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*Research Article*

# On Multiple Interpolation Functions of the Nörlund-Type $q$-Euler Polynomials

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In (2006) and (2009), Kim defined new generating functions of the Genocchi, Nörlund-type $q$-Euler polynomials and their interpolation functions. In this paper, we give another definition of the multiple Hurwitz type $q$-zeta function. This function interpolates Nörlund-type $q$-Euler polynomials at negative integers. We also give some identities related to these polynomials and functions. Furthermore, we give some remarks about approximations of Bernoulli and Euler polynomials.



## 1. Introduction, Definitions and Notations

The classical Euler numbers and polynomials have been studied by many mathematicians, which are defined as follows, respectively,

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},\quad |t|<\pi, \tag{1.1}$$

$$\frac{2e^{tx}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad |t|<\pi, \tag{1.2}$$

cf. [1–51]. Observe that $E_n(0)=E_n$.

These numbers and polynomials are interpolates by the Euler zeta function and Hurwitz-type-zeta functions, respectively,

$$\zeta_E(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \quad s \in \mathbb{C}, \tag{1.3}$$

$$\zeta_E(s,x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}, \quad s \in \mathbb{C}. \tag{1.4}$$

Let $[x] = [x : q] = 1 - q^x/1 - q$. Observe that $\lim_{q \to 1}[x] = x$, cf. [3, 47].

Various kinds of the $q$-analogue of the Euler numbers and polynomials, recently, have been studied by many mathematicians. In this paper, we use Kim's [13, 21] and Simsek's [43] methods. By using $p$-adic $q$-Volkenborn integral [12], Kim [13, 26] constructed many kinds of generating functions of the $q$-Euler numbers and polynomials and their interpolation functions. He also gave many applications of these numbers and functions. Simsek [40, 43] studied on the generating functions of the Euler numbers and Bernoulli numbers. By using these generating functions, Simsek constructed $q$-Dedekind-type sums and $q$-Hardy-type sums as well.

Recently, Cangul et al. gave higher-order $q$-Genocchi numbers and their interpolation functions. Applying $p$-adic $q$-fermionic integral on $p$-adic integers, they also gave Witt's formula of these numbers.

In [21], by using multivariate fermionic $p$-adic integral on $\mathbb{Z}_p$, Kim constructed generating function of the Nörlund-type $q$-Euler polynomials of higher order. Main motivation of this paper is to construct interpolation function of the Nörlund-type $q$-Euler polynomials. Therefore, we firstly give generating function of the Nörlund-type $q$-Euler polynomials.

Kim [21] defined Nörlund type $q$-extension Euler polynomials of higher order. He gave many applications and interesting identities. We give some of them in what follows.

Let $q \in \mathbb{C}$ with $|q| < 1$:

$$F_q(t,x) = 2\sum_{m=0}^{\infty} (-1)^m e^{[m+x]t} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{1.5}$$

Observe that $F_q(t) = F_q(t,0)$. Hence, we have $E_{n,q}(0) = E_{n,q}$. If $q \to 1$ into (1.5), then we easily obtain (1.2).

Higher-order $q$-Euler polynomials of the Nörlund type are defined by Kim [21]. He gave generating functions related to Euler numbers of higher-order. In this paper, we use generating functions in [21]. Especially, we can use the following generating function, which are proved by Kim [21, Theorem 2.3, page 5].

**Theorem 1.1** ([21, Theorem 2.3, page 5]). *For $r \in N$, and $n \geq 0$, one has*

$$F_q^{(r)}(t,x) = 2^r \sum_{m=0}^{\infty} (-1)^m \binom{m+r-1}{m} e^{[m+x]t} = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{t^n}{n!}. \tag{1.6}$$

*It is noted that if $r = 1$, then* (1.6) *reduces to* (1.5).

*Remark 1.2.* In (1.6); we easily see that

$$\begin{aligned}
\lim_{q\to 1} F_q^{(r)}(t,x) &= 2^r \sum_{m=0}^{\infty} (-1)^m \binom{m+r-1}{m} e^{(m+x)t} \\
&= 2^r e^{xt} \sum_{m=0}^{\infty} (-1)^m \binom{m+r-1}{m} e^{mt} \\
&= \frac{2^r e^{xt}}{(1+e^t)^r} \\
&= F^{(r)}(t,x).
\end{aligned} \tag{1.7}$$

From the above, we obtain generating function of the Nörlund Euler numbers of higher order. That is

$$F^{(r)}(t,x) = \frac{2^r e^{xt}}{(1+e^t)^r} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}. \tag{1.8}$$

Thus, we have

$$\lim_{q\to 1} E_{n,q}^{(r)}(x) = E_n^{(r)}(x). \tag{1.9}$$

cf.[21].

Hence, we have

$$\begin{aligned}
F^{(r)}(t,x) &= \left(\frac{2}{e^t+1}\right)\left(\frac{2}{e^t+1}\right)\cdots\left(\frac{2}{e^t+1}\right)e^{tx} \\
&= 2^r e^{tx} \sum_{n_1=0}^{\infty} e^{tn_1}(-1)^{n_1} \sum_{n_2=0}^{\infty} e^{tn_2}(-1)^{n_2} \cdots \sum_{n_r=0}^{\infty} e^{tn_r}(-1)^{n_r} \\
&= 2^r e^{tx} \sum_{n_1,n_2,\ldots,n_r=0}^{\infty} (-1)^{n_1+n_2+\cdots+n_r} e^{t(n_1+n_2+\cdots+n_r)} \\
&= \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}.
\end{aligned} \tag{1.10}$$

We now summarize the results of this paper.

In Section 2, we study on modified generating functions of higher-order Nörlund-type $q$-Euler polynomials and numbers. We obtain some relations related to these numbers and polynomials.

In Section 3, we give interpolation functions of the higher order Nörlund-type $q$-Euler polynomials.

In Section 4, we obtain some relations related to he higher order Nörlund-type $q$-Euler polynomials.

In Section 5, we give remarks and observations on an Approximation theory related to Bernoulli and Euler polynomials.

## 2. Modified Generating Functions of Higher-Order Nörlund-Type $q$-Euler Polynomials and Numbers

In this section we define generating function of modified higher order Nörlund type $q$-Euler polynomials and numbers, which are denoted by $\mathbf{E}_{n,q}^{(r)}(x)$, and $E_{n,q}^{(r)}$ respectively. We give relations between these numbers and polynomials.

We modify (1.6) as follows:

$$\mathcal{F}_q^{(r)}(t,x) = F_q^{(r)}\left(q^{-x}t,x\right), \tag{2.1}$$

where $F_q^{(r)}(t,x)$ is defined in (1.6). From the above we find that

$$\mathcal{F}_q^{(r)}(t,x) = \sum_{n=0}^{\infty} q^{-nx} E_{n,q}^{(r)}(x)\frac{t^n}{n!}. \tag{2.2}$$

After some elementary calculations, we obtain

$$\mathcal{F}_q^{(r)}(t,x) = \exp\left([x]q^{-x}t\right)\mathfrak{f}_q^{(r)}(t), \tag{2.3}$$

where

$$\mathfrak{f}_q^{(r)}(t) = \left(2^r\sum_{m=0}^{\infty}(-1)^m\binom{m+r-1}{m}e^{[m]t}\right) = \sum_{n=0}^{\infty}E_{n,q}^{(r)}\frac{t^n}{n!}. \tag{2.4}$$

From the above we have

$$\mathcal{F}_q^{(r)}(t,x) = \sum_{n=0}^{\infty}\varepsilon_{n,q}^{(r)}(x)\frac{t^n}{n!}, \tag{2.5}$$

where

$$\varepsilon_{n,q}^{(r)}(x) = q^{-nx}E_{n,q}^{(r)}(x). \tag{2.6}$$

By using Cauchy product in (2.3), we arrive at the following theorem.

**Theorem 2.1.** *For $r \in \mathbb{N}$, and $n \geq 0$, one has*

$$\varepsilon_{n,q}^{(r)}(x) = \sum_{j=0}^{n}\binom{n}{j}q^{jx}[x]^{n-k}E_{j,q}^{(r)}. \tag{2.7}$$

By using (2.7), we easily obtain the following result.

**Corollary 2.2.** *For $r \in \mathbb{N}$, and $n \geq 0$, one has*

$$\varepsilon_{n,q}^{(r)}(x) = \sum_{m=0}^{\infty}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{j, l, n-j-l}\binom{n-j+m-1}{m}(-1)^{l}q^{m+x(l+j)}E_{j,q}^{(r)}. \tag{2.8}$$

We now give some identity related to Nörlund type Euler polynomials and numbers of higher-order.

Substituting $x = 0$ into (1.10), we find that

$$E_{n}^{(r)} = 2^{r}\sum_{n_1,n_2,\ldots,n_r=0}^{\infty}\sum_{\substack{j_1,\ldots,j_r=0\\ j_1+\cdots+j_r=n}}\binom{n}{j_1,\ldots,j_r}(-1)^{n_1+n_2+\cdots+n_r}\prod_{k=0}^{r}n_k^{j_k}. \tag{2.9}$$

By (1.10) and (2.9), we arrive at the following theorem.

**Theorem 2.3.** *For $r \in \mathbb{N}$, and $n \geq 0$, one has*

$$E_{n}^{(r)} = \sum_{j=0}^{n}\binom{n}{j}(-x)^{n-j}E_{j}^{(r)}(x). \tag{2.10}$$

By using (1.10) and [35, Theorem 3.6, page 7] , we easily arrive at the following result.

**Corollary 2.4.** *For $r, v \in \mathbb{N}$, and $n \geq 0$, one has*

$$\left(E^{(r)}(x) + E^{(v)}(y)\right)^{n} = \sum_{j=0}^{n}\binom{n}{j}x^{n-j}E_{j}^{(r+v)}(y), \tag{2.11}$$

*where $(E^{(r)}(x))^{n}$ is replaced by $E_{n}^{(r)}(x)$.*

## 3. Interpolation Function of Higher-Order Nörlund-Type $q$-Euler Polynomials

Recently, higher-order Bernoulli polynomials and Euler polynomials have studied by many mathematicians. Especially, in this paper, we study on higher-order Euler polynomials which are constructed by Kim (see, e.g., [14, 17, 21, 24, 27, 28, 33]) and see also the references cited in each of these earlier works.

In [20], by using the fermionic $p$-adic invariant integral on $\mathbb{Z}_p$, the set of $p$-adic integers, Kim gave a new construction of $q$-Genocchi numbers, Euler numbers of higher order. By using $q$-Genoucchi, Euler numbers of higher order, he investigated the interesting relationship between $w$-$q$-Euler polynomials and $w$-$q$-Genocchi polynomials. He also defined the multiple $w$-$q$-zeta functions which interpolate $q$-Genoucchi, Euler numbers of higher order.

By using similar method of the papers given by Kim [20, 21], in this section, applying derivative operator $d^k/dt^k|_{t=0}$ and Mellin Transformation to the generating functions of the higher-order Nörlund-type $q$-Euler polynomials, we give interpolation function of these polynomials.

By applying operator $d^k/dt^k|_{t=0}$ to (1.6), we obtain the following theorem.

**Theorem 3.1.** *Let $r, k \in \mathbb{Z}^+$ and $x \in \mathbb{R}$ with $0 < x \leq 1$. Then one has*

$$E_{k,q}^{(r)}(x) = 2^r \sum_{m=0}^{\infty} (-1)^m \binom{m+r-1}{m} [m+x]^k. \tag{3.1}$$

Let us define interpolation function of higher-order Nörlund-type $q$-Euler numbers as follows.

*Definition 3.2.* Let $q, s \in \mathbb{C}$ with $|q| < 1$, and $0 < x \leq 1$. Then we define

$$\zeta_q^{(r)}(s,x) = 2^r \sum_{n=0}^{\infty} \frac{(-1)^n \binom{n+r-1}{n}}{[n+x]^s}. \tag{3.2}$$

*Remark 3.3.* It holds that

$$\lim_{q \to 1} \zeta_q^{(r)}(s,x) = 2^r \sum_{n=0}^{\infty} \frac{(-1)^n \binom{n+r-1}{n}}{[n+x]^s}. \tag{3.3}$$

For detail about the above function (see [5–38, 44, 47]). By applying $d^k/dt^k|_{t=0}$ derivative operator to (1.10), we easily see that

$$\zeta^{(r)}(s,x) = 2^r \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{(-1)^{n_1+\cdots+n_r}}{\left(\sum_{j=1}^{r} n_j + x\right)^s}, \tag{3.4}$$

where $s \in \mathbb{C}$.

The functions in (3.3) and (3.4) interpolate same numbers at negative integers. That is, these functions interpolate higher-order Nörlund-type Euler numbers at negative integers. So, by (3.3), we modify (3.4) in sense of $q$-analogue.

In [3–51], many authors extensively have studied on similar type of (3.4).

In (3.3) and (3.4), setting $r = 1$, we have

$$\zeta^{(1)}(s,x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s} = \zeta_E(s,x) \tag{3.5}$$

where $\zeta_E(s,x)$ denotes Hurwitz type Euler zeta function, which interpolates classical Euler polynomials at negative integers.

**Theorem 3.4.** *Let $n \in \mathbb{Z}^+$. Then one has*

$$\zeta_q^{(r)}(-n, x) = E_{n,q}^{(r)}(x). \tag{3.6}$$

*Proof.* Substituting $s = -k, k \in \mathbb{Z}^+$ into (3.2). Then we have

$$\zeta_q^{(r)}(-k, x) = 2^r \sum_{n=0}^{\infty} (-1)^n \binom{n+r-1}{n} [n+x]^k. \tag{3.7}$$

Setting (3.1) into the above, and after some elementary calculations, we easily arrive at the desired result. □

By applying the Mellin transformation to (2.5), we find that

$$\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \mathcal{F}_q^{(r)}(-t, x) dt = 2^r \sum_{m=0}^{\infty} \frac{(-1)^m \binom{m+r-1}{m} q^{nxs}}{[m+x]^s}. \tag{3.8}$$

From the above we define the following function, which interpolate $\mathbf{E}_{n,q}^{(r)}(x)$ at negative integers.

*Definition 3.5.* Let $q, s \in \mathbb{C}$ with $|q| < 1$, and $0 < x \leq 1$. Then we define

$$\mathcal{Z}_q^{(r)}(s, x) = 2^r \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} (-1)^m \binom{r+m-1}{m} \binom{s+j-1}{j} q^{x(ns+j)+mj}. \tag{3.9}$$

**Theorem 3.6.** *Let $n \in \mathbb{Z}^+$. Then one has*

$$\mathcal{Z}_q^{(r)}(-n, x) = \varepsilon_{n,q}^{(r)}(x). \tag{3.10}$$

By Theorems 5, 6, a relation between the functions $\boldsymbol{\zeta}_q^{(r)}(-n, x)$ and $\zeta_q^{(r)}(-n, x)$ is given by the following corollary.

**Corollary 3.7.**

$$\mathcal{Z}_q^{(r)}(-n, x) = q^{-nx} \zeta_q^{(r)}(-n, x). \tag{3.11}$$

*Remark 3.8.* Recently many authors have studied on the Riemann zeta function, Hurwitz zeta function, Lerch zeta function, Dirichlet series for the polylogarithm function and Dirichlet's eta function and the other functions. The *Lerch trancendent* $\Phi(z, s, a)$ is the analytic continuation of the series

$$\Phi(z, s, a) = \frac{1}{a^s} + \frac{z}{(a+1)^s} + \frac{z}{(a+2)^s} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \tag{3.12}$$

which converges for ($a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|z| < 1$; $\Re(s) > 1$ when $|z| = 1$), where as usual

$$\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}, \quad \mathbb{Z}^- = \{-1, -2, -3, \ldots\}. \tag{3.13}$$

However, $\Phi$ denotes the familiar Hurwitz-Lerch Zeta function (cf. e. g., [8], [49, page 121 et seq.]). Some special cases of the function $\Phi(z, s, a)$ are given by the following relations (e.g., and details see [8], [49, page 121 et seq.]):

(1)the Riemann zeta function

$$\Phi(1, s, 1) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1, \tag{3.14}$$

(2) the Hurwitz zeta function

$$\Phi(1, s, a) = \zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad \Re(s) > 1, \tag{3.15}$$

(3) the Dirichlet's eta function

$$\Phi(-1, s, 1) = \zeta^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \tag{3.16}$$

(4) the Dirichlet beta function

$$\frac{\Phi(-1, s, 1/2)}{2^s} = \beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}, \tag{3.17}$$

(5) the Legendre chi function

$$\frac{z\Phi(z^2, s, 1/2)}{2^s} = \chi_s(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^s}, (|z| \leq 1; \Re(s) > 1), \tag{3.18}$$

(6) the polylogarithm

$$z\Phi(z, n, 1) = Li_m(z) = \sum_{n=0}^{\infty} \frac{z^k}{n^m}, \tag{3.19}$$

(7) the Lerch zeta function (sometimes called the Hurwitz-Lerch zeta function)

$$L(\lambda, \alpha, s) = \Phi\left(e^{2\pi i \lambda}, s, \alpha\right), \tag{3.20}$$

which is a special function and generalizes the Hurwitz zeta function and polylogarithm cf. [6, 8, 20, 46, 49] and see also the references cited in each of these earlier works. Consequently, the functions $\mathfrak{Z}_q^{(r)}(-n,x)$ and $\zeta_q^{(r)}(-n,x)$ are related to the Hurwitz-Lerch zeta function and the other special functions, which are defined above:

$$2\Phi(-1,s,x) = \zeta^{(1)}(s,x) = \zeta_E(s,x). \tag{3.21}$$

## 4. Some Relations Related to Higher-Order Nörlund $q$-Euler Polynomials

In this section, by using generating function of higher-order Nörlund $q$-Euler polynomials, which is defined by Kim [20, 21], we obtain the following identities.

**Theorem 4.1.** *Let $q \in \mathbb{C}$ with $|q| < 1$. Let $r$ be a positive integer. Then one has*

$$E_{k,q}^{(r)}(x) = 2^r \sum_{j=0}^{k}\sum_{a=0}^{j}(-1)^a \binom{k}{a, j-a, k-j} \frac{q^{ja}[x]^{k-j}}{(1-q)^j\left(1+q^{k-j}\right)^{r-1}}. \tag{4.1}$$

*Proof.* By using (3.1), we have

$$\begin{aligned}
E_{k,q}^{(r)}(x) &= 2^r \sum_{m=0}^{\infty}(-1)^m \binom{m+r-1}{m}\left([m]+q^m[x]\right)^k \\
&= 2^r \sum_{m=0}^{\infty}(-1)^m \binom{m+r-1}{m}\sum_{j=0}^{k}\binom{k}{j}[m]^j q^{m(k-j)}[x]^{k-j} \\
&= 2^r \sum_{m=0}^{\infty}(-1)^m \binom{m+r-1}{m}\sum_{j=0}^{k}\binom{k}{j}\frac{(1-q^m)^j}{(1-q)^j} q^{m(k-j)}\cdot[x]^{k-j} \\
&= 2^r \sum_{m=0}^{\infty}(-1)^m \binom{m+r-1}{m}\sum_{j=0}^{k}\sum_{a=0}^{j}\frac{\binom{k}{j}\binom{j}{a}(-1)^a q^{aj+m(k-j)}\cdot[x]^{k-j}}{(1-q)^j} \\
&= 2^r \sum_{j=0}^{k}\sum_{a=0}^{j}\frac{\binom{k}{j}\binom{j}{a}(-1)^a q^{ja}\cdot[x]^{k-j}}{(1-q)^j}\sum_{m=0}^{\infty}(-1)^m \binom{m+r-1}{m} q^{m(k-j)} \\
&= 2^r \sum_{j=0}^{k}\sum_{a=0}^{j}\frac{\binom{k}{j}\binom{j}{a}(-1)^a q^{ja}\cdot[x]^{k-j}}{(1-q)^j\left(1+q^{k-j}\right)^{r-1}} \\
&= 2^r \sum_{j=0}^{k}\sum_{a=0}^{j}(-1)^a \binom{k}{a, j-a, k-j}\frac{q^{ja}[x]^{k-j}}{(1-q)^j\left(1+q^{k-j}\right)^{r-1}}.
\end{aligned} \tag{4.2}$$

Thus,we complete the proof. □

**Theorem 4.2.** *Let $q \in \mathbb{C}$ with $|q| < 1$. Let $r$ be a positive integer. Then one has*

$$E_{n,q}^{(r)}(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} q^{jx} \left(1-q^j\right)^{-r} (1-q)^{-n}. \tag{4.3}$$

*Proof.* By using (1.6)

$$\begin{aligned} F_q^{(r)}(t,x) &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m e^{[m+x]t} \\ &= 2^r \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \binom{m+r-1}{m} (-1)^m \left(\frac{1-q^{m+x}}{1-q}\right)^n \frac{t^n}{n!} \\ &= 2^r \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\binom{m+r-1}{m}(-1)^m}{(1-q)^n n!} \left(\sum_{j=0}^{n} \binom{n}{j} (-1)^j \cdot q^{jx+jm}\right) t^n \\ &= 2^r \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{n}{j}(-1)^j . q^{jx}}{(1-q)^n} \cdot \frac{t^n}{n!} \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m \cdot q^{jm} \end{aligned} \tag{4.4}$$

Thus we have;

$$\sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{t^n}{n!} = 2^r \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} (-1)^j q^{jx} \left(1-q^j\right)^{-r} \left(\frac{t}{1-q}\right)^n \frac{1}{n!} \right). \tag{4.5}$$

By comparing the coefficients $t^n/n!$ both sides in the above, we arrive at the desired result. □

**Theorem 4.3.** *Let $r, y \in \mathbb{Z}^+$. Then one has*

$$\begin{aligned} &\sum_{j=0}^{k} \binom{k}{j} E_{j,q}^{(r)}(x) E_{k-j,q}^{(y)}(x) \\ &\quad = 2^{r+y} \sum_{n=0}^{\infty}\sum_{j=0}^{n} (-1)^n \binom{j+r-1}{j} \binom{n-j+y-1}{n-j} \left([x+y] + [n-j+x]\right)^k. \end{aligned} \tag{4.6}$$

*Proof.* By using (1.6), we have

$$\begin{aligned} &\sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} E_{n,q}^{(y)}(x) \frac{t^n}{n!} \\ &\quad = 2^{r+y} \sum_{n=0}^{\infty} (-1)^n \binom{n+r-1}{n} e^{[n+x]t} \sum_{n=0}^{\infty} (-1)^n \binom{n+y-1}{n} e^{[n+x]t}. \end{aligned} \tag{4.7}$$

By using Cauchy product into the above, we obtain

$$\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}E_{j,q}^{(r)}(x)E_{n-j,q}^{(y)}(x)\frac{1}{j!(n-j)!}\right)t^{n} = 2^{r+y}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{j+r-1}{j}\binom{n-j+y-1}{n-j}(-1)^{n}e^{[j+x]t}e^{[n-j+x]t}\right), \tag{4.8}$$

From the above, we have

$$\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}E_{j,q}^{(r)}(x)E_{m-j,q}^{(y)}(x)\frac{1}{j!(m-j)!}\right)t^{m} = \sum_{m=0}^{\infty}\left(2^{r+y}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{j+r-1}{j}\binom{n-j+y-1}{n-j}\left([j+x]+[n-j+x]\right)^{m}\right)t^{m} \tag{4.9}$$

By comparing the coefficients of both sides of $t^n$ in the above we arrive at the desired result. □

*Remark 4.4.* In (4.1) setting $y = 1$, we have

$$\sum_{j=0}^{m}\binom{m}{j}E_{j,q}^{(r)}(x)E_{m-j,q}(x) = 2^{r+1}\sum_{n=0}^{\infty}(-1)^{n}\sum_{j=0}^{n}\binom{j+r-1}{j}\left([j+x]+[n-j+x]\right)^{m}. \tag{4.10}$$

The above relations give us (3.1) related to (4.1).

## 5. Further Remarks and Observations on Approximation

Apostol [1, page 481] gave Weierstrass theorem as follows.

**Theorem 5.1.** *Let $f$ be real valued and continuous on a closed interval $[a,b]$. Then, given any $\varepsilon > 0$, there exists a polynomial $p$ (which may be depend on $\varepsilon$) such that*

$$\left|f(x)-p(x)\right| < \varepsilon, \tag{5.1}$$

*for every $x \in [a,b]$.*

According to Apostol [1]; the above theorem is described by saying that every continuous function can be "uniformly approximated" by a polynomial.

We now give, more useful, and more interesting result concerning the approximation by polynomials which is related tothe Bernstein polynomials (cf. [1, 2, 4, 34, 39]).

*Definition 5.2.* ([2]) Let $f$ be a function with domain $I = [0,1]$ and range $R$. The $n$th Bernstein polynomial for $f$ is defined to be

$$B_n(x) = B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}. \tag{5.2}$$

These Bernstein polynomials are not only used with probability (the Binomial Distribution) but also used in the approximation theory.

Let $f$ be continuous on $I$ with values in $\mathbb{R}$. Then the sequence of Bernstein polynomials for $f$, defined in (5.2) converges uniformly on $I$ to $f$ (cf.[2]).

In [4], Costabile and Dell'Accio collected classical and more recent results on polynomial approximation of sufficiently regular real functions defined in bounded closed intervals by means of boundary values only. Their problem is considered from the point of view of the existence of explicit formulas, interpolation to boundary data, bounds for the remainder and convergence of the polynomial series. Applications to some problems of numerical analysis are pointed out, such as nonlinear equations, numerical differentiation and integration formulas, specially associated differential boundary value problems. Some polynomial expansions for smooth enough functions defined in rectangles or in triangles of $\mathbb{R}^2$ as well as in cuboids or in tetrahedrons in $\mathbb{R}^3$ and their applications are also discussed. They also used Bernoulli and Euler polynomials for the *polynomial approximation* cf. see for detail [4].

Lopez and Temme [34] studied on uniform approximations of the Bernoulli and Euler polynomials for large values of the order in terms of hyperbolic functions. They obtained convergent expansions for

$$B_n\left(nz + \frac{1}{2}\right), \qquad E_n\left(nz + \frac{1}{2}\right) \tag{5.3}$$

in powers of $1/n$, and coefficients are rational functions of $z$ and hyperbolic functions of argument $1/2z$, here $B_n(x)$ and $E_n(x)$ denote Bernoulli and Euler polynomials, respectively. Their expansions are uniformly valid for $|z \pm i/2\pi| > 1/2\pi$ and $|z \pm i/\pi| > 1/\pi$, respectively. For a real argument, the accuracy of these approximations is restricted to the monotonic region cf. see for detail [34].

Recently, many authors studied on very different type of the *approximation theory*. Consequently, by using the above motivations, we conclude this section by the following questions:

Bernoulli functions and Euler functions are related to trigonometric polynomials cf. [46]. Approximation by $q$-analogue of these functions may be possible.

(1) *whether or not define better uniform approximations for the Nörlund $q$-Euler polynomials higher order;*

(2) *is it possible to define uniform expansions of the Nörlund $q$-Euler polynomials higher order?*

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