches, and we show how to glue them together so that the flow is smooth. Interestingly, close to a complete bounce, the superpotential equation takes the same form as the Eikonal equation for geometric optics close to a surface with vanishing index of refraction. Thus, complete bounces are analogous to the phenomenon of total internal refraction which gives rise to mirages.

As we have mentioned, in the multi-field case the existence of a superpotential, for which the flow has the gradient form (1.2) is not obvious. The second part of this work explores the questions 1) whether a superpotential description is always possible 2) whether more general descriptions of the space of solutions in terms of non-gradient flows may sometimes be useful.

The appropriate framework to answer these questions is Hamilton-Jacobi theory, whose connection with the first order form in holography and cosmology is well known, [8, 32–35]. In Hamilton-Jacobi theory, one can always find a first order gradient flow description in the *extended* $N+1$ dimensional field space with coordinates (A, ϕ^r), generated by Hamilton's principal function $\mathcal{S}(A, \phi^r)$. However, as we will see, in order to have an N-dimensional gradient flow on the scalar manifold parametrised by $\{\phi^r\}$, $\mathcal{S}(A, \phi^r)$ must have a specific separable form [15]. Gradient flows in extended field space, with a non-separable principal function, have been shown to arise in connection with black hole solutions [36].

As we will discuss, the answer to both questions raised above is positive. The superpotential description arises from a special class of *separable* Hamilton-Jacobi (HJ) principal functions. Locally, any solution of Einstein's equation can be seen as arising from such a separable HJ function, which implies that a superpotential can always be found locally in field space. Although it may not be globally defined as a smooth function, we can use the results of the first part of this work to glue together consistently solutions in different regions. Furthermore, in the case of holography, separable solutions contain already enough integration constants to describe all possible RG flows starting from a UV maximum.

[2]This does not mean that any value of the coupling is allowed, as there could be regions in the space of couplings which are forbidden. This is true even in perturbative QFT, for example we should require $\lambda > 0$ in $\lambda\varphi^4$ theory, or $g^2 > 0$ in Yang-Mills theory. However in the allowed regions these parameters may be changed continuously.

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A situation when it may be useful *not* to use a separable function is in the presence of global symmetries: in this case, we find that if we want to classify solutions in terms of the value of the corresponding conserved charges, a gradient flow description is impossible, and a superpotential cannot be defined. Classifying solutions according to the value of conserved charges may be a useful option in multi-scalar cosmology as it may simplify the treatment.

If we restrict ourselves to holographic RG flows however one can argue that, generically, all conserved charges must vanish, and non-gradient flows are unnecessary. The reason is that, in holography, bulk symmetries should be gauged, as they also imply global symmetries of the boundary theory at the fixed point: there should be bulk gauge fields corresponding to the boundary conserved currents. As a consequence, in the absence of non-trivial gauge fields in the solution, the value of all conserved quantities must be zero by gauge invariance (or, equivalently, by Gauss's law). An important exception is the one where a continous symmetry does not leave the UV fixed point invariant, but rather it connects a family of fixed point along an exactly marginal direction (e.g. the shift symmetry of the coupling in $N = 4$ SYM). In this case the symmetry can remain ungauged in the gravity description (alhtough in the full string theory one expects it to be broken to a discrete subgroup).

This paper is organised as follows.

In section 2 we lay out our setup and introduce the first order formalism.

In section 3 we discuss holographic RG flows in terms of the superpotential. We first classify all possible solutions around generic extrema of the scalar potential, their continuous parameters, and we differentiate between UV and IR solution, in subsection 3.1. Then, in subsection 3.2, we turn to the analysis of multi-branched solutions.

In section 4 we make the connection with Hamilton-Jacobi theory. In subsection 4.1 we relate non-gradient flows to non-separable solutions of the Hamilton-Jacobi equation. We discuss global symmetries in subsection 4.2, and in subsection 4.3 we discuss the effect of the gauging of such symmetries.

Finally in section 5 we summarise our conclusion and propose further directions.

Some of the more technical details of our calculations, as well as a review of Hamilton-Jacobi theory, are left to the appendix.

2 Setup

2.1 Action, field equations and vacuum ansatz

Our starting point is the Klein-Gordon action for N self-interacting scalar fields minimally coupled to gravity, in $d+1$ dimensions,

$$S = M^{d-1} \int_{\mathcal{M}} \mathrm{d}^{d+1}x \sqrt{-g} \left[R - \frac{1}{2} \mathcal{G}_{rs} \partial_a \phi^r \partial^a \phi^s - V(\phi^r) \right] + S_{GH} \,, \tag{2.1a}$$

$$S_{GH} = -2M^{d-1} \int_{\partial\mathcal{M}} \mathrm{d}^d x \sqrt{h}\, K \,. \tag{2.1b}$$

Here $a = 0, \dots, d$ and early alphabet letters are space-time indexes, $r = 1, \dots, N$ and middle alphabet letters are field-space indexes. The field space metric $\mathcal{G}_{rs}(\phi^1, \dots, \phi^N)$ is assumed positive-definite and non-degenerate. In the Gibbon-Hawking term, S_{GH}, h_{ab} is, as usual, the induced metric on the space-time boundary $\partial\mathcal{M}$ which has extrinsic curvature K_{ab}. We will consider solutions preserving d-dimensional Poincaré invariance,

$$\mathrm{d}s^2 = \mathrm{d}u^2 + e^{2A(u)}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\,, \quad \phi^s = \phi^s(u)\,. \tag{2.2}$$

In the gauge/gravity duality, these solutions correspond to RG flows in a space of N coupling constants, each corresponding to one scalar operator. The same ansatz (2.2) can be applied to cosmology after two Wick rotations.

The Klein-Gordon and Einstein equations are

$$\ddot{\phi}^r + \widetilde{\Gamma}^r_{pq}\dot{\phi}^p\dot{\phi}^q + d\dot{A}\dot{\phi}^r - \mathcal{G}^{rp}\frac{\partial V}{\partial\phi^p} = 0\,, \tag{2.3a}$$

$$d(d-1)\dot{A}^2 - \frac{1}{2}\mathcal{G}_{rs}\dot{\phi}^r\dot{\phi}^s + V(\phi) = 0\,, \tag{2.3b}$$

$$2(d-1)\ddot{A} + \mathcal{G}_{rs}\dot{\phi}^r\dot{\phi}^s = 0\,, \tag{2.3c}$$

where $\cdot = \mathrm{d}/\mathrm{d}u$, and

$$\widetilde{\Gamma}^r_{pq} = \frac{1}{2}\mathcal{G}^{rs}\left(\partial_p\mathcal{G}_{sq} + \partial_q\mathcal{G}_{sp} - \partial_s\mathcal{G}_{pq}\right)\,,$$

with $\partial_p = \frac{\partial}{\partial\phi^p}$. In the following all field indices (middle of the alphabet) will be raised and lowered with the field metric and its inverse, and the covariant derivative compatible with $\mathcal{G}_{rs}$ will be denoted by $\tilde{\nabla}$ to distinguish it from the space-time covariant derivative.

Equation (2.3c) is redundant as it is a consequence of (2.3a) and the u derivative of eq. (2.3b). Note that $2N+1$ *integration constants* are necessary to specify a solution of (2.3): $2N$ for the Klein-Gordon equations (2.3a), and 1 for the Einstein equation (2.3b). The number of integration constants will be important later when discussing non-gradient flows in the context of Hamilton-Jacobi (HJ) theory.

If the potential has an extremum at $\phi = \phi_*$ where $V(\phi_*) = -d(d-1)/\ell^2$, then equations (2.3) have an AdS_{d+1} solution with constant scalars. This solution corresponds to a CFT. The scale factor is $A(u) = -u/\ell$, where ℓ is the curvature radius of AdS and the boundary is at $u \to -\infty$.

2.2 First order formalism

For the following analysis, it will be useful to rewrite the equations of motion in the first order formalism, first introduced in the cosmological context for multiple scalar fields by Salopek and Bond [37], and discussed in great depth in the holographic literature in e.g. [38]). An underlying assumption is that one of the fields should (at least locally) have a monotonic evolution so that it can traded for the u coordinate, thus leading to u-independent first order equations. The starting point is the 'fake superpotential' W, function of the field values ϕ^s only, and defined by

$$W(\phi^s(u)) := -2(d-1)\dot{A}(u). \tag{2.4}$$

The existence of a superpotential is guaranteed for the single-scalar case, piecewise in the regions where $\phi(u)$ is monotonic. As it has been argued [26, 39], and as we will discuss in more detail in section 3 this is true locally in field space also for the multi-field case. Throughout this section we will simply assume that, for any solution of the form (2.2), a superpotential $W(\phi)$ satisfying equation (2.4) exists. We will return to this point in section 4 where we will critically assess this assumption.

From action (2.1a), we define the field momentum densities by $\pi_r := \mathcal{G}_{rs}\dot{\phi}^s$. They differ from the canonical momenta by a factor of $-\sqrt{-g}$. Since the field-space metric is non-singular

$$\dot{\phi}^s = \mathcal{G}^{rs}(\phi)\pi_r \equiv \pi^s(\phi). \tag{2.5}$$

We can rewrite equation (2.3c) using the definitions (2.4) and (2.5), obtaining

$$\pi^r\left(\pi_r - \partial_r W\right) = 0\,. \tag{2.6}$$

This implies that $\pi_r = \partial_r W + \xi_r$, with $\pi^r\xi_r = 0$. In the special case $\xi_r = 0$, we are in the presence of a *gradient flow*.

To characterise the non-gradient part ξ_r, we express equations (2.3a) and (2.3b) in terms of W and π_r,

$$\pi^q\widetilde{\nabla}_q\pi^p - \frac{d}{2(d-1)}W\pi^p - \mathcal{G}^{ps}\partial_p V = 0, \tag{2.7}$$

$$\frac{1}{2}\pi_r\pi^r - \frac{d}{4(d-1)}W^2 - V = 0. \tag{2.8}$$

Taking a derivative of eq. (2.8) and subtracting (2.7) so as to eliminate the potential V leads to

$$\pi_p = \partial_p W + \frac{2(d-1)}{d\;W}\pi^s\mathcal{F}_{sp}\,, \qquad \mathcal{F}_{sp} \equiv \widetilde{\nabla}_s\pi_p - \widetilde{\nabla}_p\pi_s. \tag{2.9}$$

It is thus clear that

$$\xi_r = \frac{2(d-1)}{d\;W}\pi^s\mathcal{F}_{sr}, \tag{2.10}$$

and the flow is gradient iff $\pi^s\mathcal{F}_{sr}$ vanishes. If this is the case, the gradient property and the definition of $\mathcal{F}_{sp}$ in equation (2.9) also imply $\mathcal{F}_{sp} = 0$.

In this section we only restrict to gradient flows. The possibility of having non-gradient flows will be considered in section 4.2.

For gradient flows the independent equations become

$$\frac{1}{2}\mathcal{G}^{rs}\partial_r W\partial_s W - \frac{d}{4(d-1)}W^2(\phi) = V(\phi)\,, \tag{2.11a}$$

$$\dot{\phi}^r(u) = \mathcal{G}^{rs}(\phi)\partial_s W(\phi)\,, \tag{2.11b}$$

$$\dot{A}(u) = -\frac{1}{2(d-1)}W(\phi)\;. \tag{2.11c}$$

Given a superpotential $W(\phi)$, integration of (2.11b) and (2.11c) introduces $N+1$ integration constants. For a given potential $V(\phi)$, $W(\phi)$ itself is obtained by solving the partial *differential* equation (PDE) (2.11a), referred to as the *superpotential equation*. A solution $W(\phi)$ to (2.11a) is specified by an *integration function* of $N-1$ variables. On the other hand, as we have seen the total number of integration constants should be $2N+1$. Hence this formalism is highly redundant: the same flow is expected to arise from infinitely many superpotentials $W(\phi)$. It follows that it is enough to consider a subclass of solutions to the superpotential equation (2.11a) which contains N independent integration constants[3] [34].

3 Holographic flows

We now focus our attention to holographic RG flows: these are solutions which have an interpretation in gauge/gravity duality as deformations away from conformality of a UV conformal fixed point, which corresponds to an extremum of V. Solutions which are everywhere regular connect the UV extremum to a second extremum of the potential, interpreted as another conformal fixed point in the IR. For simplicity we consider potentials which are strictly negative. This avoids complications resulting from the bulk curvature becoming small along the flow, or from transitions to cosmological solutions.

The holographic β-functions of the scalar couplings are given by [8, 20]

$$\beta^r \equiv \frac{\dot{\phi}^r}{\dot{A}} = -2(d-1)\mathcal{G}^{rs}\frac{\pi_s(\phi)}{W(\phi)} \tag{3.1}$$

which, the case of gradient flows, reduces to

$$\beta^r(\phi) = -2(d-1)\mathcal{G}^{rs}\frac{\partial_s W(\phi)}{W(\phi)}. \tag{3.2}$$

Before proceeding, we list two important properties of the superpotential $W(\phi)$.

1. It follows from (2.11a) that W is bounded from below by a positive function

$$W(\phi) \geqslant B(\phi) > 0, \qquad \text{where} \qquad B(\phi) := \sqrt{-\frac{4(d-1)}{d}V(\phi)}\,. \tag{3.3}$$

   Note that we have chosen $W > 0$ without loss of generality (since the superpotential equation is invariant under $W \to -W$). This implies, through equation (2.11c), that $\dot{A}$ is always negative, so $A(u)$ *is monotonically decreasing with* u.

2. The superpotential is a monotonic function of the holographic coordinate u,

$$\frac{\mathrm{d}W}{\mathrm{d}u} = \dot{\phi}^r\partial_r W = \mathcal{G}^{rs}\partial_r W\partial_s W \geqslant 0\ . \tag{3.4}$$

Property 2 implies that W increases monotonically *along flows* and is stationary only when all the β functions, defined in (3.2), vanish simultaneously, implying $\dot{\phi}^r = 0$. However, this does not necessarily mean that the flow reaches a fixed point: for this one also needs $\ddot{\phi}^r = 0$ at the same point. From equation (2.3a), this can only happen if $\partial^r V = 0$.

[3] In Hamilton-Jacobi theory this is referred to as a *complete integral*. We will make this connection in section 4.

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Therefore, true fixed points occur only when extrema of $W(\phi)$ are also extrema of $V(\phi)$. The case of an extremum of $W(\phi)$ which is not an extremum of $V(\phi)$ corresponds to a *bounce*, i.e. a regular point of the geometry where the flow is inverted in some of the directions.[4] Bounces will be discussed in detail in subsection 3.2, and we now turn to the analysis around extrema of $V(\phi)$.

3.1 Near-extremum analysis

We start with a short review of the holographic dictionary around an extremal point of the potential V (assumed to be at the origin without loss of generality). We assume V has an analytic expansion around the extremum,

$$V(\phi) = -\frac{d(d-1)}{\ell^2} + \sum_{r=1}^{N} \frac{m_r^2}{2}(\phi^r)^2 + \sum_{r,s,p,=1}^{N} \frac{g_{rsp}}{\ell^2}\phi^r\phi^s\phi^p + \mathcal{O}(\phi^4). \tag{3.5}$$

Note that we have chosen coordinates in field-space such that the mass matrix is diagonal at the extremum [21], and we have included terms up to cubic order in the scalar fields, controlled by arbitrary dimensionless coefficients g_{rsp}.

Solutions of equations (2.3) with the potential (3.5) have the asymptotic expansion

$$A(u) = A_0 - \frac{u}{\ell} + \dots \tag{3.6a}$$

$$\phi^r(u) = \phi_-^r e^{\Delta_r^- u/\ell}(1+\dots) + \phi_+^r e^{\Delta_r^+ u/\ell}(1+\dots)\,, \quad r = 1,\dots,N, \tag{3.6b}$$

where

$$\Delta_r^\pm = \frac{d}{2} \pm \frac{1}{2}\sqrt{d^2 + 4m_r^2\ell^2}. \tag{3.7}$$

The parameters A_0, ϕ_-^r and ϕ_+^r are $2N+1$ integration constants, and hence fixing these asymptotics completely determines the solution. One usually sets $A_0 = 0$, corresponding to the boundary theory living on Minkowski space with metric $\eta_{\mu\nu}$. The parameters ϕ_-^r and ϕ_+^r are then related to the source J^r and vacuum expectation values (VEVs) of the corresponding CFT operators by:[5]

$$J^r = \ell^{-\Delta_r^-}\phi_-^r, \qquad \langle \mathcal{O}_r \rangle_J = (2\Delta_r^+ - d)\ell^{-\Delta_r^+}\phi_+^r\,, \tag{3.8}$$

and Δ_r^- and Δ_r^+ are interpreted as the conformal dimension of the source J^r and the operator $\mathcal{O}_r$, respectively.

While $\Delta_r^+ > 0$, Δ_r^- can have either sign depending on whether the operator $\mathcal{O}_r$ is relevant or irrelevant. For an extremum with M negative and $N-M$ positive mass eigenvalues m_r^2, it is convenient to split the N directions in field space into two sets, $\hat{r} = 1\dots M$ and $\check{r} = M+1\dots N$, and to introduce the notation:[6]

$$\begin{cases} \Delta_{\check{r}}^- \leqslant 0\,, & m_{\check{r}}^2 \geqslant 0\,, \quad \check{r} = M+1,\dots,N \quad \text{irrelevant operator.} \\ 0 \leqslant \Delta_{\hat{r}}^- < \frac{d}{2}\,, & -\frac{d^2}{4\ell^2} < m_{\hat{r}}^2 \leqslant 0\,, \quad \hat{r} = 1,\dots,M \quad \text{relevant operator.} \end{cases} \tag{3.9}$$

[4] In the single field case this was discussed in [22]

[5] We are working in the so-called *standard* dictionary. In the mass range $-d^2/4 < m^2 < -d^2/4+1$ there is an *alternative* dictionary, in which the roles of ϕ_+ and ϕ_- are interchanged.

[6] The symbols are chosen so that $\hat{r}$ labels direction along which $\phi = 0$ is a maximum, whereas $\check{r}$ labels those directions along which the extremal point is a minimum.

The lower bound on negative values of m^2 is the Breitenholer-Freedman bound and is required for perturbative stability of the solution.

The expansions (3.6a)–(3.6b) must hold as $\phi \to 0$. Depending on the signs of the Δ_r^-, this corresponds to either $u \to +\infty$ or $u \to -\infty$. The allowed combinations of integration constants are

UV:

$$u \to -\infty,\ \exp A(u) \to +\infty, \qquad \phi_-^{\hat{r}} \neq 0,\ \phi_-^{\check{r}} = 0,\ \phi_+^r \ \text{arbitrary}. \tag{3.10}$$

From the field theory perspective, one can only turn on the sources $\phi_-^{\hat{r}}$ corresponding to *relevant* operators, for which $\Delta_{\hat{r}}^- > 0$, and we have to set to zero those corresponding to *irrelevant* operators ($\Delta_{\check{r}}^- < 0$). Also, we are free to turn on any combination of vevs ϕ_+^r.

IR:

$$u \to +\infty,\ \exp A(u) \to 0, \qquad \phi_-^{\check{r}} \neq 0,\ \phi_-^{\hat{r}} = \phi_+^r = 0\,. \tag{3.11}$$

In this case only the sources $\phi_-^{\check{r}}$ for the *irrelevant* operators can be non-zero, because the IR fixed point is stable under deformations by such operators. However, turning on any other source or vev would make the flow miss the IR fixed point.

In the following we study flows around the extrema from the point of view of the superpotential equation rather than the equations of motion. In the single-field case, close to an extremum of V the superpotential has a universal analytic term (which can be of two different kinds), plus a sub-leading non-analytic piece which contains the single integration constant to the superpotential equation [20, 22]. As we will see in the next two subsection, this structure persists in the multi-field case, with the difference that there are many more branches of analytic solutions, and a larger class of non-analytic deformations parametrised by an integration function.

3.1.1 Analytic part of the superpotential

We start by looking for *analytic* solutions $W_0(\phi)$ for equation (2.11a) around the origin, with the potential $V(\phi)$ in equation (3.5) such that both have extrema at $\phi^r = 0$, following [21]. We use Riemann normal coordinates on the scalar manifold $\mathcal{M}_\phi$

$$\mathcal{G}_{rs}(\phi^s) = \delta_{rs} - \frac{1}{3}\mathcal{R}_{rpsq}\phi^p\phi^q - \frac{1}{6}\left(\widetilde{\nabla}_m \mathcal{R}_{rpsq}\right)\phi^p\phi^q\phi^m + \mathcal{O}(\phi)^4\,, \tag{3.12}$$

where $\mathcal{R}_{rpsq}$ is the Riemann curvature tensor associated with $\mathcal{G}_{rs}$.

Expanding both sides of equation (2.11a) in powers of ϕ^r around $\phi^r = 0$, it is easy to show that there are 2^N analytic solutions of equation (2.11a), parametrised by a string $\sigma = (\sigma_1, \ldots, \sigma_N)$ where $\sigma_i = \pm$,

$$W_0^\sigma(\phi) = \frac{2(d-1)}{\ell} + \frac{1}{2\ell}\sum_{r=1}^{N}\Delta_r^{\sigma_r}(\phi^r)^2 + \sum_{p,q,r=1}^{N}\frac{g_{pqr}\phi^p\phi^q\phi^r}{\ell(\Delta_p^{\sigma_p} + \Delta_q^{\sigma_q} + \Delta_r^{\sigma_r} - d)} + \mathcal{O}\left(\phi^4\right). \tag{3.13}$$

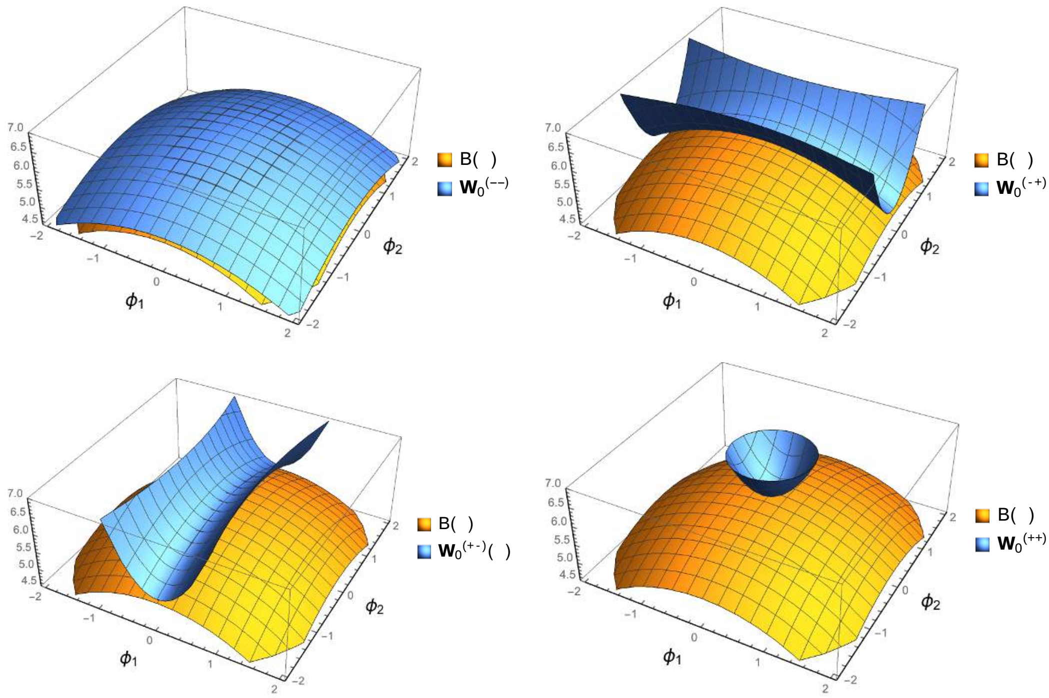

Figure 1. This figure shows the four leading solutions of the superpotential equation close to a minimum of the potential, of the form (3.13), in the two-field case. The yellow surface is the critical curve $B(\phi)$.

These 2^N solutions are the generalisation to the multi-field case of the two branches W^+, W^- for a single field [22]. When the $\Delta' s$ in the denominator of the cubic term sum up to d, logarithms will appear in the solution.

All solutions W_0^σ have an extremum, which may be a maximum, minimum, or a saddle point, at $\phi = 0$. However this extremum of W_0^σ does *not* necessarily have the same signature (of the Hessian matrix) as the extremum of V. The latter is characterised by the signature of the mass matrix, which determines whether the corresponding operators are relevant ($\phi^{\hat{r}}$) or irrelevant ($\phi^{\check{r}}$). The former, W, is determined by the signs of the $\Delta_r^\pm$. These are not necessarily the same as the signs of m_r^2 since $\Delta_r^+ > 0$ for any mass, see equation (3.9). This is illustrated in figure 1: the *same* minimum of V admits four different solutions W_0^σ with different signature of the Hessian matrix, which in our notation are $W_0^{(++)}, W_0^{(+-)}, W_0^{(-+)}, W_0^{(+-)}$.

Close to $\phi = 0$ the flow equations arising from one of the superpotentials W_0^σ become linear and decoupled,

$$\dot{\phi}^r \approx \frac{\Delta_r^{\sigma_r}}{\ell} \phi^r, \qquad \dot{A} \approx -\frac{1}{\ell}. \tag{3.14}$$

To use the language of dynamical system, we may have both *repulsive* directions ($\Delta_r^{\sigma_r} > 0$) and *attractive* directions ($\Delta_r^{\sigma_r} < 0$). A repulsive direction can correspond either to a relevant operator ($r = \hat{r}$, and $\sigma_{\hat{r}} = \pm$) or to an irrelevant operator ($r = \check{r}$ with the choice $\sigma_{\check{r}} = +$). An attractive direction always correspond to an irrelevant operator with the choice $\sigma_{\check{r}} = -$.

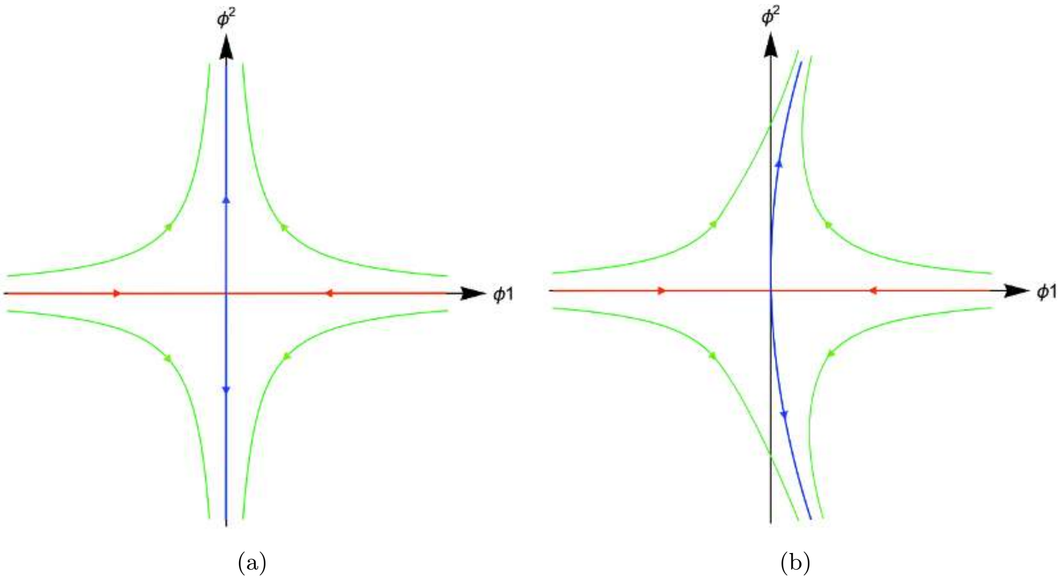

Figure 2. The left figure represents flows associated with a $W_0^{(+-)}$ solution around a local minimum of $V(\phi)$ as depicted in figure 1 *or* the flows from a $W_0^{(--)}$ solution of a saddle of $V(\phi)$ with $m_1^2 > 0$ and $m_2^2 < 0$. The right figure represents a possible effect of the non-linear terms which may or not include non-analytic terms.

When W_0^σ has both repulsive and attractive directions, generically the fixed point will not be reached, since generic initial conditions will violate both conditions (3.10) and (3.11): the flow may approach the fixed point but miss it and leave along another direction. This situation is represented in the two-field case in figure 2 (a), which represents the flow diagram of a solution of the type $W_0^{(-+)}$ with $\Delta_1^- < 0$. Nevertheless, there are special (fine-tuned) initial conditions (namely $\phi_-^1 = 0$ or $\phi_+^2 = 0$) which satisfy either (3.10) or (3.11), and such that the flow will reach $\phi = 0$ in the UV (blue line) or the IR (red line), respectively.

Notice that the same solution (3.13) with a given σ can describe both a UV and an IR fixed point, depending on the choice of initial conditions for the first order flow (3.14). This is very different from the single field case where each branch around an extremum can be unambiguously assigned to the UV or the IR, independently of the initial conditions for the first order flow [22].

3.1.2 Deformations of the analytic solution

As mentioned earlier, a solution $W(\phi)$ of equation (2.11a) is specified by an integration function of $N-1$ variables. The 2^N analytic solutions (3.13) on the contrary, do not depend on any continuous parameter. The reason is that, on any of the 2^N branches, the leading analytic behaviour close to a fixed point is universal, and the difference between solutions arises through sub-leading, non-analytic terms [18, 20]. Therefore we now consider

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subleading deformations[7] of the 2^N solutions W_0^σ of (3.13),

$$W(\phi) = W_0^\sigma(\phi) + \delta W(\phi). \tag{3.15}$$

The linearised equation for δW following from (2.11a) is

$$\mathcal{G}^{rs}\partial_r W_0^\sigma \partial_s \delta W = \frac{d}{2(d-1)} W_0^\sigma \delta W \tag{3.16}$$

which, to leading order in powers of the scalar fields, becomes

$$\sum_{r=1}^{N} \Delta_r^{\sigma_r} \phi^r \partial_r \delta W = d\delta W. \tag{3.17}$$

The general solution is a linear superposition of separable solutions

$$\delta W_S = f_1(\phi^1) f_2(\phi^2) \dots f_N(\phi^N). \tag{3.18}$$

From (3.17), the f_s are given by

$$f_s = |\phi^s|^{\kappa_s/\Delta_s^{\sigma_s}}, \qquad \sum_{s=1}^{N} \kappa_s = d \tag{3.19}$$

where κ_s are N constants. Therefore, to leading order around $\phi^r = 0$, the general solution to (3.16) is a superposition of all the special solutions (3.19), namely

$$\delta W = \frac{1}{\ell} \int d\kappa_1 \cdots \int d\kappa_N \prod_{r=1}^{N} (\phi^r)^{\kappa_r/\Delta_r^{\sigma_r}} \; K(\kappa_1, \dots, \kappa_N)\, \delta\left(\sum_{s=1}^{N} \kappa_s - d\right). \tag{3.20}$$

Although the distribution $K(\kappa_i)$ is largely arbitrary, it is subject to some constraints: for instance it must have support on those κ_r which have the same sign of the corresponding Δ_r, otherwise δW will not vanish close to $\phi^r = 0$. There will also be bounds on K at infinity from the convergence of the integral, whose detailed analysis is beside the point here. What is important is that equation (3.20) gives a huge multiplicity of deformations.

A significant qualitative difference with respect to the single field case is the following. For a single scalar, solutions of the type W^+ and infra-red solutions of the type W^- (reaching a minimum of V) are isolated and do not allow continuous deformations. As a consequence, imposing that $W(\phi)$ reaches a minimum of the potential fixes the superpotential completely, and once this choice is made all flows will automatically reach the IR fixed point. In the multi-field case instead, solutions with any number of Δ_r^+ or negative Δ_r^- still admit deformations.

There is one case in which the deformation is forbidden. When $\phi = 0$ is a *minimun* of V, all $\Delta_r^- < 0$. The solution $W_0^{(-,-,\dots-)}$ in (3.13) corresponding to the choice $\sigma_r = -$ for all components has only attractive directions, with two consequences: 1) a generic flow arising

[7] Throughout this section we have assumed $m_r^2 \neq 0$. The case when one of the masses vanishes, corresponding to $\Delta_- = 0, \Delta_+ = d$, (i.e. the operator is marginal) can be treated along the same lines, as it was done for the single field case in [22].

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from this superpotential reaches the IR fixed point; 2) the deformation is identically zero because the δ-function cannot be saturated while having all $\kappa_r < 0$ at the same time, which is required in order that only positive powers of the fields arise. These two facts have very important consequences, as we will discuss at the end of this section in subsection 3.1.3.

The deformation δW can turn on a flow velocity even in those directions which, to leading order, were attractive. This occurs by giving vevs to the corresponding operators by mixing, as we now illustrate with a simple two-field example.

Two fields example. Consider a saddle point with one irrelevant ($m_1^2 > 0$) and one relevant ($m_2^2 < 0$) direction, and choose the analytic solution with $\sigma = (--)$:

$$W_0^{(--)}(\phi) = \frac{2(d-1)}{\ell} + \frac{\Delta_1^- \left(\phi^1\right)^2}{2\ell} + \frac{\Delta_2^- \left(\phi^2\right)^2}{2\ell} + O(\phi^3), \quad \Delta_1^- < 0\,, \quad 0 < \Delta_2^- < \frac{d}{2}\,. \tag{3.21}$$

We neglect the cubic term for now (see below for a comment on its effect). The flow diagram is again represented by figure 2 (a). Close to the origin, to this order, the flow equations (2.11b) are

$$\dot{\phi}^1 = \frac{\Delta_1^-}{\ell}\phi^1 + \dots\,, \qquad \dot{\phi}^2 = \frac{\Delta_2^-}{\ell}\phi^2 + \dots\,, \tag{3.22}$$

and their general solution is

$$\phi^1 = \phi^1_- e^{\Delta_1^- u/\ell} + \dots, \qquad \phi^2 = \phi^2_- e^{\Delta_2^- u/\ell} + \dots\,. \tag{3.23}$$

We can take the limit of $u \to \pm\infty$ only if either $\phi^1_- = 0$ (UV fixed point) or $\phi^2_- = 0$ (IR fixed point). In the single field case, it would be impossible to turn on a non-analytic deformation corresponding to a vev along the attractive direction ϕ^1. Now we will show that, instead, a non-vanishing ϕ^1_+ can be can be generated by ϕ^2_- through the following non-analytic deformation:

$$\delta W = C_1 \phi^1 \left(\phi^2\right)^{\Delta_1^+/\Delta_2^-}\,, \tag{3.24}$$

which corresponds to a separable deformation of the kind (3.18) with $\kappa_1 = \Delta_1^-$, $\kappa_2 = \Delta_1^+$. Notice that this is allowed since by definition $\Delta_1^- + \Delta_1^+ = d$.

Adding the sub-leading term (3.24) to (3.21) changes the flow equations to:

$$\dot{\phi}^1 = \Delta_1^- \phi^1 + C_1 \left(\phi^2\right)^{\Delta_1^+/\Delta_2^-} \tag{3.25}$$

$$\dot{\phi}^2 = \Delta_2^- \phi^2\,. \tag{3.26}$$

Integration of (3.25) and (3.26) lead to the following expansions where only the first non-zero contribution from W_0 and from δW is presented for each field:

$$\phi^1(u) = \frac{C_1}{2\Delta_1^+ - d}\left(\phi^2_-\right)^{\Delta_1^+/\Delta_2^-} e^{\Delta_1^+ u/\ell} + \dots \tag{3.27}$$

$$\phi^2(u) = \phi^2_- e^{\Delta_2^- u/\ell} + \dots \tag{3.28}$$

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We can see from equation (3.27) that the deformation (3.24) has the effect of generating a vev-type term for ϕ^1 from a source-type term for ϕ^2,

$$\phi^1_+ = \frac{C_1}{2\Delta_1^+ - d} \left(\phi^2_-\right)^{\Delta_1^+/\Delta_2^-} . \tag{3.29}$$

Even if we start on the ϕ^2 axis, ϕ^1 will start to flow, as shown in figure (2(b)). Notice that, although it is possible to generate the same effect "perturbatively" using the cubic terms in the analytic solution, one will not obtain the correct scaling for a vev-type sub-leading term.

Notice that the same flow (3.27)–(3.28) can also be obtained starting from a solution of the kind $W_0^{(+-)}$ *without* deformation. This is an example of the redundancy of the superpotential description, and of the fact that the same flow can be obtained from different superpotentials.

We now return to the general N-field case. Given this redundancy, it is desirable to find a "minimal" set of deformations which encodes all possible flows that reach an extremum of the potential, and depends on a finite number of parameters. Such a minimal set is given by

$$\delta W = \frac{1}{\ell} \sum_{r=1}^{N} C_r \left(\phi^r\right)^{d/\Delta_r^{\sigma_r}} . \tag{3.30}$$

The expression (3.30) is essentially the sum of N single-field deformations. In this case we have to impose the restriction

$$0 < \Delta_r^{\sigma_r} < d/2, \tag{3.31}$$

because we want all terms in equation (3.30) to have powers which are both positive and sub-leading with respect to the leading quadratic terms in W_0. Since by equation (3.7) $\Delta_r^+ > d/2$, we conclude that the deformation of the form (3.30) have to satisfy:

$$\text{either} \quad \left\{\Delta_r^{\sigma_r} = \Delta_r^- \quad \text{and} \quad \Delta_r^- > 0\right\} \qquad \text{or} \quad C_r = 0, \tag{3.32}$$

i.e, as in the single field case, it can only appear on for *relevant* operators for which the leading term in W_0^σ corresponds to turning on a *source* ϕ_-. The constants C_r, as in the single field case, determine the sub-leading (*vev*) terms in these directions,

$$\phi^r_+ = \frac{C_r d}{\Delta_r^- \Delta_r^+} \left(\phi^r_-\right)^{\Delta_r^+/\Delta_r^-} . \tag{3.33}$$

It is easy to check that, for a generic UV extremum with M relevant and $N - M$ irrelevant directions, the deformations in the class (3.30) are enough to have as many integration constants as there are boundary conditions satisfying the constraint (3.10): there are M $C_{\hat{s}}$ in δW, plus N initial conditions for the flow equations. These correspond to the M source terms $\phi_-^{\hat{r}}$ and N vev terms $\phi_+^{\hat{r}}$ which are needed to fix the asymptotics in the UV, see equation (3.10). The remaining $N - M$ integration constants (which should in principle be there) are necessarily set to zero if the flow has to reach the UV fixed point. Thus the deformations of the type (3.30), constrained by (3.32), are enough to obtain all (non-generic) solutions which reach an extremum.

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Δ	m^2	type of fixed point	type of operator	source	vev	Deformation
$+$	$>$	UV	Irrelevant	0	$\neq 0$	No
$+$	$<$	UV	Relevant	0	$\neq 0$	No
$-$	$>$	IR	Irrelevant	$\neq 0$	0	No
$-$	$<$	UV	Relevant	$\neq 0$	$\neq 0$	Yes

Table 1. A classification of the deformation of the type (3.30) in terms of the signs of the squared masses and the choice of Δ_r^+ or Δ_r^-. The nature of the fixed point when reached along ϕ^r, i.e., where all the ϕ^s with $s \neq r$ are set to zero, is indicated. When all the masses have the same sign and the dimensions are of the same kind, we can unambiguously classify the fixed points, otherwise a single extremum may belong to different categories above.

If we start from an extremum which is a local *maximum* of the potential V, all the Δ_r^- are positive and the deformation (3.30) contains N arbitrary integration constants C_r, which is the number needed to describe a generic solution. This is what we will refer to in the next section as a *complete integral*, to use the terminology of Hamilton-Jacobi theory. Therefore solutions of this form which arrive at UV maxima are generic.[8] In the case of a *minimum*, all m_r^2 are positive and all the C_r in (3.30) are constrained to vanish.

In table (3.1.2) we summarise the allowed deformations of the special type (3.30) and the interpretation of the corresponding operator in the holographic language.

To summarise, around an extremum of V with M negative and $N-M$ positive mass eigenvalues, the set of superpotentials

$$\begin{aligned} W^\sigma = \; & \frac{2(d-1)}{\ell} + \frac{1}{2\ell}\left(\sum_{r=1}^{N} \Delta_r^{\sigma_r} \left(\phi^r\right)^2 + \mathcal{O}\left(\phi\right)^3\right) \\ & + \sum_{\hat{r}=1\ldots M \text{ and } \sigma_{\hat{r}}=-} C_{\hat{r}} \left(\phi^{\hat{r}}\right)^{d/\Delta_{\hat{r}}^-} \left(1+\mathcal{O}(\phi^3)\right) \end{aligned} \tag{3.34}$$

encodes all flows arriving to or departing from the extremum. It is important to remark that, when considering solutions which connect different extrema, generically one can choose the special form (3.34) only close to one of them. For the others, the deformation will have a more general form of the type (3.20).

If we include the sub-leading cubic term in W_0^σ from equation (3.13) in solving equation (3.16), the corresponding solution reads

$$\delta W(\phi) = \sum_{\Delta=\Delta_{\hat{r}}^-} C_s \left(\phi^s\right)^{d/\Delta_s - 1}\left[\phi^s + \sum_{p,q=1}^{N} \frac{3\; d\; g_{spq}\; \phi^p\; \phi^q}{\Delta_s\left(\Delta_s+\Delta_p+\Delta_q-d\right)\left(\Delta_s-\Delta_p-\Delta_q\right)} + \mathcal{O}(\phi)^3\right]. \tag{3.35}$$

The denominators of the cubic term can vanish for certain combinations of dimensions, in which case the corresponding terms in the expansion are replaced by terms containing

[8] This does not mean that *all* solutions will arrive at a given maximum, e.g. if there are multiple local maxima. However one does not need to tune the integration constants to arrive there, unlike the case for generic extrema.

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logarithmic contributions. There are further corrections to this expression, organised in a double series expansion in ϕ^r *and* C_r. The general form in the single field case can be found in [21].

3.1.3 Lifting the arbitrariness in W: IR regularity

As we have seen, the solution $W^{(-,-\dots-)}$ at a minimum of V does not admit continuous deformations. This fact has very far reaching consequences. In fact, as we will argue below, $W^{(-,-\dots-)}$ is the *only* one among all the W^σ which fulfills an important regularity condition. If this condition is imposed, the uniqueness of $W^{(-,-\dots-)}$ lifts the redundancy in the superpotential description and completely fixes the solution W everywhere in field space. Thus, the theory has only one physical vacuum (or at most a finite number, if there is more than one local minimum of V) for any value of the UV initial conditions for the flow (which do not enter in the superpotential).

We now discuss the regularity condition. In holography, not all bulk solutions are allowed, but only those which satisfy certain conditions in the IR. If one requires strict regularity (finiteness of the curvature invariants), then in the class of solutions (2.2), the only possibility is that the flow reaches an IR asymptotically AdS fixed point. However, one does not want to check regularity individually for each flow. This is where the superpotential formulation is useful: it gives a way of imposing regularity for whole classes of flows at the same time. With this in mind, the most economic regularity requirement is that *all* flows around a minimum of V reach the minimum as an IR fixed point. This is the case if, around the minimum, W is chosen to be $W^{(-,-\dots-)}$. This choice is reasonable, because it leaves the possibility of slightly deforming the initial conditions in the UV (i.e. the values of the UV sources) without spoiling the IR behavior. This is crucial from the dual QFT point of view: for example, every time we compute a correlation function we perturb the sources, and we do not want to worry about the fact that a small deformation may render the theory inconsistent. Of course, a big change in the source may still lead the flow elsewhere, therefore ultimately the range of allowed values may be restricted. But this is not unusual even in perturbative field theories, where it is known that certain theories make sense only in certain *continuous* ranges of couplings (e.g. for $\lambda\phi^4$ theory we must restrict to $\lambda > 0$). The fact that not all ranges of couplings lead to IR-regular solutions was also observed in holographic RG flows in [22]: there, we saw an example where for negative UV source an IR fixed point may be reached, whereas for positive UV source no regular solution exists.

To summarize, imposing that the superpotential around a minimum of V has the form $W^{(-,-,\dots,-)}$ completely fixes the full W while at the same time not restricting the values of the UV sources.

Of course, if we impose this condition, there is no guarantee that the solution in the UV will look like one with the special class of deformation, (3.34). In general, the subleading non-analytic part will have the form (3.20), with some fixed function $K(\kappa_r)$.

3.2 Bounces

We now study the geometry close to an extremum of W which is *not* an extremum of V. At these points, the flow inverts its direction (it "bounces"). In the single-field case this behavior was analysed in detail in [22] where it was shown to lead to a breakdown of the first-order formalism and to multi-branched superpotentials. In the multi-field scenario, there is a much richer a variety of bounces. We begin with a brief review of the single field case.

Review of single-field bounces. Here we set $N = 1 = \mathcal{G}_{11}$, as can always be done for a single-field. The superpotential equation is

$$V = \frac{1}{2}W'^2 - \frac{d}{4(d-1)}W^2 \tag{3.36}$$

where $W' = dW/d\phi$. Suppose W' vanishes at a point $\phi = \phi_B$ where the potential has a regular expansion and non-vanishing first derivative. Then, expanding the superpotential equation (3.36) around ϕ_B one finds two branches of $W(\phi)$ that meet at ϕ_B. For concreteness consider $V'(\phi_B) > 0$. There are two solutions connecting to ϕ_B,

$$\begin{aligned} W_\uparrow(\phi) &\simeq W_B + \frac{2}{3}\sqrt{2V'(\phi_B)}\,(\phi-\phi_B)^{3/2}, \\ W_\downarrow(\phi) &\simeq W_B - \frac{2}{3}\sqrt{2V'(\phi_B)}\,(\phi-\phi_B)^{3/2}, \end{aligned} \qquad \phi > \phi_B, \tag{3.37}$$

where

$$W_B \equiv W(\phi_B) = \sqrt{-\frac{4(d-1)}{d}V(\phi_B)} \equiv B(\phi_B). \tag{3.38}$$

Equation (3.38) means that bounces occur when the flow reaches the critical curve which bounds the forbidden region defined in equation (3.3). The two branches can be merged into a fully regular solution for the metric and the scalar: integrating the flow equations we obtain

$$\phi(u) = \phi_B + \frac{V'(\phi_B)}{2}(u-u_B)^2 + \mathcal{O}(u-u_B)^3 = \begin{cases} \phi_\uparrow(u) & \text{for } u > u_B, \\ \phi_\downarrow(u) & \text{for } u < u_B, \end{cases} \tag{3.39}$$

$$A(u) = A_B - \sqrt{\frac{V(\phi_B)}{d(d-1)}}(u-u_B) + \mathcal{O}(u-u_B)^4. \tag{3.40}$$

It is clear from (3.39) and (3.40) that at the point ϕ_B both the scalar field and the metric are regular. The bounce corresponds to a point where the first (but not the second) derivative $\dot\phi(u)$ vanishes at some point $u = u_B$, and the superpotential becomes double-valued because ϕ ceases to be a good coordinate. The two branches correspond to $u > u_B$ and $u < u_B$, and $\phi < \phi_B$ on both branches.

Multi-field bounces. When many flows of a family have a field component which is reversed along its flow, the velocity field is multi-valued. This means that a superpotential required to describe such flows has different branches. We will call a *bounce* a point (or a set of points) in field space around which a solution $W(\phi)$ to the superpotential equation (2.11a) has more than one branch, i.e. some of the scalars reverse their direction along the flow. When n scalars reverse direction simultaneously we call this an *order n bounce*. The special case of $n = N$ will be called a *complete bounce* to distinguish it from the cases where $n < N$, which we will call *partial bounces*.

3.2.1 Complete bounces

Complete bounces are defined as loci $\phi = \{\phi^r_B\}$ where a solution $W(\phi)$ of the superpotential equation (2.11a) reaches its lower bound,

$$W_B \equiv W(\phi_B) = \sqrt{-\frac{4(d-1)}{d}V(\phi_B)} \equiv B(\phi_B)\ . \tag{3.41}$$

These may be isolated points, or may form a co-dimension p sub-manifold, Σ_{N-p}, of the scalar manifold $\mathcal{M}_\phi$. In the latter case Σ_{N-p} lies on an equipotential surface of $V(\phi)$, since by the superpotential equation (2.11a) all derivatives of W must vanish at all points on Σ_{N-p}. Hence W, and thus V from (3.41) is constant.

Because an equipotential is a co-dimension-one sub-manifold of $\mathcal{M}_\phi$, the simplest example of a bounce is obtained by giving the superpotential equation initial condition (3.41) on an entire connected piece of an equipotential,[9] which we will denote Σ_{N-1}. There remains just one direction orthogonal to Σ_{N-1} which we call ψ, along which W must vary. The superpotential equation then becomes effectively one-dimensional leading to a two-branched superpotential as in the single-field case. More generally, we can impose that (3.41) holds only on a sub-manifold $\Sigma_{N-p} \subset \Sigma_{N-1}$, with $1 \leqslant p \leqslant N$.

In order to write explicit solutions it will be convenient to choose coordinates such that Σ_{N-p} corresponds to fixing p of the coordinates, which we label $\psi \equiv \psi^i$, $i = 1, \ldots, p$, to their values ψ^i_B. The remaining coordinates will be denoted by Λ^α with $\alpha = 1, \ldots, N-p$. In other words,

$$\phi = (\psi^1, \ldots, \psi^p, \Lambda^\alpha, \ldots, \Lambda^{N-p}) \equiv (\psi, \Lambda), \tag{3.42}$$

$$\Sigma_{N-p} = \left\{ \phi \in \mathcal{M}_\phi \mid \phi = (\psi^1_B, \ldots, \psi^p_B, \Lambda^\alpha, \ldots, \Lambda^{N-p}) \equiv (\psi_B, \Lambda)\,,\ \left.\frac{\partial V}{\partial \Lambda^\alpha}\right|_{(\psi_B,\Lambda)} = 0 \right\}, \tag{3.43}$$

where the second condition is the equipotential condition. We will also often use the notation $\phi_B = (\psi_B, \Lambda)$. In the vicinity of Σ_{N-p}, the superpotential can be expanded as

$$W(\phi) = W_B + \delta W, \qquad \delta W \xrightarrow{\psi \to \psi_B} 0. \tag{3.44}$$

As we assume $V(\phi)$ is analytic, we can write

$$V(\psi, \Lambda) = V_B + \sum_{i=1}^{p} \left.\frac{\partial V}{\partial \psi^i}\right|_{(\psi_B,\Lambda)} \delta\psi^i + \mathcal{O}(\delta\psi)^2 \tag{3.45}$$

where $\delta\psi = \psi - \psi_B$. Inserting the expansions (3.44)–(3.45) in the superpotential equation gives, to lowest order in $\delta\psi$ and δW,

$$\frac{1}{2}\partial_r \delta W \partial^r \delta W - \frac{d}{4(d-1)}\left(2W_B + \delta W\right)\delta W = \sum_{i=1}^{p} \left.\frac{\partial V}{\partial \psi^i}\right|_{(\psi_B,\Lambda)} \delta\psi^i\,, \tag{3.46}$$

[9] Typically, an equipotential is the union of disjoint hyper-surfaces and to consider bounces in the sequence we will only consider connected equipotentials.

where we have kept linear terms in $\delta\psi$ but up to *quadratic* terms in δW, because as we will see in a moment some of these terms are of the same order as $\delta\psi$. Indeed, suppose that δW is of order $(||\delta\psi||)^\gamma$, $\gamma > 0$. It follows that the three terms on the left hand side of equation (3.46) scale as

$$W_B \delta W \sim (||\delta\psi||)^\gamma, \qquad (\delta W)^2 \sim (\partial_\Lambda \delta W)^2 \sim (||\delta\psi||)^{2\gamma}, \qquad (\partial_\psi \delta W)^2 \sim (||\delta\psi||)^{2\gamma-2}. \tag{3.47}$$

Of the three terms above, if $\gamma < 2$ the third is the dominant one for small $\delta\psi$, whereas the first term dominates if $\gamma > 2$. In the latter case however it is impossible to match the linear term on the right hand side. Therefore it is the third term in equation (3.47) which dominates over the first two, and we conclude that $\gamma = 3/2$ as in the single-field bounce case. Equation (3.46) can then be approximated as,

$$\partial_r \delta W \partial^r \delta W = 2 \sum_{i=1}^{p} \left.\frac{\partial V}{\partial \psi^i}\right|_{(\psi_B,\Lambda)} \delta\psi^i \ . \tag{3.48}$$

Notice that equation (3.48) is similar to the Eikonal equation of geometrical optics, with the left-hand side playing the role of the square of the refractive index. Bounces are analogous to total internal refraction (i.e. mirages) off a region with vanishing refraction index.

There are many types of solutions to equation (3.48). Those which are closest to the single-field case (3.37) is the following set of solutions, differing by 2^p sign combinations:

$$\delta W(\psi_B, \Lambda) = \frac{2}{3} \sum_{i=1}^{p} (-1)^{s_i} \sqrt{\left[\frac{2}{\mathcal{G}^{ii}} \frac{\partial V}{\partial \psi^i}\right]_{(\psi_B,\Lambda)} \delta\psi^i}\, \delta\psi^i + \mathcal{O}(\delta\psi)^2 \, , \quad s_i = 0, 1 \ . \tag{3.49}$$

We have written the solution so that the expression under the square root is positive for both signs of $\partial V / \partial \psi_i$. Although all solutions (3.49) are valid close to the bounce surface Σ_p, only certain combinations can be glued together, as in the single field case, to obtain regular flows $\phi^r(u)$. To find which ways of gluing are consistent, we write the flow equation (2.11b) using equation (3.49)

$$\dot\psi^i = (-1)^{s_i} \sqrt{\left[2\mathcal{G}^{ii} \frac{\partial V}{\partial \psi^i}\right]_{(\psi_B,\Lambda)} \delta\psi^i} + \mathcal{O}(\delta\psi) \, , \ s_i = 0, 1 \, , \tag{3.50a}$$

$$\dot\Lambda^\alpha = \mathcal{O}(\delta\psi)^{3/2} \ . \tag{3.50b}$$

Integration of (3.50), with the initial condition $\phi^r(u_b) = (\psi_B, \Lambda_B)$, leads to

$$\psi^i(u) = \psi^i_B + \frac{1}{2} \left[\frac{\mathcal{G}^{ii}}{2} \frac{\partial V}{\partial \psi^i}\right]_{(\psi_B,\Lambda)} (u - u_B)^2 + \mathcal{O}(u - u_B)^3 \, , \tag{3.51a}$$

$$\Lambda^\alpha(u) = \Lambda^\alpha_B + \mathcal{O}(u - u_B)^3, \tag{3.51b}$$

where in each direction we need to impose the consistency condition

$$(-1)^{s_i} \left[\frac{\partial V}{\partial \psi^i}\right]_{(\psi_B,\Lambda)} (u - u_B) > 0 \, . \tag{3.52}$$

As in the single field case, the two sign choices $s_i = 0, 1$ in each direction give the solution for $u > u_B$ and $u < u_B$, respectively. However, all fields $\psi_i(u)$ must be in the same range ($u > u_B$ and $u < u_B$) for the solution to be smooth. Therefore the condition (3.52) fixes all but an overall sign choice, which out of the set (3.49) leaves only two solutions which glue consistently across the bounce,

$$W(\psi, \Lambda) = W_B \pm \frac{2}{3} \sum_{i=1}^{p} \text{sign} \left(\left[\frac{\partial V}{\partial \psi^i} \right]_{\psi_B} \right) \sqrt{\left[\frac{2}{\mathcal{G}^{ii}} \frac{\partial V}{\partial \psi^i} \right]_{\psi_B} \delta\psi^i \, \delta\psi^i} + \mathcal{O}(\delta\psi)^2 \; . \tag{3.53}$$

The $\pm$ sign actually characterise two branches of the same multi-valued superpotential, as follows from the fact that both branches describe the same set of flows but for different ranges of u.

There exist other consistent solutions to (3.48) and many of them share an interesting feature: they extend Σ_{N-p} to a higher-dimensional sub-manifold of Σ_{N-1}. For example, take the following solution,

$$\begin{aligned} W(\psi, \Lambda) = W_B \pm \frac{2\sqrt{2}}{3} \Bigg\{ & \left[\sum_{j,k=p-n+1}^{p} \mathcal{G}^{ij} \frac{\partial V}{\partial \psi^j} \frac{\partial V}{\partial \psi^k} \right]_{(\psi_B, \Lambda)}^{-1/2} \left(\sum_{l=p-n+1}^{p} \left[\frac{\partial V}{\partial \psi^l} \right]_{(\psi_B, \Lambda)} \delta\psi^l \right)^{3/2} \\ & + \sum_{i=1}^{p-n} \sqrt{\left[\frac{1}{\mathcal{G}^{ik}} \frac{\partial V}{\partial \psi^i} \right]_{(\psi_B, \Lambda)} \delta\psi^i \, \delta\psi^i} \Bigg\} + \mathcal{O}(\delta\psi)^2 \; . \end{aligned} \tag{3.54}$$

For this solution there is a bounce at $\psi = \psi_B$ as before, but now this point belongs to a line of bounces defined by

$$\sum_{l=1}^{n \leqslant p} \left[\frac{\partial V}{\partial \psi^l} \right]_{(\psi_B, \Lambda)} \delta\psi^l = 0 \; . \tag{3.55}$$

The bounce manifold is now $N - p + 1$ dimensional. By a change of coordinates, we can recast the solution (3.54) in the form (3.49) by defining one more Λ-coordinates for the directions satisfying (3.55) and removing one ψ-coordinate.

Below we illustrate these features by considering complete bounces in a simple two-scalar example.

Two-field example. We set $\phi = (\phi^1, \phi^2)$, and to make things simple we consider a flat field space metric[10] $\mathcal{G}_{rs} = \delta_{rs}$. The equipotentials are one-dimensional, therefore by our general discussion we can have bounces at isolated points or along lines. We first assume that ϕ_B is an isolated *point*, thus all coordinates will be of the ψ^i type. We will see below that the solutions in which bounces occurs on a line naturally appear in this framework.

For a generic complete bounce,

$$\partial_1 V(\phi_B) \neq 0, \qquad \partial_2 V(\phi_B) \neq 0. \tag{3.56}$$

[10] By the use of Riemann normal coordinates, (3.12), this assumption turns out to be a general at the order in ϕ^r in which we will work.

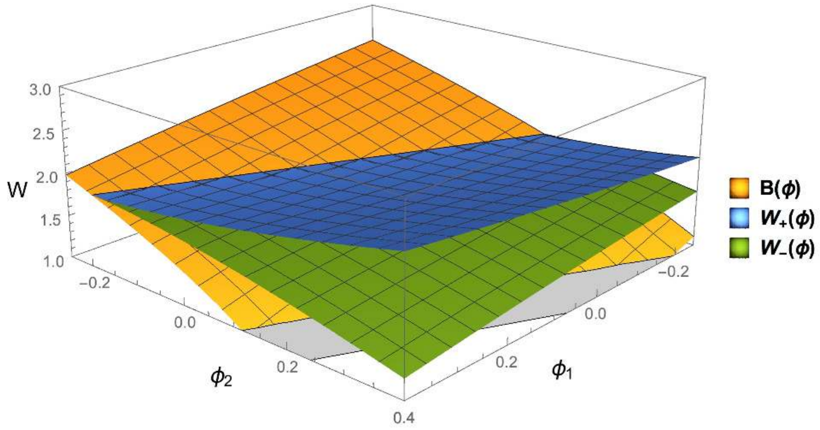


Figure 3. Two fields — a complete bounce given, to leading order in $\phi - \phi_B$, by equation (3.60). The associated vector field is depicted in figures 4, with the lower (upper) branch in green (blue).

Another way of writing the Eikonal equation (3.48) is

$$\vec{\nabla}(\delta W) = \sqrt{2\sum_{r=1}^{2} \partial_r V|_B\, \delta\phi^r\, \vec{m} + \mathcal{O}(\delta\phi)^2} \equiv \sqrt{X}\,\vec{m} + \mathcal{O}(\delta\phi)^2 \tag{3.57}$$

for some, generally ϕ^1 and ϕ^2-dependent, unit vector $\vec{m}$. One can write this as $\vec{m} = d\vec{r}/du$ where $\vec{r}(u)$ is the 'light ray' with u an affine parameter along the path. An equivalent form of (3.57) is

$$\partial_1 \delta W(\phi) = \sqrt{X}\cos(g(\phi)) \qquad \partial_2 \delta W(\phi) = \sqrt{X}\sin(g(\phi)) \tag{3.58}$$

where g is a function satisfying, to leading order in $\delta\phi = (\phi - \phi_B)$, a constraint equation derived from $\partial_{[1}\partial_{2]}W = 0$, namely

$$(X\partial_2 g + \partial_1 V|_B)\sin g + (X\partial_1 g - \partial_2 V|_B)\cos g = 0. \tag{3.59}$$

One solution is

$$W(\phi) = W_B \pm \frac{2\sqrt{2}}{3}\frac{\left[\partial_1 V(\phi_B)\delta\phi^1 + \partial_2 V(\phi_B)\delta\phi^2\right]^{3/2}}{\|\partial_s V|\,(\phi_B)\,\|} + \mathcal{O}(\delta\phi^r)^2 \tag{3.60}$$

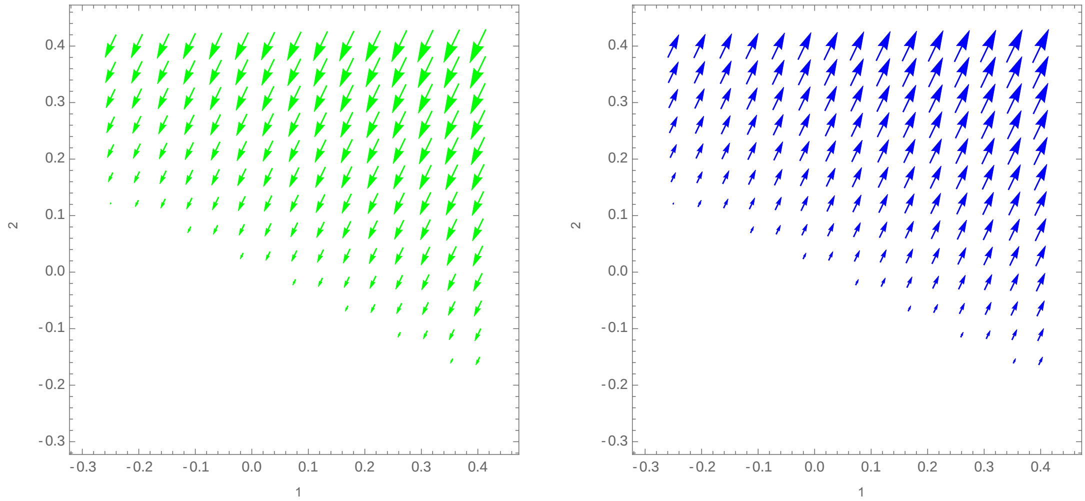
corresponding to $\tan g = [\partial_2 V/\partial_1 V]_B$. The solution (3.60) is represented in figure 3 and the gradients of its two branches are shown figures 4. Notice that the bounce occurs along a line given by the solution to $X = 0$ with X defined in equation (3.57). It corresponds, therefore, to the maximal dimension of an equipotential in two dimensions.

Another solution to equation (3.59) is given by

$$W = W_B \pm \frac{2}{3}\sum_{r=1}^{2} \mathrm{sign}\left(\partial_r V|_{\phi_B}\right)\delta\phi^r\sqrt{2\,\partial_r V|_{\phi_B}\,\delta\phi^r} + \mathcal{O}(\delta\phi^s)^2\,, \tag{3.61}$$

where the choice of signs was fixed using the consistent solutions (3.53) and the $\pm$ sign is related to the range of u in the expansion (3.51a) by

$$\pm = \mathrm{sign}\left[(u - u_B)\,\partial_r V|_{\phi_B}\right]. \tag{3.62}$$



(a) Upper branch of $\partial_r W$ from (3.60). (b) Lower branch of $\partial_r W$ from (3.60).

Figure 4. The superpotential (3.60) gives rise, to leading order in $(\phi^r - \phi^r_B)$, to the vector fields depicted above. In figure (a) the green arrows correspond to the gradient of the lower branch of $W(\phi)$ depicted in figure 3, the choice of a negative sign in (3.60). Figure (b) represents the gradient of the upper branch from figure 3, a positive sign in (3.60). Each flow line goes back along itself close to this complete bounce.

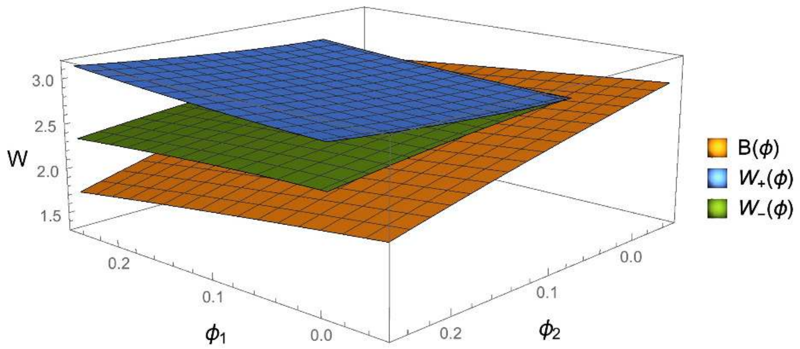


Figure 5. Two fields — a complete bounce given, to leading order in $\delta\phi^r = \phi^r - \phi^r_B$, by equation (3.61). The two branches meet at a point.

While for the first solution, (3.60), the bounce occurs on a line, for this second solution, (3.61), the bounce takes place *only* at the point $\phi = \phi_B$ because this is the only place where $\|\partial_r W\|$ in equation (3.61) vanishes.

One apparent drawback of the expansion (3.61) is that the the reality condition for the superpotential, $V_r\,(\phi^r - \phi^r_B) > 0$, defines a quarter of the plane with origin at the point ϕ_B and we cannot obtain the full solution which extends outside of this region from this expansion. However, from figure 6 we observe that the velocity field is contained in this range, meaning that no solution of the form (3.61) has flows escaping this region of the plane and an extension is, in practice, unnecessary.

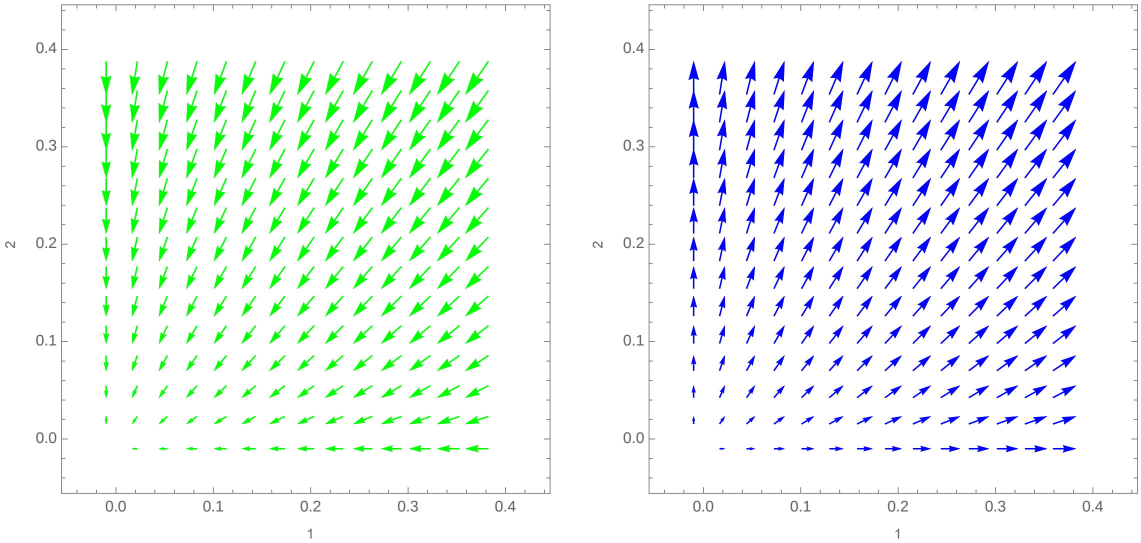


Figure 6. The gradient of the superpotential given by equation (3.61) and depicted in figure 5. The velocity field focus on the point $\phi = \phi_B = 0$. The fact that the $W(\phi)$ cannot be continued for negative $\phi^r - \phi^r_B$, as follows from (3.61) and from the signs of $\partial_r V|_{\phi_B}$ for our potential, is not a problem since the paths this superpotential generates do not cross the lines $\delta\phi^1 = 0$ and $\delta\phi^2 = 0$.

Thinking in terms of the Eikonal equation (3.48), complete bounces can be seen as a total internal reflection on a meta-material with a refractive index that vanishes linearly as we approach the critical curve.

3.2.2 Partial bounces

Partial bounces occur on loci where some, but not all field components have vanishing speed, in contrast with the complete bounces treated in subsection 3.2.1. A more precise definition is the following:

> *A partial bounce corresponds to a change of branches of the superpotential occurring on a sub-manifold Σ_{N-p} of $\mathcal{M}_\phi$, such that $\|\partial_r V|_{\Sigma_{N-p}}\|$ is non-zero and the superpotential $W(\phi)$ does not equal $B(\phi)$ on Σ_{N-p}.*

At a partial bounce, as at a complete bounce, a single-valued superpotential fails $W(\phi)$ to describe locally geodesically complete solutions. A bounce is also characterised by the existence of a second superpotential which coincides with $W(\phi)$ on Σ and which makes the flows locally geodesically complete. The superpotential should be seen as locally double-branched and its integral lines fail to define local coordinates around the bounce.

For concreteness, we start by determining the behaviour of $W(\phi)$ around a partial bounce occuring on a co-dimension one sub-manifold. Bounces will no longer correspond to equipotentials and we will see below which changes this implies with respect to the complete bounces of subsection 3.2.1. We compute the explicit form of the holographic β-function perturbatively near the bounce. We show that the β-function has two branches, as in the single-field case [22] and we highlight the differences from that case.

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Our choice of coordinates on an open neighbourhood of a co-dimension one bouncing manifold, Σ_{N-1}, is such that

$$\|\partial W\|^2 = \mathcal{G}^{\psi\psi}\left(\frac{\partial W}{\partial\psi}\right)^2 + \mathcal{G}^{ij}\frac{\partial W}{\partial\Lambda^i}\frac{\partial W}{\partial\Lambda^j}\,, \tag{3.63}$$

where Σ_{N-1} in these coordinates is characterised by $\psi = \psi_B$. We can rewrite the superpotential equation (2.11a), in these coordinates, as

$$\frac{\partial W}{\partial\psi} = \pm\sqrt{\frac{1}{\mathcal{G}^{\psi\psi}}\left(2V(\psi,\Lambda) + \frac{d}{2(d-1)}W^2(\psi,\Lambda) - \mathcal{G}^{\alpha\beta}\frac{\partial W}{\partial\Lambda^\alpha}\frac{\partial W}{\partial\Lambda^\beta}\right)}\,, \tag{3.64}$$

with $\alpha = 1,\ldots,N-1$. Equation (3.64) becomes degenerate for a given $\psi = \psi_B$ in the following if

$$\left.\frac{\partial W}{\partial\psi}\right|_{(\psi_B,\Lambda)} = 0 \quad , \quad \mathcal{G}_{\psi\psi}(\psi_B,\Lambda)\neq 0 \quad \text{and} \quad \partial_\psi V(\psi_B,\Lambda)\neq 0. \tag{3.65}$$

The condition that $\mathcal{G}_{\psi\psi}$ is non-zero is, for metrics of the form (3.63) equivalent to the non-degeneracy of $\mathcal{G}$. This ensures that the vanishing of $\partial_\psi W$ at Σ_{N-1} is not a consequence of ill-defined coordinates. By analogy with the one-dimensional case, we expect a pair of solutions to the superpotential equation that meet at Σ_{N-1}, corresponding to the $\pm$ sign in (3.64).

From (3.64), (3.65) and our definition of a bounce, it follows that when $W(\phi)$ is restricted to Σ_{N-1} it must satisfy:

$$V_B(\Lambda) \equiv V(\psi_B,\Lambda) = \frac{1}{2}\left[\mathcal{G}^{ij}\frac{\partial W}{\partial\Lambda^i}\frac{\partial W}{\partial\Lambda^j}\right]_{(\psi_B,\Lambda)} - \frac{d}{4(d-1)}W^2(\psi_B,\Lambda)\,. \tag{3.66}$$

This is nothing but the $N-1$ dimensional superpotential equation (2.11a). Therefore, partial bounces occur along solution of the lower-dimensional superpotential equation restricted to the directions Λ^i.

Around $\psi = \psi_B$ we can expand V and W as follows:

$$V(\psi,\Lambda) = V_B(\Lambda) + \delta\psi\left.\frac{\partial V}{\partial\psi}\right|_{(\psi_B,\Lambda)} + \mathcal{O}\left(\delta\psi\right)^2, \quad \text{with} \quad \delta\psi \equiv (\psi-\psi_B)\,, \tag{3.67a}$$

$$W(\psi,\Lambda) = W_B(\Lambda) + \delta W(\psi,\Lambda) \equiv W_B(\Lambda) + \delta W. \tag{3.67b}$$

where

$$W_B(\Lambda) \equiv W(\psi_B,\Lambda) \tag{3.68}$$

is a solution to (3.66). Proceeding in the same way as for the complete bounce, we arrive at the solution close to a partial bounce,

$$W(\psi,\Lambda) = W(\psi_B,\Lambda) \pm \frac{2}{3}\delta\psi\sqrt{2\left[\mathcal{G}_{\psi\psi}\frac{\partial V}{\partial\psi}\right]_{(\psi_B,\Lambda)}\delta\psi} + \mathcal{O}(\delta\psi)^2\,. \tag{3.69}$$

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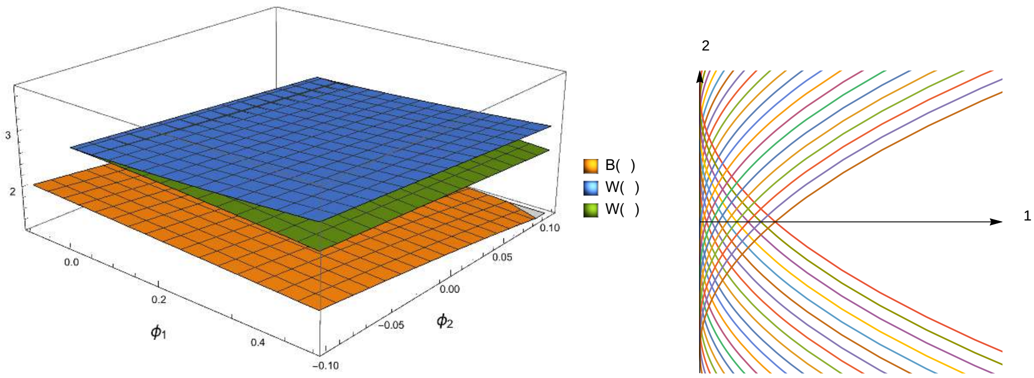

(a) (b)

Figure 7. Figure (a) represents in yellow the curve $B(\phi)$ and a two-branched superpotential W in green and blue. $B(\phi)$ is associated with the potential $V(\phi)$ through equation (3.3). The plot is in the linear regime (3.71) with parameters $V_0 = -1$, $V_1 = 0.44$, $V_2 = 3.31$. The two branches of $W(\phi)$ are given by equations (3.72) and (3.73). The upper branch of the superpotential corresponding to the + sign in (3.72) is in blue and the lower branch of W is in green. Figure (b) shows the flow lines around a bounce, from equation (3.74), differing by the choice of ϕ_0^1. Even though no individual line is self-intersecting, the ensemble of the flow lines gets superposed after the bounce.

The similarity with the single-field case and with the complete bounces of the previous subsection is manifest. The dependence of W on $\delta\psi$ has the same power 3/2 and the solutions are restricted to $\psi < \psi_B$ *or* $\psi > \psi_B$, with two possible signs defining branches. Both branches should again be seen as belonging to the same superpotential in order to make the flow $\phi^r(u)$ locally geodesically complete. The difference lies in the fact that $W(\psi_B, \Lambda)$ and the term under the square root are functions of Λ^α and not constants. Therefore, along a partial bounce the flow has non-vanishing speed along the surface Σ_{N-1}. When all velocities vanish at the bounce we are back to the case of a complete bounce which we analysed in subsection 3.2.1.

As in the case of complete bounces, a partial bounce can also occur on a sub-manifold of $\mathcal{M}_\phi$ with co-dimension $p > 1$, Σ_{N-p}. The solutions (3.53) generalise to partial bounces by the simple replacement:

$$W_B \to W_B(\Lambda^\alpha) \quad \text{with} \quad i = 1, \ldots, p \quad \text{and} \quad \alpha = 1, \ldots, N-p \,. \tag{3.70}$$

as long as $W_B(\Lambda^\alpha) \equiv (\psi_B^i, \Lambda^\alpha)$ solves (3.66) and, of course, as long as the potential and the metric to be such that $\mathcal{G}^{ir}\partial_r V$ is non-zero on Σ_{N-p}.

One important distinction needs to be made between partial and complete bounces: a flow with a partial bounce can also be derived from a superpotential with no bounces, as a consequence of the large redundancy of the superpotential formalism in the multi-field case. On the other hand, in a complete bounce the flow has to reach the critical curve, and this will be the case for any choice of superpotential.

To conclude this section we give an example of a partial bounce in the case of two scalar fields.

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Two-field example. Consider two scalars with a flat metric and a potential in the linear approximation such that:

$$V(\phi^1, \phi^2) = C + V_1\phi^1 + V_2\phi^2 + \mathcal{O}(\phi)^2, \quad V_1 > 0,\; V_2 > 0\,. \tag{3.71}$$

For simplicity we will consider a pair of superpotentials of the form:

$$W = W_0 + W_2\phi^2 \pm \sqrt{2V_1\phi^1}\,\phi^1 + \mathcal{O}(\phi)^{5/2} \tag{3.72}$$

The expansion (3.72) means that we have set $\left[\partial^2 W/\partial(\phi^2)^2\right]_{(0,0)}$ to zero. Equation (3.66) in this case is a single-field superpotential equation which has a continuum of solutions parametrised by an integration constant. Imposing $W(\phi)$ is of the form of (3.72) already fixes this constant, fixing also the relation between W_0, W_2 and the coefficients appearing in (3.71):

$$W_0 = \sqrt{\frac{2(d-1)}{d}\left(\sqrt{V_0^2 + \frac{2(d-1)}{d}V_2^2} - V_0\right)} \tag{3.73a}$$

$$W_2 = -\sqrt{\frac{V_0}{V_2} + \sqrt{\left(\frac{V_0}{V_2}\right)^2 + \frac{2(d-1)}{d}}} \tag{3.73b}$$

The flows coming from the superpotential (3.72) after solving (2.11b) are of the form:

$$\phi^1(u) = \phi_0^1 + \frac{V_1}{2}(u-u_B)^2 + \mathcal{O}(u-u_B)^3 \tag{3.74a}$$

$$\phi^2(u) = W_2(u-u_B) + \mathcal{O}(u-u_B)^3 \tag{3.74b}$$

Figure 7(a) shows the two branches of the superpotential (3.72) plotted together with the potential (3.71) for a specific choice of parameters. Figure 7(b) displays the flows (3.74) for different values of ϕ_0^1. The flow lines return to themselves after the bounce, so we cannot use the flow lines to define one of the coordinates unless we choose a single branch of the superpotential. This figure suggests pursuing the analogy with geometrical optics which is suggested by the Eikonal equation (3.48) and the flow equation (2.11b). As noted previously, a complete bounce is the analogous of a total reflection. What figure 7 suggests is that a partial bounce can be thought of the analogue of a mirage.

4 Comments on the canonical formalism for multi-field gravity

The superpotential formalism used in the previous section is closely related to the Hamilton-Jacobi formalism. In this section we will turn to the general formulation in terms of the Hamilton-Jacobi principal function to gain a more precise understanding why the *gradient* flows assumption is generic. We will also argue that a description in terms of non-gradient flows may be useful in some situations, like in the presence of a global symmetry (this situation however does not apply in the context of holography).

We begin by recalling a few facts from Hamilton-Jacobi (HJ) theory. This provides a procedure to map solutions $(q(t), p(t))$ of the equations of motion into constants (β, α)

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which are functions of the initial conditions (q_0, p_0) by means of a canonical transformation. The generating function of such a canonical transformation is called *Hamilton's principal function* and is usually denoted by $\mathcal{S}(q^i, \alpha_i, t)$ where α_i, $i = 1, \ldots, n$, are independent integration constants which define the new constant momenta. The canonical transformation defined by the principal function is such that:

$$p_i = \frac{\partial \mathcal{S}}{\partial q^i}, \tag{4.1a}$$

$$\beta^i = \frac{\partial \mathcal{S}}{\partial \alpha_i}, \tag{4.1b}$$

$$H\left(q^i, \frac{\partial \mathcal{S}}{\partial q^i}, t\right) + \frac{\partial \mathcal{S}}{\partial t} = 0, \tag{4.1c}$$

where $H(q, p, t)$ is the Hamiltonian of the mechanical system and (4.1c) is called the *Hamilton-Jacobi equation.*

Only derivatives of $\mathcal{S}$ appear in (4.1c) implying that $\mathcal{S}$ is determined only up to an additive constant. $\mathcal{S}$ is a function of $n + 1$ variables and equation (4.1c) places a single constraint on this function. To integrate (4.1c) it is therefore necessary to specify an *integration function* of n variables. However, for $\mathcal{S}$ to define a canonical transformation to new constant coordinates (α, β) it is necessary that it contains n *independent, non-additive* integration *constants*, showing that this formalism is highly redundant. The remaining n integration constants which are necessary to specify a solution are obtained by integrating (4.1a), where the momenta p_i are written in terms of the velocities $\dot{q}^i$.

One very useful aspect of Hamilton-Jacobi theory is that it straightforwardly incorporates the relationship between conserved quantities and symmetries. For example, when the Hamiltonian H does not depend explicitly on time, energy is conserved and we can write:

$$\mathcal{S}(q, P, t) = \mathcal{W}(q, P) - Et \ . \tag{4.2}$$

The function $\mathcal{W}(q, P)$ is called Hamilton's characteristic function. When (4.2) holds, equation (4.1c) takes the following form:

$$H\left(q^i, \frac{\partial \mathcal{W}}{\partial q^i}\right) = E. \tag{4.3}$$

In other words, principal functions of the form (4.2) group solutions to the equations of motion which have the same energy. Similarly, when one of the momenta $p_{\hat{i}}$ is conserved and has the specific value $\widehat{P}_{\hat{i}}$, by integrating (4.1a) one can isolate the dependence of $\mathcal{S}$ on $q^{\hat{i}}$:

$$\mathcal{S}(q^i, p_i) = \widehat{\mathcal{S}} + \widehat{P}_{\hat{i}} q^{\hat{i}} \tag{4.4}$$

where $\widehat{\mathcal{S}}$ is independent of $q^{\hat{i}}$.

4.1 Gradient flows revisited

We will now implement the HJ formalism outlined above in the case of gravity coupled to multiple scalars. This is a well known procedure [33, 34] and here we want to focus on the

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question of what sets apart gradient flows with respect to the more general non-gradient flows mentioned in section 2.2. More specifically, we would like to answer the question of when is it necessary to consider the non-gradient case. As we will see, this question is related to the *separability* of the A-dependence in the HJ principal function $\mathcal{S}(A,\phi^r)$.

An effective Lagrangian which leads to the equations of motion (2.3) is given by

$$L(A,\phi^r,\dot{A},\dot{\phi}^r,N) = Ne^{dA}\left[\frac{d(d-1)}{N^2}\dot{A}^2 - \frac{1}{N^2}\mathcal{G}_{rs}\dot{\phi}^r\dot{\phi}^s - V(\phi)\right] . \tag{4.5}$$

where $\dot{} = \frac{d}{du}$: the "time" for the evolution of the scalars and the scale factor is the holographic coordinate. In (4.5) the variable N does not have a u-derivative and represents a constraint. The equations of motion following from (4.5) are

$$\frac{d(d-1)}{N^2}\dot{A}^2 - \frac{1}{2N^2}\mathcal{G}_{rs}\dot{\phi}^r\dot{\phi}^s + V(\phi) = 0\,, \tag{4.6a}$$

$$\left[\frac{\mathcal{G}_{rs}}{N}\left(\ddot{\phi}^r + \widetilde{\Gamma}^s_{pq}\dot{\phi}^p\dot{\phi}^q + d\dot{A}\dot{\phi}^s - \frac{\dot{N}}{N}\dot{\phi}^s\right) - N\partial_r V\right] = 0\,, \tag{4.6b}$$

$$\left[\frac{d(d-1)}{N}\left(2\ddot{A} + d\dot{A}^2 - 2\dot{A}\frac{\dot{N}}{N}\right) + \frac{d}{2N}\dot{\phi}^2 + d\ N\ V\right] = 0\ . \tag{4.6c}$$

Because N is not dynamical, we can set $N=1$, in which case equations (4.6) become the original equations of motion (2.3). The canonical momenta are

$$p_s = -e^{dA}\mathcal{G}_{rs}\dot{\phi}^r\,, \tag{4.7a}$$

$$p_A = e^{dA}2d(d-1)\dot{A}\ . \tag{4.7b}$$

and the Hamiltonian is given by

$$H(A,\phi^r,p_A,p_r) = e^{-dA}\left(\frac{p_A^2}{4d(d-1)} - \frac{1}{2}p_r p^r\right) + e^{dA}V(\phi)\ . \tag{4.8}$$

Written in terms of the H, equation (4.6a) is just the Hamiltonian constraint,

$$H = \dot{\phi}^r p_r + \dot{A}p_A - L = 0. \tag{4.9}$$

This reduces the number of independent integration constants from $2N+2$ to $2N+1$ (the dynamics is constrained to stay on the zero-energy surface).

The HJ principal function $\mathcal{S}$ is such that

$$p_A = \frac{\partial\mathcal{S}}{\partial A}, \quad p_r = \frac{\partial\mathcal{S}}{\partial\phi^r}, \tag{4.10}$$

and it satisfies the HJ equation following from the Hamiltonian constraint (4.9):

$$e^{-dA}\left[\frac{1}{4d(d-1)}\left(\frac{\partial\mathcal{S}}{\partial A}\right)^2 - \frac{1}{2}\mathcal{G}^{rs}\frac{\partial\mathcal{S}}{\partial\phi^r}\frac{\partial\mathcal{S}}{\partial\phi^s}\right] + e^{dA}V(\phi) = 0. \tag{4.11}$$

Comparing (4.11) with (4.2) and (4.1c) we see that Hamilton's principal function $\mathcal{S}$ is the same as the characteristic function $\mathcal{W}$, since $E=0$. This also means that we need N

independent integration constants in $\mathcal{S}$ in order to have a well defined map from (q, p) to the initial conditions $\mathcal{S}$. Together with the $N + 1$ additional ones coming from the first order flow equations (4.10), this makes up the needed $2N + 1$ integration constants.

A *special class* of Hamilton's principal functions $\mathcal{S}$ is of the following separable form between A and ϕ^r,

$$\mathcal{S} = -e^{dA} W(\phi) \ . \tag{4.12}$$

With this ansatz, equation (4.11) reduces to the superpotential equation (2.11a) and equations (4.7) become the flow equations (2.11b) and (2.11c). Moreover, the flow is automatically gradient, as $\pi_r = \partial_r W$.

The converse statement is slightly more subtle: given a solution which can be written in terms of a gradient flow, it does not follow that the associated principal function has the separable form (4.12). On the other hand, one can show that in this case *there exists* at least one separable solution to Hamiton-Jacobi equation (4.11). Moreover, all non-separable solutions coincide with the separable one on-shell. This point is developed in more detail in appendix A, where we also show how to recover non-gradient flows of the form (2.5) with non-zero curl (2.9)–(2.10) from a non-separable solution $\mathcal{S}(A, \phi^r)$.

We can conclude that, if solutions of equation (4.11) of the separable form (4.12) contain enough independent integration constants, one can be sure that any solution of the full system can be written in the form of a gradient flow. In the context of holography, when solutions connect to a fixed point in the UV which is a local *maximum* of the potential, this is indeed the case, as we have seen in section 3: close to the UV fixed point, the set of solutions with asymptotic form given in equation (3.34) provides all integration constants to reproduce any near-boundary asymptotics of the form (3.6). Moreover, if the solution goes through a bounce, there is a unique way to connect the two branches into a regular solutions, meaning that one does not introduce possible non-gradient flows at bounces.

It has also been noted (see e.g. [26, 39]) that, away from a bounce, locally *any* solution can be embedded in a non-gradient flow with an appropriate superpotential (which however may not be globally defined on field space). We show how to perform this local reconstruction of the superpotential to lowest order in In appendix C.

It follows from the discussion above that, at least in holography, one may safely forget non-gradient flows. However, there are situations in which using a non-separable HJ function may actually be convenient: this is the case when there are global symmetries on the scalar manifold and we may decide to classify solutions in terms of the corresponding conserved quantities. This will be the subject of the next subsection.

4.2 Symmetries vs. gradient flows

As we have discussed in the previous section, locally all flows can be put in a gradient form, and in holography gradient flows are enough to generate all bulk solutions which connect to a maximum of the potential in the UV. However, in this section we argue that in some contexts it can be useful to consider *non-separable* principal functions, giving rise to non-gradient flows, in particular when symmetries of the scalar sector are present.

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Consider a system in our setup characterised by an $O(2)$ isometry in field space with Killing vector field $k^s(\phi)$. We associate a coordinate Θ to the motion along the integral lines of this vector field. For an infinitesimal transformation that takes Θ to $\Theta+\delta\Theta$ we have

$$\delta_\Theta \phi^s := k^s(\phi)\delta\Theta. \tag{4.13}$$

Invariance of the potential translates into:

$$\delta_\Theta V(\phi) = \delta\Theta \; k^s \partial_s V(\phi) = 0. \tag{4.14}$$

As a result of the symmetry (4.14) and the fact that it is an isometry, the momentum conjugate to Θ, p_Θ, is a constant of the motion:

$$p_\Theta = -e^{dA}\mathcal{G}_{\Theta s}(\phi^\alpha)\dot{\phi}^s, \qquad \dot{p}_\Theta = 0, \quad \alpha = 1, \ldots, N-1, \quad \phi^N = \Theta \tag{4.15}$$

It is therefore, possible to write a Hamilton's principal function that groups flows with the same angular momentum $p_\Theta = L$ as follows:

$$\mathcal{S} = \mathcal{S}_L(A, \phi^1, \ldots, \phi^{N-1}) + L\Theta. \tag{4.16}$$

The principal function (4.16) is clearly non-factorisable but is advantageous in practice as it allows to group solutions according the value of conserved charge L. From (4.16) we define $\widehat{W}_L(\phi^\alpha, A)$ by

$$\widehat{W}_L(\phi^\alpha, A) \equiv -e^{-dA}\left(\mathcal{S}_L(\phi^\alpha, A) + L\Theta\right), \quad \alpha = 1, \ldots, N-1, \quad \phi^N = \Theta \,. \tag{4.17}$$

The canonical momenta associated with A and ϕ^r can be obtained from the principal function (4.16) by the use of equation (4.10). However, in order to know if a flow is gradient *on the scalar manifold*, it is necessary to eliminate the A-dependence in the momenta. This can be achieved by finding a function $\mathcal{A}(\phi)$ which equals $A(u)$ on flows and, as explained in appendix A, is a solution to the differential equation:

$$\mathcal{G}^{rs}\left[\partial_r \mathcal{S}(\phi, A)\right]_{A=\mathcal{A}(\phi)} \partial_s \mathcal{A} = \frac{1}{2d(d-1)} \partial_A \mathcal{S}(\phi, A)\big|_{A=\mathcal{A}(\phi)} \,. \tag{4.18}$$

Once a solution $\mathcal{A}(\phi)$ is provided, we can use it to "project" the momenta onto the scalar manifold. It is convenient to define

$$W(\phi) := -d^{-1} e^{-d\mathcal{A}(\phi)}\left[\partial_A \mathcal{S}(\phi, A)\right]_{\mathcal{A}} \,, \tag{4.19a}$$

$$\pi_r(\phi) := -e^{-d\mathcal{A}(\phi)}\left[\partial_r \mathcal{S}(\phi, A)\right]_{\mathcal{A}} \,, \tag{4.19b}$$

because when equations (4.19) are combined with (4.10) we obtain

$$\dot{A} = -\frac{W(\phi)}{2(d-1)} \,, \tag{4.20a}$$

$$\dot{\phi}^r = \mathcal{G}^{rs}\pi_s = \pi^r \tag{4.20b}$$

which are precisely equations (2.4) and (2.5).

Clearly, the procedure we have defined to project on the scalar manifold is not unique, as it depends on the choice of $\mathcal{A}(\phi)$ among the infinite family of solutions to equation (4.18). However, the gradient property does not depend on the particular choice of projection, but only on how $\mathcal{S}$ depends on A. This point is explored in detail in appendix A.

By projecting in the same way the Klein-Gordon equations and the Einstein equations which are derived from (4.7), (4.10) and (4.11) one obtains, as shown in appendix A,

$$\pi_p = \partial_p W + \frac{2(d-1)}{d\, W}\pi^s\left(\widetilde{\nabla}_s\pi_p - \widetilde{\nabla}_p\pi_s\right). \tag{4.21}$$

which are nothing but equations (2.9). The term in parenthesis in (4.21) is the curl of the vector field π^r, which we can compute from $\mathcal{S}(A,\phi)$ after solving (4.18). In other words, *we can map the principal function $\mathcal{S}(A,\phi)$ to the velocity field $\pi_r(\phi)$ it generates on field space and determine the curl of π^r.*

For the special case in which one wants to *classify solutions in terms of conserved charges*, equation (4.21) is necessary if working in field-space only. An example is provided by the principal function (4.16) which is associated with the conserved charge L. The symmetries of the problem imply that a solution $\mathcal{A}(\phi)$ to (4.18) can be taken to be independent of Θ. It is then possible to define $W_L(\phi)$, the function $W(\phi)$ associated with a given value of the conserved charge L from (4.17) and (4.19a).

$$W_L(\phi^\alpha) := \widetilde{W}_L(\phi^\alpha, \mathcal{A}(\phi^\alpha)), \qquad \alpha = 1,\ldots,N-1\,, \quad \phi^N = \Theta. \tag{4.22}$$

Once this is done, the component of the velocity field associated with the conserved charge, π_Θ, satisfies

$$\partial^\alpha W_L \partial_\alpha \pi_\Theta = \frac{d}{2(d-1)} W_L\; \pi_\Theta\;. \tag{4.23}$$

If W_L is non-zero, equation (4.23) implies that a non-vanishing π_Θ will necessarily have a ϕ^α dependence. We also know from (4.16) and the Θ-independence of $\mathcal{A}(\phi^\alpha)$ that the other components of the velocity field, π_α, are also Θ-independent. As a consequence, the velocity field will have a non-zero curl

$$\mathcal{F}_{\alpha\Theta} \equiv \partial_\alpha\pi_\Theta - \partial_\Theta\pi_\alpha = \partial_\alpha\pi_\Theta \neq 0. \tag{4.24}$$

The principal function (4.16) leads to a *non-gradient velocity field* over $\mathcal{M}_\phi$

It is important to remark that the conservation of angular momentum does not forbid the existence of factorisable principal functions, i.e. of the form (4.12). A factorisable solution corresponds in this case to a family of flows in which different flows have different angular momenta, while in the non-separable solutions of the form (4.16) all the flows have the same angular momentum L.

UV behaviour. It is instructive to make the above discussion explicit close to a UV fixed point holographic RG flows.

Consider the case of two scalar fields (ϕ_1,ϕ_2) with flat field-metric and a potential $V(R)$, which has a maximum at $R=0$ (which will serve as a UV fixed point), where

$$\begin{cases} \phi^1 = R\cos\Theta \\ \phi^2 = R\sin\Theta \end{cases} \tag{4.25}$$

The metric components are:

$$\mathcal{G}_{RR} = 1, \quad \mathcal{G}_{\Theta\Theta} = R^2, \quad \mathcal{G}_{R\Theta} = 0. \tag{4.26}$$

We are interested in constructing paths which define Archimedean spirals:

$$R = \alpha\,(\Theta - \Theta_0), \qquad \Theta_0 \in [0, 2\pi)\ . \tag{4.27}$$

Close to $R = 0$ the solution has the expansion (3.6) with $\Delta_1 = \Delta_2 > 0$ and $u \to -\infty$. If we now write the conserved angular momentum using (4.13), (4.25) and (4.26) we obtain:

$$L = \phi^1 p_2 - \phi^2 p_1 = e^{dA(u)} \left(\phi^1 \dot{\phi}^2 - \phi^2 \dot{\phi}^1 \right) \tag{4.28}$$

Using the asymptotic expansions (3.6) of ϕ^r and A we obtain

$$L = (2\Delta^+ - d) \left(\phi^1_- \phi^2_+ - \phi^2_- \phi^1_+ \right), \tag{4.29}$$

Therefore, fixing L places a constraint relating the sources ϕ^r_- and the VEVs ϕ^r_+ in the UV, in contrast with the usual procedure of keeping *only* the sources fixed.

Close to the UV we can also obtain an approximate expression for the non-separable Hamilton principal function with fixed L. The computation is presented in appendix D. The final result takes the form of an expansion in e^{-dA}, which tends to zero in the UV,

$$\mathcal{S}_L(A, R, \Theta) = -e^{dA} W_0(R) + L\Theta + e^{-dA} W_2(L, R) + \mathcal{O}\left(e^{-2dA}\right), \tag{4.30}$$

where $W(R)$ satisfies a radial superpotential equation which is *independent* of L. The last term is proportional to L^2 and its explicit form is given in equation (D.7).

Notice that L enters at sub-leading order in the expansion, and in the UV $\mathcal{S}$ is again separable. This is consistent with the fact that L is proportional to the VEV terms in the solution.

4.3 Back to holography: gauging global symmetries

From a gravitational perspective, the solutions in the previous sub-section coming from a non-gradient velocity field over the scalar manifold are perfectly acceptable. However, as we will explain below, in holography these solutions are unphysical, since all symmetries of the scalar manifolds must be gauged. As we will see, this implies that Poincaré-invariant solutions have necessarily zero charge.

It has been argued that quantum gravity does not allowed for exact global symmetries (see e.g. [40]). In the context of the AdS/CFT correspondence, this argument can be made precise. Indeed, suppose that we are in the presence of an isometry of $\mathcal{G}_{rs}(\phi)$ which leaves the potential $V(\phi)$ invariant, and suppose the UV is realized at an isolated maximum of the potential (say at $\phi_r = 0$). Then, the UV exremum must be mapped into itself under the action of the symmetry, meaning that the latter is also a global symmetry of the corresponding CFT, which then possesess the corresponding conserved currents among its operators. This in turn requires the existence of bulk gauge fields corresponding to these currents, and this translates into the requirement that any isometry of $\mathcal{G}_{rs}(\phi)$ which leaves the potential $V(\phi)$ invariant should be gauged.[11]

We proceed now to gauge the $O(2)$ isometry of a two-dimensional and flat scalar manifold. In cartesian coordinates the action of the $O(2)$ isometry on fields ϕ^r is linear and its infinitesimal form is given by:

$$\delta_\Theta \phi^s := \Theta \epsilon^{sp} \phi_p. \tag{4.31}$$

where, ϵ^{sp} is the Levi-Civita symbol. When the potential is a function only of $|\phi|$, the transformation (4.31) leaves the action (2.1) invariant. In order to gauge the transformation (4.31), we introduce the gauge field $A_a(x)$ which transforms as:

$$\delta_\Theta A_a = \partial_a \Theta \tag{4.32}$$

and is minimally coupled to ϕ. Our convention for the covariant derivative is:

$$D_a \phi^s(x) = \partial_a \phi^s(x) - A_a(x) \epsilon^{sp} \phi_p. \tag{4.33}$$

The associated field strength is denoted by F_{ab}, and the general form of the action is

$$S = M^{d-1} \int_{\mathcal{M}} \mathrm{d}^{d+1}x \sqrt{-g} \left[\frac{1}{2\kappa} R - \frac{1}{2} \mathcal{G}_{ps} D_a \phi^p D^a \phi^s - V(\phi^q) - \frac{Z(\phi)}{4} F_{ab} F^{ab} \right] + S_{GH} \,, \tag{4.34a}$$

where the function $Z(\phi)$ gives an extra coupling of the scalars to the gauge fields.

The equations of motion following from the action (4.34) are:

$$Z \nabla_b F^{ba} + F^{ba} \partial_b \phi^s \partial_s Z - \epsilon_{ps} \phi^p D^a \phi^s = 0 \,, \tag{4.35a}$$

$$\mathcal{D}_a (D^a \phi^s) - \mathcal{G}^{sp} \left(\partial_p V + \frac{1}{4} F_{ab} F^{ab} \partial_p Z \right) = 0 \,, \tag{4.35b}$$

$$\begin{aligned} R_{ab} - \frac{1}{2} g_{ab} R = \kappa \Big[&\mathcal{G}_{pq} \Big(D_a \phi^p D_b \phi^q - g_{ab} \frac{1}{2} D_c \phi^p D^c \phi^q \Big) \\ &+ g_{ab} V(\phi) + Z(\phi) \mathfrak{g}_{IJ} \Big(F^I_{ac} F^{J\cdot c}_{b\cdot} - \frac{g_{ab}}{4} F^I_{cd} F^{Jcd} \Big) \Big]. \end{aligned} \tag{4.35c}$$

where $\mathcal{D}_a$ is a gauge and field-space covariant derivative. The most general ansatz preserving d-dimensional Poincaré invariance is

$$\mathrm{d}s^2 = \mathrm{d}u^2 + e^{2A(u)} \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \,, \tag{4.36a}$$

$$\phi^s = \phi^s(u) \,, \tag{4.36b}$$

$$A_a(u) = \delta^u_a A_u(u) \,. \tag{4.36c}$$

[11]The above argument applies to *isolated* UV fixed points, but does not extend to the case in which there is a continous manifold of fixed points connected by exactly marginal directions, and we thank Thomas Van Riet for pointing this out to one of the authors. In such cases the symmetry is not an invariance of the boundary CFT, but rather a transformation connecting different inequivalent UV CFTs. From the bulk point of view, this is related to the breaking of the bulk symmetry by a choice of the UV boundary conditions. It is not obvious whether the global symmetry should be gauged in this case, from the supergravity perspective. In this work we will not consider the possibility of exactly marginal deformations, but we will assume UV fixed point to be isolated.

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The independent equations of motion (4.35) for the ansatz (4.36) are:

$$0 = \phi^p \epsilon_{ps} D_u \phi^s, \tag{4.37a}$$

$$0 = \left[\delta^p_q(\partial_u + d\dot{A}) - A_u \epsilon^p{}_q\right] D_u \phi^q + \widetilde{\Gamma}^p_{sq} D_u \phi^s D_u \phi^q - \mathcal{G}^{pq}\partial_q V, \tag{4.37b}$$

$$0 = \frac{d(d-1)}{2\kappa}\dot{A}^2 - \frac{1}{2}(\dot{\phi} - A_u \phi)^2 + V(\phi). \tag{4.37c}$$

Equations (4.37a) means that the angular component of the covariant derivative is zero for the pure gauge configuration (4.36c) and it can be rewritten as:

$$\dot{\Theta}(u) = A_u(u). \tag{4.38}$$

This implies that if we gauge the $O(2)$ isometry, any motion along the angular direction becomes a gauge artefact. Equivalently, we can go to the unitary gauge $A = 0$ and find that Gauss's law (4.37a) implies $L = 0$.

5 Conclusion

In this work we have performed a systematic analysis of Einstein gravity coupled to N scalar fields, using the first order formalism of flow equations. Although our results were mostly framed in the language of the gauge/gravity duality, they can also be used in the context of cosmology if one trades the holographic coordinate for time and flips a few signs [33, 38]

Our analysis shows that in holography, when bulk solutions have a maximum of the potential to connect to in the UV, one can always write the solution as gradient flows coming from a superpotential function. The latter may have branch points corresponding to bounces, when one or more of the coordinates on field space invert their flow direction. We have written the general solution close to an extremum of the bulk potential in terms a universal analytic superpotential plus and a set of continuous sub-leading deformations, which carry the integration constants which ultimately determine the fate of the solution in the infrared.

As in the single-field case, an appropriate regularity condition in the IR at minima of the scalar potential determines the full superpotential completely.

We have extended to the multi-field case the analysis of bouncing solutions, for which the superpotential becomes multi-branched. We have shown that complete bounces (all fields turning around) can occur on any equipotential hyper-surface on dimension ranging from zero to $N-1$, and lying on the critical curve $B(\phi) \propto \sqrt{-V(\phi)}$. Partial bounces can also occur away from the critical curve, when only a subset of the fields inverts its flow direction.

We have pointed out that one *may* choose not to work with gradient flows, for example if one wants to classify bulk solutions according to some conserved charges. Rather than the superpotential, one is then led to use a non-separable Hamilton principal function as a generating function of a first order flow in the *full* parameter space of the metric plus scalar fields. Although the case with global symmetries is not relevant for holography, since there all symmetries must necessarily be gauged, it can be useful to consider for cosmological applications, something that we think would be worth exploring further.

This work may be extended in several ways. In holography, as we have explained, all symmetries of the scalar manifold must be gauged, and in order for a solution with a non-zero charge one must turn on a gauge field. This generically breaks Poincaré invariance, which means a departure from a vacuum solution (e.g. by turning on a chemical potential). It would be desirable to have a general first order formalism in the presence of multiple scalars *and* gauge fields, as it would be of interest both for condensed matter applications and for holographic models of QCD.

Another way of obtaining non-vacuum solutions is by going to finite temperature and considering black hole solutions. In this context, already in the case of pure AdS gravity it was shown how to write the flow equations in terms of a non-separable Hamilton principal function [36]. It would be interesting to work out the extension of to single- or multi-field black hole solutions.

Finally, as we have mentioned above, many models of scalar field inflation have multiple fields, and are intrinsically not reducible to the single field case. Multi-field inflationary models were recently discussed in connection with holography in [41]. It would be interesting to investigate whether the formalism we have developed here can be used to describe near-de Sitter solutions and whether using non-gradient flows associated with conserved quantities can simplify the analysis of the space of solution.

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A Non-gradient flows from a non-separable principal function

In this appendix the main equations we will need will be (4.11)

$$e^{-dA}\left[\frac{1}{4d(d-1)}\left(\frac{\partial\mathcal{S}}{\partial A}\right)^2-\frac{1}{2}\mathcal{G}^{rs}\frac{\partial\mathcal{S}}{\partial\phi^r}\frac{\partial\mathcal{S}}{\partial\phi^s}\right]+e^{dA}V(\phi)=0\,, \tag{A.1}$$

together with the following combination of (4.7) and (4.10):

$$\dot{\phi}^r = -\,e^{-dA}\mathcal{G}^{rs}\frac{\partial\mathcal{S}}{\partial\phi^s}\,, \tag{A.2a}$$

$$\dot{A} = \frac{e^{-dA}}{2d(d-1)}\frac{\partial\mathcal{S}}{\partial A}\,. \tag{A.2b}$$

One simple equation which follows immediately from the HJ (A.1) and will used below can be derived by first multiplying the HJ equation by $\exp(dA)$ and differentiating the resulting expression with respect to A:

$$0 = \partial_A \left[\frac{\left(e^{-dA}\partial_A \mathcal{S}\right)^2}{4d(d-1)} - \frac{1}{2} e^{-2dA} \left(\partial^r \mathcal{S} \partial_r \mathcal{S}\right) \right] \tag{A.3}$$

The first equation we can derive from (A.1) and (A.2) and which will be useful below is (2.3c). We start by differentiating (A.2b) with respect to u

$$\begin{aligned} \ddot{A} &= \frac{1}{2d(d-1)} \left[\dot{A}\partial_A \left(e^{-dA} \frac{\partial \mathcal{S}}{\partial A} \right) + \dot{\phi}^r \partial_r \left(e^{-dA} \frac{\partial \mathcal{S}}{\partial A} \right) \right] \\ &= -\frac{e^{-2dA}}{2(d-1)} \left(\partial^r \mathcal{S} \partial_r \mathcal{S}\right) \\ &= -\frac{1}{2(d-1)} \mathcal{G}_{rs} \dot{\phi}^r \dot{\phi}^s \end{aligned} \tag{A.4}$$

where we used (A.3).

Assume we have one solution $\mathcal{S}(\phi, A)$ of equation (A.1). From (A.2a) it is clear that in general the flows are not given by functions of the scalar fields alone. However, it is possible to *project* the flows on field space by the following procedure. We start by deriving the key element of this procedure, the function $\mathcal{A}(\phi)$ defined over the scalar manifold such that it coincides with $A(u)$ on solutions:

$$A(u) = \mathcal{A}(\phi(u)) \tag{A.5}$$

The existence of $\mathcal{A}(\phi)$ on a neighbourhood of any point where $\mathcal{S}$ is locally single-valued is guaranteed provided that one can solve (A.2b) for $\mathcal{A}(\phi)$ satisfying (A.5):

$$e^{dA}\dot{A} = \mathcal{G}^{rs} \left[\partial_r S(\phi, A)\right]_{A=\mathcal{A}(\phi)} \partial_s \mathcal{A} = \frac{1}{2d(d-1)} \partial_A \mathcal{S}(\phi, A)\big|_{A=\mathcal{A}(\phi)} \tag{A.6}$$

The non-linear, first-order partial-differential equation on the scalar manifold (A.6) is expected to include an integration function of $N-1$ variables. Once a solution to (A.6) is given we can start locally projecting the solution associated with $\mathcal{S}(\phi, A)$ on the scalar manifold.[12] As there will be many projected quantities, we define the following simplified notation:

$$f(\phi, A)\big|_{A=\mathcal{A}(\phi)} \equiv f\big|_{\mathcal{A}} \tag{A.7}$$

We define the projected functions $W(\phi)$, $\widehat{W}(\phi)$ and the projected vector field $\pi_r(\phi)$ through:

$$W(\phi) := -\, d^{-1} e^{-d\mathcal{A}(\phi)} \left[\partial_A S(\phi, A)\right]_{\mathcal{A}} \;. \tag{A.8a}$$

$$\widehat{W}(\phi, A) := -\, e^{-dA} S(\phi, A)\,, \tag{A.8b}$$

$$\pi_r(\phi) := \left[\partial_r \widehat{W}\right]_{\mathcal{A}} = -\, e^{-d\mathcal{A}(\phi)} \left[\partial_r S(\phi, A)\right]_{\mathcal{A}} \;. \tag{A.8c}$$

[12] As a consequence of this integration function there are infinitely many solutions to (A.6), each one leading to a different projection. However the results in this appendix do not depend on the specific choice.

When equations (A.8) are combined with (A.2) we obtain:

$$\dot{\phi}^r = \mathcal{G}^{rs}\pi_s = \pi^r \,, \tag{A.9a}$$

$$\dot{A} = -\frac{W(\phi)}{2(d-1)} \,. \tag{A.9b}$$

The HJ equation (A.1) assumes the same form as (2.8) with π_r playing the role of π_r:

$$\frac{1}{2}\pi_r\pi^r - \frac{d}{4(d-1)}W^2 - V = 0, \tag{A.10}$$

A complete equivalence of a generic non-separable principal function S and of non-gradient flows can be explicitly shown by deriving equation (2.9) from the Hamilton-Jacobi formalism together with (A.5) and (A.6). We will now prove this equivalence. We start rewriting (A.4) using the definitions (A.8):

$$\pi^r\left(\pi_r - \partial_r W\right) = 0 \,. \tag{A.11}$$

In other words, we can write π_r as a sum of a gradient and a non-gradient part:

$$\pi_r = \partial_r W + \xi_r \quad , \quad \pi^r\xi_r = 0 \,. \tag{A.12}$$

With the definitions (A.8) we can explicitly write ξ in terms of $\widehat{W}$ by a direct comparison of π_r and $\partial_r W$. It is convenient to first write $W(\phi)$ in terms of $\widehat{W}$ by substituting the definition (A.8b) into (A.8a):

$$W(\phi) = \left[\widehat{W}(\phi, A) + d^{-1}\partial_A\widehat{W}(\phi, A)\right]_{\mathcal{A}} \tag{A.13}$$

By deriving the expression (A.13) with respect to ϕ^r we obtain:

$$\begin{aligned}
\partial_r W(\phi) &= \left[\partial_r\widehat{W}(\phi, A) + \partial_A\widehat{W}(\phi, A)\partial_r\mathcal{A} + d^{-1}\left(\partial_r\partial_A\widehat{W}(\phi, A) + \partial_A^2\widehat{W}(\phi, A)\partial_r\mathcal{A}\right)\right]_{\mathcal{A}} \\
&= \left[\partial_r\widehat{W}(\phi, A)\right]_{\mathcal{A}} + d^{-1}e^{-d\mathcal{A}}\partial_r\left[e^{dA}\partial_A\widehat{W}(\phi, A)\right]_{\mathcal{A}} && \text{(A.14)} \\
&= \pi_r + d^{-1}e^{-d\mathcal{A}}\partial_r\left[e^{dA}\partial_A\widehat{W}(\phi, A)\right]_{\mathcal{A}} && \text{(A.15)}
\end{aligned}$$

where in the last line we used the definition (A.8c) of π_r in terms of $\partial_r\widehat{W}(\phi, A)$. By comparing (A.12) and (A.15) we obtain the following two equivalent expressions for ξ:

$$\begin{aligned}
\xi_r &= -\frac{1}{d}e^{-d\mathcal{A}}\partial_r\left[e^{dA}\partial_A\widehat{W}(\phi, A)\right]_{\mathcal{A}} \\
&= -d^{-1}\left[\partial_r\partial_A\widehat{W}(\phi, A)\right]_{\mathcal{A}} - \left[\partial_A\widehat{W}(\phi, A) + d^{-1}\partial_A^2\widehat{W}(\phi, A)\right]_{\mathcal{A}}\partial_r\mathcal{A} && \text{(A.16)}
\end{aligned}$$

Contracting (A.16) with π^r gives back, after some algebra, equation (A.3), showing that ξ_r is indeed orthogonal to π^r. To compute the curl of π_r we notice that, from (A.12), it equals the curl of ξ_r and using (A.16), we obtain:

$$\partial_{[r}\xi_{s]} = \partial_{[r}W_{s]} = -\left[\partial_{[r}\partial_A\widehat{W}(\phi, A)\right]_{\mathcal{A}}\partial_{s]}\mathcal{A} = d\;\xi_{[r}\partial_{s]}\mathcal{A} \tag{A.17}$$

We can contract (A.17) with π^r and use (A.12) both to eliminate the term proportional to $\pi^r\xi_r$ and to rewrite the result solely in terms of π_r and $\mathcal{A}$. The results is the following expression:

$$\pi^r\left(\partial_r W_s - \partial_s W_r\right) = -d\ \left(\pi^r\partial_r\mathcal{A}\right)\xi_s = -d\ \left(\pi^r\partial_r\mathcal{A}\right)\left(W_s - \partial_s W\right) \tag{A.18}$$

Because $\mathcal{A}$ is a solution to equation (A.6), we can rewrite this equations in terms of the quantities defined in (A.8) leading to the following identity on field space:

$$\pi^r\partial_r\mathcal{A} = -\frac{1}{2(d-1)}W(\phi)\ . \tag{A.19}$$

Substituting (A.19) into (A.18) leads to the expression:

$$\pi^r\left(\partial_r W_s - \partial_s W_r\right) = \left(\frac{d}{2(d-1)}W\right)\left(W_s - \partial_s W\right) \tag{A.20}$$

When W is non-zero we can rearrange equation (A.20) in the following form:

$$\pi_s = \partial_s W + \frac{2(d-1)}{d\ W}\pi^r\left(\partial_r\pi_s - \partial_s\pi_r\right)\ . \tag{A.21}$$

Equation (A.21) is nothing but equation (2.9), proving that when a solution $\mathcal{S}$ to the HJ equations is non-separable and there is a function $\mathcal{A}(\phi)$ satisfying (A.5)–(A.6), the flows can be projected on field space and result in a non-gradient velocity field for ϕ^r.

The well known fact that when the principal function $\mathcal{S}$ factorises, i.e., when $\widehat{W}$ defined on (A.8b), is independent of A follows here from the first line of (A.16) where we see that ξ vanishes identically and from equation (A.12). Imposing that the flow is gradient, i.e., ξ vanishes identically, tells us that $\widehat{W}$ *can* depend on A but only in specific way that we show below.

A gradient flow given by $W(\phi)$ does not necessarily imply a separable principal function $\mathcal{S}(\phi, A)$. From now on *assume that we know $W(\phi)$ for a curl-free velocity field π^r but do not know $S(\phi, A)$ and we want to determine it.* Equation (A.21) imply immediately that π_r is the gradient of W, so the flows are gradient flows. From the definition of ξ in equation (A.12) we know that it must vanish and, using the first line of (A.16) for finite $\mathcal{A}$ this means:

$$0 = \partial_r\left[e^{dA}\partial_A\widehat{W}(\phi, A)\right]_{\mathcal{A}} \implies \left[e^{dA}\partial_A\widehat{W}(\phi, A)\right]_{\mathcal{A}} = C\ d \tag{A.22}$$

where the factor of d is chosen for later convenience and $\mathcal{A}$ is a solution of equation (A.19) which can be written for gradient flows as:

$$-2(d-1)\partial^r\log(W)\partial_r\mathcal{A} = 1. \tag{A.23}$$

We can write the most general solution to (A.23) in terms of an arbitrary vector field $\eta^r(\phi)$ which contains the integration function which specifies a given $\mathcal{A}(\phi)$ as follows:

$$\mathcal{A}(\phi) = \int (Y_r + \eta_r) d\phi^r , \tag{A.24a}$$

$$Y_r(\phi) \equiv -\frac{W}{2(d-1)} \frac{\partial_r W}{\partial_s W \partial^s W} , \tag{A.24b}$$

$$\partial_{[r} \left(Y_{s]} + \eta_{s]} \right) = 0 , \tag{A.24c}$$

$$Y^s \eta_s = 0 . \tag{A.24d}$$

Equation (A.24c) is integrability condition which is equivalent to the requirement of vanishing torsion in field space. Expression (A.24a) naturally incorporates the inhomogeneous solution to equation (A.23). Equation (A.24c) should be solved for η_s subject to the orthogonality constraint (A.24d) and a choice of a field η_s among all possible solutions corresponds to a choice of a homogeneous solution to (A.23).

Let $g(\phi, A)$ be *any* regular function of the scalar field and the scale factor and $\mathcal{A}(\phi)$ any function satisfying (A.24). Equation (A.22) implies, at most, that

$$\partial_A \widehat{W}(\phi, A) = e^{-dA} \left[C + g(\phi, \mathcal{A}(\phi)) - g(\phi, A) \right] d . \tag{A.25}$$

Integration of (A.25) demands the choice of an arbitrary function of the scalar fields, here called $f(\phi)$:

$$\widehat{W}(\phi, A) = - e^{-dA} \left[g(\phi, \mathcal{A}(\phi)) + C \right] - \int^A \mathrm{d}B e^{-dB} g(\phi, B)\ d\ + f(\phi). \tag{A.26}$$

Since we have started with the assumptions that we know π_r which is closed, i.e. $\partial_{[r} W_{s]}$ vanishes, and we know $W(\phi)$, we can use equation (A.13) to relate $W(\phi)$ and $f(\phi)$:

$$f(\phi) = W(\phi) + \int^{\mathcal{A}} \mathrm{d}B e^{-dB} g(\phi, B)\ d + e^{-d\mathcal{A}} g(\phi, \mathcal{A}) \tag{A.27}$$

Substitution of (A.27) into (A.26) yields:

$$\widehat{W}(\phi, A) = - e^{-dA} C + W(\phi) + \int_A^{\mathcal{A}} \mathrm{d}B e^{-dB} \left[g(\phi, B) - g(\phi, \mathcal{A}) \right]\ d. \tag{A.28}$$

Substituting (A.27) into (A.26) we obtain: having the general expression (A.26) we can express Hamilton's principal function in terms of the unknown functions $g(\phi, A)$ and $f(\phi)$ by multiplying (A.25) by $-\exp(dA)$, as prescribed by (A.8b).

$$\mathcal{S}(\phi, A) = C - e^{dA} W(\phi) + \int_A^{\mathcal{A}} \mathrm{d}B e^{d(A-B)} \left[g(\phi, B) - g(\phi, \mathcal{A}) \right]\ d. \tag{A.29}$$

It can be convenient to relate the function $f(\phi)$ to the function $W(\phi)$ defined in (A.8a).

For the special case in which $g(\phi, A)$ vanishes, substitution of (A.26) in equation (A.13) implies that $f(\phi)$ equals $W(\phi)$:

$$g(\phi, A) \implies \widehat{W}(\phi, A) = W(\phi) - C e^{-dA} . \tag{A.30}$$

The principal function can be reconstructed via (A.8b) and the result is:

$$\mathcal{S}(\phi, A) = -e^{dA}W(\phi) + C \sim -e^{dA}W(\phi) \ . \tag{A.31}$$

The last step means that a vanishing and a non-vanishing C are equivalent as $\mathcal{S}$ is defined up to an arbitrary additive constant. Therefore, the absence of a curl implies the existence of a $\mathcal{S}$ which factorises as in (A.28), as $\widehat{W}$ can be chosen from (A.27) with vanishing C, but other, non-separable principal functions exist which yield the same gradient flows.

B A non-gradient velocity field for two scalars: analytic example

We can build a non-gradient solution by first choosing the paths that the we want in such a way that the velocity field they generate has the desired profile. In this example we proceed with Consider two scalar fields on a flat field-space and we define polar coordinates on $\mathcal{M}_\phi$:

$$\begin{cases} \phi^1 = R\cos\Theta \\ \phi^2 = R\sin\Theta \end{cases} \tag{B.1}$$

so that the metric components are:

$$\mathcal{G}_{RR} = 1, \quad \mathcal{G}_{\Theta\Theta} = R^2, \quad \mathcal{G}_{R\Theta} = 0. \tag{B.2}$$

We are interested in constructing paths which define Archimedean spirals:

$$R = \alpha\ (\Theta - \Theta_0), \qquad \Theta_0 \in [0, 2\pi) \ . \tag{B.3}$$

Each angle in (B.3) defines a curve of the family, as shown in figure 8 for a particular choice of α. The corresponding velocity field can be written as:

$$\dot{R}(u) = W^R(R) \equiv R\ \gamma(R), \tag{B.4a}$$
$$\dot{\Theta}(u) = W^\Theta(R) := \alpha\ R\ \gamma(R) \ . \tag{B.4b}$$

The function $\gamma(R)$ encodes the rate of change of R through (B.4a) and the factor of R relating γ to W^R is chosen for later convenience in order to make the power series expansion of γ regular in R. The curl of this velocity field is

$$\mathcal{F}_{R\Theta} = \partial_R \pi_\Theta = \alpha\left(R^3\gamma(R)\right)' . \tag{B.5}$$

where we used the metric (B.2) to lower the Θ index. This velocity field must be accompanied by a function $W(\phi)$ which satisfies equation (2.9). We choose it to be rotationally invariant: $W = W(R)$. With this assumption and (B.5), equation (2.9) become the following pair of equations:

$$W_R = \partial_R W - \frac{2(d-1)}{d\ W} W^\Theta \alpha\left(R^3\gamma(R)\right)', \tag{B.6a}$$
$$\pi_\Theta = +\frac{2(d-1)}{d\ W} W^R \alpha\left(R^3\gamma(R)\right)' \tag{B.6b}$$

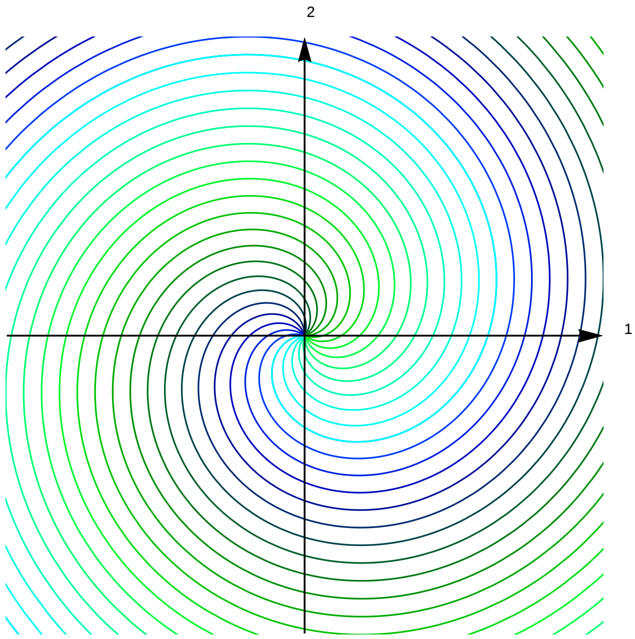

Figure 8. Archimedean spirals (B.3) with differerent values of Θ_0 for α given by (B.10). The corresponding velocity field (B.4) has a non-zero curl (B.5).

Equation (B.6b) can be seen as defining $W(R)$ in terms of the unknown function γ, as W^R and W^Θ are related to it via (B.4):

$$W(R) = +\frac{2(d-1)}{d\ R^2}\left(R^3\gamma(R)\right)' \tag{B.7}$$

With (B.7) providing the functional form of $W(R)$, equation (B.6a) provides a differential equation to be solved for $\gamma(R)$:

$$0 = \partial_R\left(\frac{\partial_R\left(R^3\gamma(R)\right)}{R^2}\right) - \frac{d}{2(d-1)}\left(1+\alpha^2R^2\right)R\gamma(R). \tag{B.8}$$

Among the two linearly independent solutions of equation (B.8) the one which provides a regular velocity field at the origin is

$$\begin{aligned}\gamma(R) =& \frac{\left(2\sqrt{[}4]2\Gamma\left(\frac{3}{2}-\frac{\alpha+\sqrt{\frac{d}{2d-2}}}{4\alpha}\right)\right)\left(e^{\frac{1}{2}\alpha\sqrt{\frac{d}{2d-2}}r^2}U\left(-\frac{\alpha+\sqrt{\frac{d}{2d-2}}}{4\alpha},-\frac{1}{2},-\sqrt{\frac{d}{2d-2}}r^2\alpha\right)\right)}{\sqrt{\pi}\left(\sqrt{[}4]2r^3\right)} \\ &- \frac{\sqrt{[}4]2\left(e^{\frac{1}{2}\alpha\sqrt{\frac{d}{2d-2}}r^2}L^{-\frac{3}{2}}_{\frac{\alpha+\sqrt{\frac{d}{2d-2}}}{4\alpha}}\left(\alpha\left(-\sqrt{\frac{d}{2d-2}}\right)r^2\right)\right)}{\left(\sqrt{[}4]2r^3\right)L^{-\frac{3}{2}}_{\frac{\alpha+\sqrt{\frac{d}{2d-2}}}{4\alpha}}(0)}\end{aligned} \tag{B.9}$$

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However, when α has the following value:

$$\alpha = -\sqrt{\frac{d}{2(d-1)}} \tag{B.10}$$

the regular solution (B.9) vanishes identically and the regular solution to (B.8) is given by a simple expression in integral form:

$$\gamma(R) := \frac{(d-1)}{\ell R^2} e^{\frac{d}{4(d-1)}R^2} \left(1 - \int_0^1 \exp\left(\frac{d\,(x^2-1)}{2(d-1)}R^2\right) dx\right) \tag{B.11}$$

For the choice (B.10), it is simple to substitute γ from (B.11) into (B.4) and (B.7) in order to obtain π^r and W, the later simplifying to:

$$W(R) = 2(d-1)\frac{1}{\ell}\exp(\frac{dR^2}{4(d-1)}) - R^2\gamma(R) \tag{B.12}$$

With these quantities, we can construct the potential $V(R)$ that has these flows as solutions, by the use of the algebraic equation (2.8). The result is:

$$V(R) = \frac{1}{2}\left(\gamma(R)R\right)^2 - \frac{d}{\ell}\left(\frac{(d-1)}{\ell} - \gamma(R)R^2\right)\exp(\frac{d}{2(d-1)}R^2) \tag{B.13}$$

where the function $\gamma(R)$ is given by (B.11). The first terms of the series expansion of the potential (B.13) around the origin are:

$$V(R) = -\frac{(d-1)d}{l^2} - \frac{d^2R^2}{9l^2} - \frac{7d^3R^4}{360\left((d-1)l^2\right)} + O\left(R^5\right) \tag{B.14}$$

Equation (2.11c) with W given by (B.12) leads to a scale factor that diverges as an asymptotically AdS warp factor as R tends to zero. Through the holographic dictionary this would naively means that the we should associate a UV fixed point at $R = 0$ for the flows solving (B.3). There is of a curl, as follows from (4.24), but in the R expansion the curl is sub-leading close to the origin. In cartesian coordinates:

$$\dot{\phi}^1(u) = \frac{d}{3}\phi^1(u) + \mathcal{O}(\phi)^2, \tag{B.15a}$$

$$\dot{\phi}^2(u) = \frac{d}{3}\phi^2(u) + \mathcal{O}(\phi)^2. \tag{B.15b}$$

With (B.11) the curl acquires a simpler form in terms of γ and its first derivative:

$$\mathcal{F}_{R\Theta} = -\sqrt{\frac{d}{2(d-1)}}R^2\left[3\gamma(R) + R\gamma'(R)\right]. \tag{B.16}$$

is therefore a sub-leading property of the flows close to the UV, showing that for these solutions the first term on the right-hand side of equation (4.21) dominates over the second. In other words, (4.21) mixes leading and sub-leading terms of the R expansion. We will see now that if we try to interpret this in holographic way, this mixing amounts to *fix a relation between sources and VEVs* at the UV and not by fixing the source.

C Local reconstruction of the superpotential

In this appendix we show how to locally reconstruct a superpotential, given a solution. The idea is that, around a generic point along a given flow $(A(u), \phi^r(u))$ we can make an appropriate coordinate transformation in field space from $\{\phi_r\} \to (\xi, \eta_\alpha)$ with only ξ changing along the flow and $N-1$ "spectator" coordinates η_α, which parametrise the directions orthogonal to the flow. In these coordinates one can reduce locally to a single-field flow, which therefore admits a local superpotential $W_{loc}(\xi)$, independent of η_α, for which the flow equation simply become

$$\dot{A} = -2(d-1)W_{loc}, \quad \dot{\xi} = W'_{loc}, \quad \dot{\eta_\alpha} = 0. \tag{C.1}$$

Below we show how this construction works explicitly at lowest order in the u-dependence of the solution.

We from a solution of equations (2.3), and we expand it in u around a point u_0 where the gradient vector $\dot{\phi}_r$ is non-vanishing. Without loss of generality we can set $u_0 = 0$ and $A(0) = \phi_r(0) = 0$ by coordinate transformations and field redefinitions. To lowest order around $u = 0$ we can write:

$$A(u) = A_1 u + \frac{1}{2}A_2 u^2 + O(u^3), \quad \phi^r(u) = \bar{\phi}^r u + O(u^2), \tag{C.2}$$

where A_1, A_2 and $\bar{\phi}^r$ are constants and by assumption not all the $\bar{\phi}^r$ vanish. We need to consider the term A_2 because it enters in equation (2.3c) at the same order as $\dot{\phi}^2$, which starts as a constant as $u \to 0$. Using equations (2.3b)–(2.3c) we can determine A_1 and A_2 to be

$$A_1 = -\frac{1}{2(d-1)}W_1, \qquad A_2 = -\frac{1}{2(d-1)}\mathcal{G}_{rs}\bar{\phi}^r\bar{\phi}^s \tag{C.3}$$

where W_1 is determined algebraically by the equation:

$$\frac{d}{4(d-1)}W_1^2 - \frac{1}{2}\mathcal{G}_{rs}\bar{\phi}^r\bar{\phi}^s = -V(0). \tag{C.4}$$

The velocities $\bar{\phi}^r$ are then determined by equation (2.3a) from the derivatives of the potential and of the metric.

We now define the new field variable ξ by

$$\xi = \left(\mathcal{G}_{rs}\phi^r\phi^s\right)^{1/2}. \tag{C.5}$$

Along the flow, around $u = 0$ it behaves as

$$\xi(u) = \bar{\xi}u + O(u^2), \quad \bar{\xi} \equiv \left(\mathcal{G}_{rs}\bar{\phi}^r\bar{\phi}^s\right)^{1/2} \neq 0. \tag{C.6}$$

We can in principle define a coordinate transformation around $\xi = 0$ by adding $N-1$ coordinates η_α orthogonal to the flow, which to this order will be independent of u along the solution.

We now *define* the local superpotential around the point $\xi = 0$ by:

$$W(\xi) = W_1 + \bar{\xi}\xi + O(\xi^2), \tag{C.7}$$

where W_1 is the same constant determined from equation (C.4). It is now straightforward to check that (C.7) serves, locally, as a superpotential for the flow close to $u = 0$,

$$\dot{\phi}^r = \mathcal{G}^{rs}\partial_s W_{loc}, \qquad \dot{A} = -2(d-1)W_{loc} \tag{C.8}$$

up to terms of order u. This construction works as along as $\bar{\xi} = 0$, i.e. if the starting point around which we expand is *not* a complete bounce.

D Non-separable solution with angular momentum

We consider the two-field case, with radial and angular variables R and Θ, diagonal and rotationally invariant metric $(\mathcal{G}_{RR}(R), \mathcal{G}_{\Theta\Theta}(R))$ and rotationally invariant potential $V = V(R)$. We start from HJ equation (4.11) and look for solution of the form (4.16),

$$\mathcal{S}(A, R, \Theta) = \mathcal{S}_0(A, R) + L\Theta, \tag{D.1}$$

for which equation (4.11) becomes,

$$e^{2dA}V(R) + \left[\frac{1}{4d(d-1)}\left(\frac{\partial \mathcal{S}_0}{\partial A}\right)^2 - \frac{1}{2}\mathcal{G}^{RR}\left(\frac{\partial \mathcal{S}_0}{\partial R}\right)^2 - \frac{L^2}{2}\mathcal{G}^{\Theta\Theta}\right]. \tag{D.2}$$

We now look for solutions in an expansion in large e^{dA},

$$\mathcal{S}_0(A, R) = -e^{dA}W_0(R) + W_1(R) + e^{-dA}W_2(R) + \ldots . \tag{D.3}$$

Inserting this ansatz in equation (D.2) we find:

- *Order e^{2dA}:*

  $$\frac{d}{4(d-1)}W_0^2 - \frac{1}{2}\mathcal{G}^{RR}\left(\frac{dW_0}{dR}\right)^2 + V = 0 \tag{D.4}$$

  i.e. the "reduced" superpotential equation with respect depending only on R;

- *Order e^{dA}:*

  $$\mathcal{G}^{RR}\frac{dW_0}{dR}\frac{dW_1}{dR} = 0 \tag{D.5}$$

  which implies constant $W_1 = C_1$;

- *Order e^0:*

  $$\frac{d}{2d(d-1)}W_0W_2 + \mathcal{G}^{RR}\frac{dW_0}{dR}\frac{dW_2}{dR} = \frac{L^2}{2}\mathcal{G}^{\Theta\Theta} \tag{D.6}$$

  This is the lowest order at which L enters in $\mathcal{S}_0$ (notice that it enters in $\mathcal{S}$ in equation (D.1) at one order higher through the $L\Theta$ term). The solution of equation (D.6) is

  $$W_2 = \exp\left(-\frac{d}{2(d-1)}\int\frac{W_0}{\mathcal{G}^{RR}W_0'}\right)\left[C_2 + L^2\int\frac{\mathcal{G}^{\Theta\Theta}}{\mathcal{G}^{RR}W_0'}\exp\left(\frac{d}{2(d-1)}\int\frac{W_0}{\mathcal{G}^{RR}W_0'}\right)\right] \tag{D.7}$$

  where C_2 is another integration constant.

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References

[1] J.M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [hep-th/9711200] [INSPIRE].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109] [INSPIRE].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150] [INSPIRE].

[4] H.J. Boonstra, K. Skenderis and P.K. Townsend, *The domain wall/QFT correspondence*, *JHEP* **01** (1999) 003 [hep-th/9807137] [INSPIRE].

[5] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *Novel local CFT and exact results on perturbations of $N = 4$ superYang-Mills from AdS dynamics*, *JHEP* **12** (1998) 022 [hep-th/9810126] [INSPIRE].

[6] V. Balasubramanian and P. Kraus, *Space-time and the holographic renormalization group*, *Phys. Rev. Lett.* **83** (1999) 3605 [hep-th/9903190] [INSPIRE].

[7] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Renormalization group flows from holography supersymmetry and a c theorem*, *Adv. Theor. Math. Phys.* **3** (1999) 363 [hep-th/9904017] [INSPIRE].

[8] J. de Boer, E.P. Verlinde and H.L. Verlinde, *On the holographic renormalization group*, *JHEP* **08** (2000) 003 [hep-th/9912012] [INSPIRE].

[9] M. Bianchi, D.Z. Freedman and K. Skenderis, *How to go with an RG flow*, *JHEP* **08** (2001) 041 [hep-th/0105276] [INSPIRE].

[10] S. de Haro, S.N. Solodukhin and K. Skenderis, *Holographic reconstruction of space-time and renormalization in the AdS/CFT correspondence*, *Commun. Math. Phys.* **217** (2001) 595 [hep-th/0002230] [INSPIRE].

[11] I. Papadimitriou and K. Skenderis, *Correlation functions in holographic RG flows*, *JHEP* **10** (2004) 075 [hep-th/0407071] [INSPIRE].

[12] D. Langlois, S. Renaux-Petel, D.A. Steer and T. Tanaka, *Primordial fluctuations and non-Gaussianities in multi-field DBI inflation*, *Phys. Rev. Lett.* **101** (2008) 061301 [arXiv:0804.3139] [INSPIRE].

[13] D. Langlois, S. Renaux-Petel, D.A. Steer and T. Tanaka, *Primordial perturbations and non-Gaussianities in DBI and general multi-field inflation*, *Phys. Rev.* **D 78** (2008) 063523 [arXiv:0806.0336] [INSPIRE].

[14] S. Renaux-Petel and K. Turzyński, *Geometrical destabilization of inflation*, *Phys. Rev. Lett.* **117** (2016) 141301 [arXiv:1510.01281] [INSPIRE].

[15] A. Celi et al., *On the fakeness of fake supergravity*, *Phys. Rev.* **D 71** (2005) 045009 [hep-th/0410126] [INSPIRE].

[16] K. Skenderis and P.K. Townsend, *Gravitational stability and renormalization group flow*, *Phys. Lett.* **B 468** (1999) 46 [hep-th/9909070] [INSPIRE].

JHEP07(2018)022

[17] P. Binetruy et al., *Universality classes for models of inflation*, *JCAP* **04** (2015) 033 [arXiv:1407.0820] [INSPIRE].

[18] I. Papadimitriou, *Multi-trace deformations in AdS/CFT: exploring the vacuum structure of the deformed CFT*, *JHEP* **05** (2007) 075 [hep-th/0703152] [INSPIRE].

[19] D. Martelli and A. Miemiec, *CFT/CFT interpolating RG flows and the holographic c function*, *JHEP* **04** (2002) 027 [hep-th/0112150] [INSPIRE].

[20] E. Kiritsis, W. Li and F. Nitti, *Holographic RG flow and the quantum effective action*, *Fortsch. Phys.* **62** (2014) 389 [arXiv:1401.0888] [INSPIRE].

[21] J. Bourdier and E. Kiritsis, *Holographic RG flows and nearly-marginal operators*, *Class. Quant. Grav.* **31** (2014) 035011 [arXiv:1310.0858] [INSPIRE].

[22] E. Kiritsis, F. Nitti and L. Silva Pimenta, *Exotic RG Flows from Holography*, *Fortsch. Phys.* **65** (2017) 1600120 [arXiv:1611.05493] [INSPIRE].

[23] U. Gürsoy, E. Kiritsis and F. Nitti, *Exploring improved holographic theories for QCD: Part II*, *JHEP* **02** (2008) 019 [arXiv:0707.1349] [INSPIRE].

[24] M. Bianchi, D.Z. Freedman and K. Skenderis, *Holographic renormalization*, *Nucl. Phys.* **B 631** (2002) 159 [hep-th/0112119] [INSPIRE].

[25] I. Papadimitriou, *Holographic renormalization of general dilaton-axion gravity*, *JHEP* **08** (2011) 119 [arXiv:1106.4826] [INSPIRE].

[26] J. Sonner and P.K. Townsend, *Axion-dilaton domain walls and fake supergravity*, *Class. Quant. Grav.* **24** (2007) 3479 [hep-th/0703276] [INSPIRE].

[27] U. Gürsoy, E. Kiritsis, L. Mazzanti and F. Nitti, *Holography and thermodynamics of 5D dilaton-gravity*, *JHEP* **05** (2009) 033 [arXiv:0812.0792] [INSPIRE].

[28] J.K. Ghosh, E. Kiritsis, F. Nitti and L.T. Witkowski, *Holographic RG flows on curved manifolds and quantum phase transitions*, arXiv:1711.08462 [INSPIRE].

[29] W. Boucher, *Positive energy without supersymmetry*, *Nucl. Phys.* **B 242** (1984) 282 [INSPIRE].

[30] P.K. Townsend, *Positive energy and the scalar potential in higher dimensional (super)gravity theories*, *Phys. Lett.* **B 148** (1984) 55.

[31] D.Z. Freedman, C. Núñez, M. Schnabl and K. Skenderis, *Fake supergravity and domain wall stability*, *Phys. Rev.* **D 69** (2004) 104027 [hep-th/0312055] [INSPIRE].

[32] I. Papadimitriou and K. Skenderis, *AdS/CFT correspondence and geometry*, *IRMA Lect. Math. Theor. Phys.* **8** (2005) 73 [hep-th/0404176] [INSPIRE].

[33] K. Skenderis and P.K. Townsend, *Hamilton-Jacobi method for curved domain walls and cosmologies*, *Phys. Rev.* **D 74** (2006) 125008 [hep-th/0609056] [INSPIRE].

[34] I. Papadimitriou, *Holographic renormalization as a canonical transformation*, *JHEP* **11** (2010) 014 [arXiv:1007.4592] [INSPIRE].

[35] I. Papadimitriou, *Lectures on holographic renormalization*, *Springer Proc. Phys.* **176** (2016) 131.

[36] J. Lindgren, I. Papadimitriou, A. Taliotis and J. Vanhoof, *Holographic Hall conductivities from dyonic backgrounds*, *JHEP* **07** (2015) 094 [arXiv:1505.04131] [INSPIRE].

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[37] D.S. Salopek and J.R. Bond, *Nonlinear evolution of long wavelength metric fluctuations in inflationary models*, *Phys. Rev.* **D 42** (1990) 3936 [INSPIRE].

[38] K. Skenderis and P.K. Townsend, *Hidden supersymmetry of domain walls and cosmologies*, *Phys. Rev. Lett.* **96** (2006) 191301 [hep-th/0602260] [INSPIRE].

[39] J. Diaz Dorronsoro, B. Truijen and T. Van Riet, *Comments on fake supersymmetry*, *Class. Quant. Grav.* **34** (2017) 095003 [arXiv:1606.07730] [INSPIRE].

[40] T. Banks and L.J. Dixon, *Constraints on string vacua with space-time supersymmetry*, *Nucl. Phys.* **B 307** (1988) 93 [INSPIRE].

[41] J. Garriga, K. Skenderis and Y. Urakawa, *Multi-field inflation from holography*, *JCAP* **01** (2015) 028 [arXiv:1410.3290] [INSPIRE].

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