It has also been found that the distribution of intensity found in certain celestial spectra can be approximately reproduced in the laboratory.

In any attempt to interpret the phenomena observed in connection with the Balmer series of hydrogen, it is necessary to know the particular type to which this series belongs. In order to decide this point a study has been made of the separations of the components of lines of the Balmer series of hydrogen, and the mean values of the separations of the doublets constituting the lines H_α and H_β have been found to be respectively 0·132 Å.U. and 0·033 Å.U. These values are consistent with the separations appropriate to a Principal series, and the first is in precise agreement with the value deduced by Buisson and Fabry.

These results have been obtained by crossing a Lummer Gehrcke plate with the neutral wedge, and submitting the contours obtained to mathematical analysis, by means of which the distribution of intensity in the individual components, and the separation of the components, can be determined.

On Multiple Integrals.

By Prof. W. H. Young, Sc.D., F.R.S.

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§ 1. In the modern theory of absolutely convergent integrals, as distinct from the older Riemann theory, the difference between multiple and repeated integration falls to the ground. Every multiple integral is equal to the corresponding repeated integrals, and the formulæ involving such multiple integrals, even when expressed without the repeated integral notation, can be obtained by means of the repeated integration process. To really grasp the distinction* between multiple and repeated integrals we have to take as independent variables functions of bounded variation; more precisely, we have to replace the ensemble of variables $(x, y, z, \ldots)$ by a single function of bounded variation, which, indeed, takes the place of the product of the variables $xyz\ldots$, the more general integration reducing to the ordinary when this product replaces the function in question.

I have already had occasion elsewhere to explain what is meant by a function of bounded variation of any number of variables, and to define

* In the French language there is said to be, as yet, no word expressing this distinction.

integration of any bounded function, and absolutely convergent integration of unbounded functions, with regard to such functions of bounded variation It will be sufficient here to call special attention to the fact that a function of bounded variation, for simplicity, let us say, of two variables, x and y, is expressible, to a value of the function *près*, as the difference of two functions, which are monotone increasing in x, in y, and in (x, y), representing the positive and negative variations in the plane, increased by the differences of the variations along the axis of x and along the axis of y respectively. There is, in consequence of the definition, a unique limit at each point for each of the four standard modes of approach, these four limits coinciding, except at points lying on a countable set of lines parallel to the axes at most. The modes of approach here considered correspond to the four quadrants when the point is the origin, the lines corresponding and parallel to the axes being excluded. The limits along these excluded lines may be different from one another and from the other four.*

Such functions, then, cannot in general be double integrals of ordinary functions of two variables, even if they are continuous. The simplest case, one which constantly occurs in practice, is that in which the functions are not integrals, solely because they are not continuous. This is the case, for instance, with the sum-function of the double Fourier series whose general term is $\sin mx \sin ny/mn$, which is discontinuous at the origin and along the axes, and is the simplest type of double integral elsewhere.

The step taken in extending the theory of integration to that in which the ordinary variables are replaced by one or more functions of bounded variation, is for this very reason a matter to which no one who has occasion to use mathematical analysis can be indifferent, though he may be quite unaware of the existence of continuous functions of bounded variation which are not integrals.

The original idea of introducing integration with respect to a monotone function of a single independent variable is due to Stieltjes, and was employed by him exclusively in the case of continuous integrands. Indeed, his definition fails in general with a less restricted hypothesis. On the other hand, Lebesgue has given a definition,† depending on change of the independent variable, for integration of any bounded function with respect to a continuous function of bounded variation. In my own theory, in which the treatment runs precisely parallel to that I have employed when the variable itself is the integrator (that is, the function with respect to which we integrate), the integrand and integrator are both perfectly general, the latter being any function of

* This could only be at the exceptional points already mentioned.

† Lebesgue's method fails when there are more independent variables than one.

bounded variation, not necessarily continuous, and the former any bounded or unbounded function to which the process is applicable. In the case in which the integrator is an absolutely convergent integral, we obtain ordinary Lebesgue integrals, and I have utilised this fact on more than one occasion to obtain new theorems in ordinary integration.

In a recent paper, I have thus obtained the generalisation for any number of variables of the second theorem of the mean in Bonnet's and other forms, as well as the formula for integration by parts in its most general form.

In such applications, certain fundamental theorems connecting ordinary integration with the more general type I have here in mind, are necessary. Moreover, in the development of the new theory itself, a new type of theorem is required. Whereas absolutely convergent multiple integration with respect to the product $xyz...$ is always expressible in the form of a repeated integral, this ceases in general to be the case when the integrator is no longer expressible as the product of factors, each involving a single variable only. It becomes important, therefore, to formulate tolerably wide conditions under which such a reduction of difficulty is possible.

In the present communication to the Society, I propose to give a brief account of some of the formulæ, which, from these two different, but closely allied points of view, are fundamental.

I begin by explicitly stating certain underlying facts on which the theory of integration with respect to a function of bounded variation in two or more variables is based.

I have not introduced any discussion of integration with respect to a continuous function which is not of bounded variation. Many of the formulæ might be deduced from those of the present paper in the case when the integrand is a function of bounded variation, but it seems advisable to reserve the discussion of this subject for a separate communication.

I have also thought it unnecessary to give the formulæ for functions of more variables than two.

§ 2. The function $g(x, y)$ with respect to which we integrate is supposed to be any function of bounded variation with respect to (x, y). Every such function is, however, expressible in the form

$$g(x, y) = g(0, 0) + \pi(x, y) - \nu(x, y),$$

where $\pi(x, y)$ is the sum of the positive variations of $g(x, y)$ with respect to (x, y), to x and to y respectively, each taken in the rectangle $(0, 0; x, y)$, the last two variations being taken along the axes of x and y, and $\nu(x, y)$ being similarly defined for the negative variations.

From the definition it follows at once that only $P(x, y)$ and $N(x, y)$, the positive variation and negative variation, taken positively, with respect to (x, y), contribute anything to the double integral with respect to $g(x, y)$, so that $g(x, y)$ is virtually expressible as the difference of $P(x, y)$ and $N(x, y)$, that is the difference of two functions, each of which is monotone increasing with respect to (x, y), to x and to y.

It will be convenient, therefore, to avoid repetition, to suppose that in the enunciation of our theorems $g(x, y)$, the function with respect to which we integrate (integrator) is itself such a function (monotonely monotone increasing function). It is clear from what has been said that there will be no loss of generality in adopting such a course, except in cases where some additional special hypothesis is made with respect to the integrator, which does not *ipso facto* hold for the functions of which $g(x, y)$ is in general the difference.

The condition that $g(x, y)$ is monotonely increasing with respect to (x, y) is expressed as follows:—

$$\Big[g(x, y)\Big]_{x,\, y}^{x+h,\, y+k} \equiv g(x+h, y+k) - g(x, y+k) - g(x+h, y) + g(x, y) \geq 0$$

$$(0 < h,\ 0 < k), \quad (1)$$

and may be expressed in words by saying that the plane increment round any rectangle whose sides are parallel to the axes is positive, or more precisely by saying that *the monotone increase with respect to either of the variables has an increasing rate as the other variable increases.*

From these properties it follows that $g(x, y)$ has a unique limit as we approach any particular point from one side or the other along the axes, or in any manner in each of the completely open quadrants. Denoting the former four limits by v_1, v_2, v_3, v_4, beginning with the horizontal right hand axis, and proceeding round clockwise, and the latter four by u_1, u_2, u_3, u_4, beginning with the first, or $(+, +)$-quadrant and proceeding clockwise, we have the inequalities

$$u_3 \quad v_2 \leq u_2 \leq v_1 \leq u_1; \qquad u_3 \;\; v_3 \leq u_4 \leq v_4 \leq u_1.$$

Hence also, remembering that $g(x, y)$ lies between v_2 and v_4 or v_1 and v_3, we can show that $g(x, y)$ *is continuous with respect to* (x, y), *except possibly at points lying on a countable set of parallels to the axes of* x *and* y.

§ 3. The last result enables us to prove, as in the theory of functions of a single real variable,* *mutatis mutandis*, that any simple l or u-function of (x, y) can be expressed as the limit of a monotone sequence of simple l or

* W. H. Young, "Integration with Respect to a Function of Bounded Variation," 'Lond. Math. Soc. Proc.,' Ser. 2, vol. 13, p. 139 (1913).

u-functions* none of whose discontinuities coincide with discontinuities of $g(x, y)$. We have accordingly, as in the original theory, three methods of starting:—

(1) *By defining the integral of a general simple l or u-function $f(x, y)$.* This will not be used in the present note.

(2) *By defining the integrals of special simple l and u-functions, whose discontinuities are not discontinuities of $g(x, y)$. The formula is*

$$\int_{0,0}^{a,b} f(x, y)\, dg(x, y) = \int \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} f_{i,j} \Big[g(x, y) \Big]_{c_i, k_j}^{c_{i+1}, k_{j+1}},$$

that is

$$\int_{0,0}^{a,b} f(x, y)\, dg(x, y) = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} f_{i,j} (g_{i,j} - g_{i,j+1} - g_{i+1,j} + g_{i+1,j+1}), \qquad (2)$$

where the fundamental rectangle $(0, 0;\ a, b)$ is supposed divided up by lines parallel to the axes, the feet of which on the x-axis have abscissæ $c_1 = 0, c_2, c_3, \ldots, c_m = a$, and on the y-axis the ordinates $k_1 = 0, k_2, k_3, \ldots, k_n = b$, in any convenient way such that all the discontinuities of $g(x, y)$ lie on the dividing lines, none of which coincide with any of the countable set of lines parallel to the axes on which the discontinuities of $g(x, y)$ lie; $f_{i,j}$ then denotes the constant value of $f(x, y)$ in the interior of the rectangle whose left-hand bottom corner is (c_i, k_j), and $g_{i,j}$ is the value of $g(x, y)$ at (c_i, k_j).

(3) *By defining the integrals of continuous functions.* The formula is

$$\int_{0,0}^{a,b} f(x, y)\, dg(x, y) = \text{Lt} \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} f(\text{P}_{i,j}) (g_{i,j} - g_{i,j+1} - g_{i+1,j} + g_{i+1,j+1}), \qquad (3)$$

where $\text{P}_{i,j}$ is any convenient point in the rectangle whose left-hand bottom corner is (c_i, k_j), and the rest of the notation has been already explained.

It may be shown that the integrals we obtain are the same, whichever of the three methods we adopt. The definition of the integral of a function which is the limit of a monotone sequence of functions whose integrals have been previously defined is that it is the limit of the integrals of the functions constituting the sequence. The validity of such a definition follows as in one dimension, and we have, as in one-dimensional theory, the theorem that the integral of $f(x, y)$ is at the same time the lower bound of the integrals of l-functions greater than $f(x, y)$ and the upper bound of the integrals of u-functions less than $f(x, y)$.

§ 4. We can now prove certain formulæ for our double integrals with respect to $g(x, y)$. The method of proof will in general consist in proving the

* Such functions are constant in the interior of a finite number of rectangles into which the region of integration is divided by lines parallel to the axis.

formulæ from first principles for one or other of our two standard forms (2) and (3) of the preceding article, and then deducing the result by the method of monotone sequences, either by using generalised induction, or by deducing the truth of the formula for l- and u-functions and applying the theorem mentioned at the conclusion of the preceding article.

We begin by establishing the following extremely important result:—

THEOREM.—*If $g(x, y)$ is an absolutely convergent integral with respect to (x, y), then*

$$\int_{0,0}^{a,b} f(x, y)\, dg(x, y) = \int_{0,0}^{a,b} f(x, y)\, \phi(x, y)\, d(xy), \tag{4}$$

where $\phi(x, y)$ is any function of which $g(x, y)$ is the integral, for example the function equal to the repeated differential coefficient of $g(x, y)$, wherever this differential coefficient exists, and equal to zero elsewhere.*

Since the rectangle $(0, 0\,;\, x, y)$ is the sum of the rectangles $(\lambda, \mu\,;\, x, y)$ and $(\lambda, \mu\,;\, 0, 0)$ minus the sum of the rectangles $(\lambda, \mu\,;\, 0, y)$ and $(\lambda, \mu\,;\, x, 0)$, it follows that

$$\int_{0,0}^{a,b} \phi(x, y)\, dxdy = \left[\int_{\lambda,\mu}^{x,y} \phi(x, y)\, dxdy\right]_{0,0}^{a,b} = \Big[g(x, y)\Big]_{0,0}^{a,b}.$$

If then $f(x, y)$ is a constant, say c, the left-hand side of (4) is by definition equal to $c\Big[g(x, y)\Big]_{0,0}^{a,b}$, while the right-hand side, being $c\int_{0,0}^{a,b}\phi(x,y)\, dxdy$, has as has just been pointed out, the same value. Thus the formula is true when f is a constant.

Again, if $f(x, y)$ is a simple l or u-function of the special type, its integral with respect to $g(x,y)$ is the sum of the finite number of terms, by the formula (2), each of which is the integral of a constant, and therefore, by what has been proved, is equal to the ordinary double integral of that constant multiplied by $\phi(x, y)$ over the corresponding rectangle. Summing all these terms, the equation (4) follows in this case.

But if the equation (4) holds for each of the members $f_1(x, y)$, $f_2(x, y)$,... of a monotone sequence, it holds for their limiting function $f(x, y)$, since by definition $\int_{0,0}^{a,b} f(x, y)\, dg(x, y)$ is the limit of $\int_{0,0}^{a,b} f_n(x, y)\, dg(x, y)$, and $\int_{0,0}^{a,b} f(x, y)\, \phi(x, y)\, dx\, dy$ is the limit of $\int_{0,0}^{a,b} f_n(x, y)\, \phi(x, y)\, dx\, dy$ by a known theorem.

It follows, therefore, that the theorem is true for general simple l and u-functions, as the limits of monotone sequences of these special functions,

* See a forthcoming paper by the author.

and is therefore true for general l and u-functions, as the limits of monotone sequences of simple l and u-functions.

Hence by generalised induction the theorem is true for all functions which can be obtained as the limits of monotone sequences of functions, starting with simple l and u-functions. This proves the truth of the formula (4).

Similarly we may prove the more general theorem:

THEOREM.—*If* $G(x, y)$ *is the integral of* $r(x, y)$ *with respect to* $s(x, y)$, *where* $s(x, y)$ *is a function of* (x, y) *of bounded variation, we have*

$$\int_{0,\,0}^{a,\,b} f(x, y)\, dG(x, y) = \int_{0,\,0}^{a,\,b} f(x, y)\, r(x, y)\, ds(x, y). \tag{5}$$

We have in fact only to start from the identity

$$\int_{0,\,0}^{a,\,b} r(x, y)\, ds(x, y) = \left[\int_{\lambda,\,\mu}^{a,\,b} r(x, y)\, ds(x, y)\right]_{0,\,0}^{a,\,b} = \left[G(x, y)\right]_{0,\,0}^{a,\,b},$$

and the argument is otherwise the same as before.

This general theorem, which I have not hitherto stated, even for functions of a single variable, is of very great use in practice, as it enables us to make in a very simple manner certain transformations in our integrals, which would otherwise scarcely occur to the worker.

§ 5. THEOREM.—*If the integrand is independent of one of the variables, the integration reduces to integration with respect to a function of the other variable, in accordance with the formula*

$$\int_{0,\,0}^{a,\,b} f(x)\, dg(x, y) = \int_0^a \left[f\, dg\right]_{y=0}^{b} = \int_0^a f(x)\, d\left(g(x, b) - g(x, 0)\right). \tag{6}$$

To prove this, let $f(x)$ be a continuous function. We then have, by (3),

$$\int_{0,\,0}^{a,\,b} f(x)\, dg(x, y) = \text{Lt} \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} f(c_i)\left(g_{i,\,j} - g_{i,\,j+1} - g_{i+1,\,j} + g_{i+1,\,j+1}\right)$$

$$= \text{Lt} \sum_{i=1}^{m-1} f(c_i)\left(g_{i,\,1} - g_{i,\,2} - g_{i,\,n} + g_{i+1,\,n}\right) = \int_0^a f(x)\left\{dg(x, b) - dg(x, 0)\right\}.$$

Thus the formula is verified in the case considered. Since, therefore, the left-hand and right-hand sides continually reproduce themselves, when we let $f(x)$ describe monotone sequences, first of continuous functions, then of l and u-functions, which are the limits of such sequences, and so on, it follows that the formula holds for all functions $f(x)$ mathematically definable.

This proves the formula, using generalised induction.

If, however, we prefer to use the theorem alluded to at the end of §3, we only use the above method to prove the formula for l and u-functions, and then remark that $\int_{0,\,0}^{a,\,b} f(x)\, dg(x, y)$ is the upper bound of the integrals of all

u-functions $a(x)$ less than $f(x)$, and $\int_0^a f(x)(dg(x, b) - dg(x, 0))$, being the integral of $f(x)$ with respect to $g(x, b) - g(x, 0)$, is the upper bound of the integrals of the functions $a(x)$ with respect to the same function; since the formula (4) holds for $a(x)$ in place of $f(x)$, it therefore holds as it stands.

This proves the formula also. Thus by either method the formula is verified by monotone sequences.

§6. THEOREM.—*If the integrator $g(x, y)$ contains a parameter t, the result of integration with respect to this parameter is given by the following formula:—*

$$\int_0^c dt \int_{0,\,0}^{a,\,b} f(x, y)\, dg(x, y, t) = \int_{0,\,0}^{a,\,b} f(x, y)\, d\left(\int_0^c g(x, y, t)\, dt\right), \tag{5}$$

provided only $g(x, y, t)$ possesses an absolutely convergent integral with respect to t.

Suppose first that $f(x, y)$ is* a simple l or u-function of the special type, whose discontinuities are different from any of those of $g(x, y, t)$. Then, denoting $\int_0^c g(x, y, t)$ by $G(x, y)$, we have by (2),

$$\int_0^c dt \int_{0,\,0}^{a,\,b} f(x, y)\, dg(x, y, t) = \int_0^c dt \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} f_{i,j}(g_{i,j} - g_{i,j+1} - g_{i+1,j} + g_{i+1,j+1})$$

$$= \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} f_{i,j}(G_{i,j} - G_{i,j+1} - G_{i+1,j} + G_{i+1,j+1})$$

$$= \int_{0,\,0}^{a,\,b} f(x, y)\, dG(x, y),$$

since $G(x, y)$ is easily seen to be a simple function of the same special type as $g(x, y)$. This proves the formula in this case.

But if $f_1(x, y), f_2(x, y), \ldots$, are functions forming a monotone sequence with $f(x, y)$ as limit, we may integrate this sequence term-by-term with respect to the function $\int_0^c g(x, y, t)\, dt$, so that

$$\int_{0,\,0}^{a,\,b} f(x, y)\, dG(x, y) = \operatorname*{Lt}_{n\to\infty} \int_{0,\,0}^{a,\,b} f_n(x, y)\, dG(x, y).$$

And also similarly

$$\int_{0,\,0}^{a,\,b} f(x, y)\, dg(x, y, t) = \operatorname*{Lt}_{n\to\infty} \int_{0,\,0}^{a,\,b} f_n(x, y)\, dg(x, y, t).$$

Denoting the function of t represented by either side of the last equation by $q(t)$, and the function whose limit is taken on the right by $q_n(t)$, $q_n(t)$

* It is convenient here as elsewhere to suppose f positive in the proof. This is no restriction, as we may always break up f into the difference of two such functions.

increases or decreases with n, according as $f_n(x, y)$ does so, and therefore also describes a monotone sequence with $q(t)$ as limit. Therefore, as before, we may integrate term-by-term and write

$$\int_0^c q(t)\,dt = \underset{n\to\infty}{\mathrm{Lt}} \int_0^c q_n(t)\,dt.$$

Now, supposing the equation (5) to hold for each of the functions $f_n(x, y)$, we have, with our present notation,

$$\int_0^c q_n(t)\,dt = \int_{0,\,0}^{a,\,b} f_n(x, y)\,d\mathrm{G}(x, y).$$

Proceeding to the limit with n, and using the results just obtained, we get therefore

$$\int_0^c q(t)\,dt = \int_{0,\,0}^{a,\,b} f(x, y)\,d\mathrm{G}(x, y),$$

which is identical with (5).

If therefore the equation (5) holds for all the members of any monotone sequence of functions, it holds for the limiting function. Thus, as before, the theorem is true, since it has been shown to hold for simple l- and u-functions of special type.

THEOREM.—*If the integrand contains a parameter t, we may integrate with respect to t inside the integral, that is*

$$\int_0^c dt \int_{0,\,0}^{a,\,b} f(x, y, t)\,dg(x, y) = \int_{0,\,0}^{a,\,b} \left(\int_0^c f(x, y, t)\,dt\right) dg(x, y).$$

The proof of this theorem is *mutatis mutandis* the same as that in the ordinary theory of absolutely convergent integrals.

§ 7. We have already given in § 5 the simplest case in which integration with respect to a function of bounded variation reduces to ordinary integration. We now prove the following further theorems:—

THEOREM.—*If* $\mathrm{F}(x, y)$ *is an integral* with respect to* x, *then*

$$\int_{0,\,0}^{a,\,b} \mathrm{F}(x, y)\,dg(x, y) = \int_{y=0}^{b} \Big[\mathrm{F}(x, y)\,dg(x, y)\Big]_{x=0}^{a} - \int^{a} dx \int_{y=0}^{b} \frac{d\mathrm{F}}{dx} dg(x, y), \quad (6)$$

where $d\mathrm{F}/dx$ is any one of the derivates of $\mathrm{F}(x, y)$ with respect to x. In fact, denoting by $f(x, y)$ the function which is equal to the differential coefficient with respect to x of $\mathrm{F}(x, y)$ wherever this exists and is finite, and is zero, or has any other convenient values, at the remaining set of values of x of content zero, we have

$$\mathrm{F}(x\ \ y) = \int_0^x f(t, y)\,dt + \mathrm{F}(0, y).$$

* Absolutely convergent, or Lebesgue, integral.

Now by the formula (4),

$$\int_{0,0}^{a,b} F(0, y)\, dg(x, y) = \int_0^b F(0, y)\,(dg(a, y) - dg(0, y)),$$

and $$\int_0^a dt \int_{t,0}^{a,b} f(t, y)\, dg(x, y) = \int_0^a dt \int_{y=0}^{b} f(t, y)\,(dg(a, y) - g(t, y)).$$

But change of order of integration is allowable on each side of the last equation*; thus

$$\int_{0,0}^{a,b} dg(x, y) \int_0^x f(t, y)\, dt = \int_{y=0}^{b} dg(a, y) \int_0^a f(t, y)\, dt - \int_0^a dt \int_{y=0}^{b} f(t, y)\, dg(t, y),$$

which may be written

$$\int_{0,0}^{a,b} (F(x, y) - F(0, y))\, dg(x, y)$$
$$= \int_0^b \{F(a, y) - F(0, y)\}\, dg(a, y) - \int_0^a dx \int_{y=0}^{b} f(x, y)\, dg(x, y).$$

Using the expression already given for the integral of $F(0, y)$, this gives

$$\int_{0,0}^{a,b} F(x, y)\, dg(x, y)$$
$$= \int_0^b \{(F(a, y)\, dg(a, y) - F(0, y)\, dg(0, y)\} - \int_0^a dx \int_0^b f(x, y)\, dg(x, y),$$

which is identical with the formula to be proved.

In the theorem just proved the integration is reduced to repeated integration first with respect to a function of bounded variation of one variable, and then with respect to the other variable. From which we get

THEOREM.—*If* $F(x, y)$ *is an integral with respect to* x, *and* $G(x, y)$ *is an integral with respect to* y, *and a function of bounded variation with respect to* (x, y), *then*

$$\int_{0,0}^{a,b} F(x, y)\, dG(x, y) = \int_0^b \left[F\frac{dG}{dy}\right]_0^a dy - \int_0^a dx \int_0^b \frac{dF}{dx}\frac{dG}{dy}\, dy.$$

Here we have been able to go still farther; all the integrations employed are ordinary.

Again, without any assumption as to the integrator, we can reduce to ordinary integration when the integrand is a double integral. The theorem, which is an immediate consequence of (4) when the formula for integration by parts for multiple integrals is employed, is as follows:—

* "Integration with respect to a Function of Bounded Variation," § 32, p. 148.

THEOREM.—*If $f(x, y)$ is a double integral with respect to (x, y), then*

$$\int_{0,\,0}^{a,\,b} f(x, y)\, dg(x, y)$$

$$= \Big[fg\Big]_{0,\,0}^{a,\,b} - \int_{x=0}^{a} \left[\frac{df}{dx} g\right]_{y=0}^{b} dx - \int_{y=0}^{b} \left[\frac{df}{dy} g\right]_{x=0}^{a} dy + \int_{0,\,0}^{a,\,b} \frac{d^2 f}{dx\,dy}\, g\, dx\, dy. \quad (8)$$

This theorem also follows from (6) by the formula for integration by parts for integration of functions of a single variable.

§ 8. We shall now find it convenient to prove the following property of our integrator $g(x, y)$. This is that *the derivates of $g(x, y)$ with respect to either variable are monotone ascending functions of the other variable.*

In fact the relation (1) gives

$$\frac{g(x+h, y+k) - g(x+h, y)}{k} \geq \frac{g(x, y+k) - g(x, y)}{k} \; (0 < h;\ 0 < k).$$

Let k describe such a sequence of positive values with zero as limit that the right-hand side approaches its upper limit, the upper right-hand derivate of $g(x, y)$ with respect to y, say $g_{(y)}{}^{+}(x, y)$. Then the left-hand side approaches some limit or limits which are not greater than $g_{(y)}{}^{+}(x+h, y)$. Thus

$$g_{(y)}{}^{+}(x+h, y) \geq g_{(y)}{}^{+}(x, y).$$

Similarly, choosing the sequence of values of k so that the left-hand side approaches its lower limit $g_{+}{}^{(y)}(x+h, y)$, the lower right-hand derivate of g with respect to y, we get

$$g_{+}{}^{(y)}(x+h, y) \geq g_{+}{}^{(y)}(x, y).$$

This shows that both the right-hand derivates of $g(x, y)$ with respect to y are monotone ascending functions of x. Similarly, taking h positive and k negative, in which case the symbol $\geq$ in (1) must, of course, be changed to $\leq$, in order that the plane increment may begin with the right-hand top corner, and proceed clockwise, we find that both the left-hand derivates of $g(x, y)$ with respect to y are monotone ascending functions of x.

By symmetry the same result holds when x and y are interchanged. This, therefore, proves the required result.

§ 9. If $g(x, y)$, in addition to having the property of being what we have called *a monotonely monotone function of* (x, y), (§ 2), is an integral with respect to x, it is the integral of any one of its derivates with respect to x. These derivates agree and are finite except at a set of content zero of values of x *for each fixed value of* y and, therefore, agree and are finite except at a set of values of (x, y) of plane content zero. Hence they agree and are finite except at a set of content zero of values of y *for each fixed value of* x not belonging to a certain set of content zero of values of x. Putting aside this exceptional set of values of x, therefore, it follows from §8 that, x being constant, dg/dx exists and is finite except at a set of content zero of values of y, and *the values at the exceptional points may be so assigned that dg/dx becomes a monotone ascending function of y, defined for all values of y.** With this understanding therefore we may, when $g(x, y)$ is an integral with respect to x, not only write $g(x, y) = \int (dg/dx)\,dx$, but also we may integrate a function with respect to dg/dx, x being constant, and then integrate the result with respect to x, the exceptional set of content zero of values of x at which the inside integral is undefined not affecting the second integration.

With this understanding, we have the following theorem on the reduction to a repeated integral :—

THEOREM.—*If $g(x, y)$ is a monotonely monotone ascending function of (x, y) and an integral with respect to x, and therefore an integral with respect to x of a monotone increasing function of y, we may write*

$$\int_{0,\,0}^{a,\,b} f(x, y)\,dg(x, y) = \int_0^a dx \int_0^b f(x, y)\,d\,(dg/dx). \tag{7}$$

If $f(x, y)$ is a constant, say c, the theorem is at once seen to be true, for, by definition, the left-hand side of the equation is

$$c\,(g(a, b) - g(a, 0) - g(0, b) + g(0, 0)),$$

and the right-hand side is the same, since, performing the inside integration, we get, writing $g'(x, y)$ for dg/dx,

$$\int_0^a c\,(g'(x, b) - g'(x, 0))\,dx.$$

Hence, by the formula (2), the theorem holds when $f(x, y)$ is a simple l- or u-function of the special type, for the integral is then a double summation of a finite number of such integrals with constant integrands.

But any simple l- or u-functions can be expressed as the limit of a monotone sequence of simple function of the special types just contemplated, say,

$$f_1(x, y) \leq f_2(x, y) \leq \ldots\ldots \to f(x, y),$$

and we may integrate this sequence term-by-term either with respect to $g(x, y)$, or first with respect to $g'(x, y)$, x being constant, and then with respect to x.

Thus $$\int_{0,\,0}^{a,\,b} f(x, y)\,dg(x, y) = \mathop{\mathrm{Lt}}_{n\to\infty} \int_{0,\,0}^{a,\,b} f_n(x, y)\,dg(x, y),$$

and also $$\int_0^a dx \int_0^b f(x, y)\,dg'(x, y) = \mathop{\mathrm{Lt}}_{n\to\infty} \int_0^a dx \int_0^b f_n(x, y)\,dg'(x, y).$$

* This mode of regarding the matter is sufficient for the purpose of justifying the notation dg/dx in equation (7). For fuller information with regard to the derivates and repeated derivates of functions of bounded variation, see the forthcoming paper referred to in §4.

But, by what we have already proved,

$$\int_{0,}^{a,\,b} f_n(x, y)\,dg(x, y) = \int_0^a dx \int_0^b f_n(x, y)\,dg'(x, y).$$

From these three relations the required relation immediately follows.

Similarly, it now follows generally for any l- and u-functions, these being the limits of monotone sequences of simple l- and u- functions.

But this proves the theorem generally by generalised induction, or, if we prefer, as follows: the left-hand side of (7) is the upper bound of $\int_{0,\,0}^{a,\,b} \phi(x, y)\,dg(x, y)$, where $\phi(x, y)$ is any u-function less than $f(x, y)$; and the right-hand side of (7) is the integral with respect to x of the upper bound of $\int_0^b \phi(x, y)\,dg'(x, y)$, which is the same as the upper bound of $\int_0^a dx \int_0^b \phi(x, y)\,dg'(x, y)$.

For $\int_0^b \phi\,dg' \leq \int_0^a f\,dg'$, so that integrating, with respect to x, and taking the upper bound, the upper bound of $\int_0^a dx \int_0^b \phi\,dg' \leq \int_0^a dx \int_0^b f\,dg'$. But we can find a ϕ for which $\int_0^b \phi\,dg'$ differs by as little as we please from $\int_0^b f\,dg'$, so that integrating with respect to x and taking the upper bound, we see that the upper bound of $\int_0^a dx \int_0^b \phi\,dg' \geq \int_0^a dx \int_0^b f\,dg' - ea$, where e is as small as we please. From this the truth of the statement made at once follows.

But by what has been proved

$$\int_{0,\,0}^{a,\,b} \phi(x, y)\,dg(x, y) = \int_0^a dx \int_0^b \phi(x, y)\,dg'(x, y),$$

so that the upper bounds of the two sides of this equation are also equal. This gives us the equation (7), and proves the theorem.

§ 10. From the theorem of the preceding article, we at once deduce the following:—

THEOREM.—*If $g(x, y)$ is a monotonely monotone ascending function of (x, y), and an integral with respect to x, and therefore an integral with respect to x of a monotone increasing function of y, we have*

$$\frac{d}{dx}\int_{0,\,0}^{x,\,y} f(u, v)\,dg(u, v) = \int_0^y f(x, v)\,d\left(\frac{dg(x, v)}{dx}\right), \tag{8}$$

provided the right-hand side is a continuous function of x.

§ 11. When the integrator, instead of the integrand, describes a sequence, the mere monotony of the sequence is not enough to ensure that the limiting process is allowable, that is, that the integral with respect to the limit is the same as the limit of the integral.

In this connection the following theorem is of importance :—

THEOREM.—*If $g(x, y, z)$ is a function of (x, y), which is monotonely increasing with respect to x, to y, and to (x, y), and has its triple increment always of the same sign, while it is monotone with respect to z, then*

$$\operatorname*{Lt}_{z \to c+0} \int_{0,0}^{a,b} f(x, y)\, dg(x, y, z) = \int_{0,0}^{a,b} f(x, y)\, dg(x, y, c+0). \tag{9}$$

From the hypothesis in the enunciation it follows that

$$g(x, y, c+k) - g(x, y, c+0), \qquad (0 < k),$$

is monotonely monotone increasing with respect to (x, y), supposing the triple increment of g to be positive. For the function here considered is the limit, as $e \to 0$ through positive values, of

$$g(x, y, c+k) - g(x, y, c+e), \qquad (e < k),$$

which is monotonely monotone increasing with respect to (x, y).

Hence if $f(x, y)$ is a bounded function of (x, y), and is numerically $\leq$ M,

$$\left| \int_{0,0}^{a,b} f(x, y)\, d\big(g(x, y, c+k) - g(x, y, c+0)\big) \right|$$
$$\leq M \Big[g(x, y, c+k) - g(x, y, c+0) \Big]_{0,0}^{a,b},$$

and hence has the unique limit zero as $k \to 0$. This proves the theorem when $f(x, y)$ is bounded.

When $f(x, y)$ is not bounded, we denote by $f_m(x, y)$ the function which is equal to f, where this is less than m, and is elsewhere equal to m (f being as usual taken to be positive). Making then $m \to \infty$, we see, by monotone sequences, that the theorem is true.

As a particular case of the theorem just proved, we have the following :—

THEOREM.—*If $g_n(x, y)$ is a monotonely monotone increasing function of (x, y), and has, as $n \to \infty$, $g(x, y)$ as limit, and generates such a monotone increasing or decreasing sequence that its double increment always increases, or always decreases, with n, then*

$$\operatorname*{Lt}_{n \to \infty} \int_{0,0}^{a,b} f(x, y)\, dg_n(x, y) = \int_{0,0}^{a,b} f(x, y)\, dg(x, y).* \tag{10}$$

* This theorem may be utilised, among other ways, in extending the theorem of Integration by Parts.