On Multiple Simple Recourse models

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Abstract

We consider multiple simple recourse (MSR) models, both continuous and integer versions, which generalize the corresponding simple recourse (SR) models by allowing for a refined penalty cost structure for individual shortages and surpluses.

It will be shown that (convex approximations of) such MSR models can be represented as explicitly specified continuous SR models, and thus can be solved efficiently by existing algorithms.

Keywords: simple recourse, integer recourse, convex approximation

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1. Introduction

Consider the linear programming problem

$$\begin{array}{ll} \min\limits_{x} & cx \\ \text{s.t.} & Ax = b \\ & Tx = \omega \\ & x \in \mathbb{R}^n_+. \end{array} \tag{1}$$

Such models are successfully applied to analyze a wide variety of practical problems. However, the implicit assumption that all parameters of the model are known (deterministic) is often not satisfied in practice. For example, the constraints $Tx = \omega$ in model (1) could represent the goal to satisfy future demand ω for certain goods at minimal costs cx; however, the exact amount of this demand is not known at the time that the decisions x need to be made. In stochastic programming models, uncertainty about the value of parameters is modeled by assuming that they can be represented by *random variables*, with known distribution.

Obviously, (1) is not well defined if the right-hand side vector $\omega \in \mathbb{R}^m$ is random, as we will assume. To arrive at a meaningful model, it needs to be extended. A well-known way to do this, is by formulating a (two-stage) *recourse model*, see e.g. [1, 5, 13]. The underlying idea is that the decision x is made before knowing the realization of ω, and then, once the realization is known, deviations $\omega - Tx$ have to be corrected at certain (minimal) costs. The costs of these corrections or *recourse actions*, given by the *recourse cost function* v as $v(\omega - Tx)$, are computed for each possible realization of the random parameters ω; weighted with their respective probabilities, they constitute the *expected recourse costs* $\mathcal{Q}(x) := \mathbb{E}_\omega [v(\omega - Tx)]$ corresponding to a given decision x. The criterion to choose x thus becomes minimal total expected costs, consisting of direct costs cx and expected recourse costs $\mathcal{Q}(x)$. The conceptual model is

$$\min_{x \in X} cx + \mathcal{Q}(x),$$

with $X := \{x \in \mathbb{R}^n_+ : Ax = b\}$, which is specified except for the recourse cost function v. This function is commonly defined as the value function of a second linear program, called the *second-stage* problem: for $s \in \mathbb{R}^m$,

$$\begin{array}{lll} v(s) := & \min\limits_{y} & qy \\ & \text{s.t.} & Wy = s \\ & & y \in Y, \end{array}$$

where W is called the *recourse matrix*, q specifies the recourse unit cost parameters,

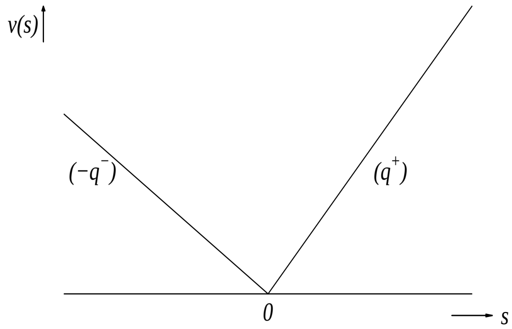

Figure 1.1: Simple recourse penalty costs ($m = 1$).

and Y describes the feasible set of recourse actions.

This specification of the recourse model allows for a lot of flexibility: many choices for the recourse structure (q, W, Y) are possible. A well-known special case is *simple recourse*, defined by the recourse structure $W = (I_m, \ -I_m)$, with I_m the $m \times m$ identity matrix, $q = (q^+, q^-)$, and $Y = \mathbb{R}^{2m}_+$, so that v is given by

$$v(s) = \begin{array}{ll} \min\limits_y & q^+ y^+ + q^- y^- \\ \text{s.t.} & y^+ - y^- = s \\ & y^+, y^- \in \mathbb{R}^m_+ \end{array} , \qquad s \in \mathbb{R}^m.$$

Using the *separability* of the function v, and under the usual assumption that the cost parameters satisfy $q^+ + q^- \geq 0$ (component wise), it holds $v(s) = \sum_{i=1}^m v_i(s_i)$, where each function v_i is given in closed form:

$$v_i(s_i) = q_i^+ (s_i)^+ + q_i^- (s_i)^-,$$

with $(u)^+ := \max\{u, 0\}$, $(u)^- := \max\{-u, 0\}$, $u \in \mathbb{R}$.

The interpretation of this recourse structure is that *linear penalty costs* are assigned to both shortages and surpluses with respect to each constraint $T_i x = \omega_i$, $i = 1, \ldots, m$, *individually*. See Figure 1.1.

Separability of the function v is the key to very efficient algorithms [1, 5, 13], allowing to solve simple recourse models of high dimensions. Indeed, for problems with hun-

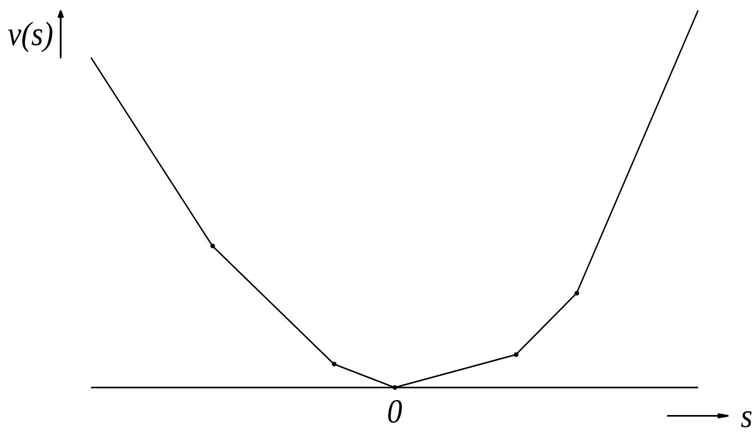

Figure 1.2: Multiple simple recourse penalty costs ($m = 1$).

dreds of random variables, as for example is the case in some engineering applications, a simple recourse formulation is the only feasible choice from a computational point of view.

On the other hand, in many applications linear penalty costs are not realistic or desirable. In [6] Klein Haneveld proposed the *multiple simple recourse* model, which is a generalization of the simple recourse model allowing for *piecewise linear* convex penalty cost functions, see Figure 1.2. Although attractive from a modeling point of view (see Section 2), this model has not been used much in practice because no (efficient) algorithms were available.

Below we will show that every multiple simple recourse model can be transformed to an equivalent simple recourse model. Consequently, such models can be solved by existing algorithms for simple recourse models.

Following the discussion on continuous multiple simple recourse models, we present corresponding results for the integer version, which is obtained by setting $Y \subset \mathbb{Z}_+^{2m}$ instead of $Y \subset \mathbb{R}_+^{2m}$. Properties and algorithms for simple integer recourse models are discussed in [12, 15, 8, 10, 9].

2. Examples of piecewise linear penalty cost functions

Before going into technical details, we motivate our interest in multiple simple recourse models by sketching some applications which call for piecewise linear penalty cost functions.

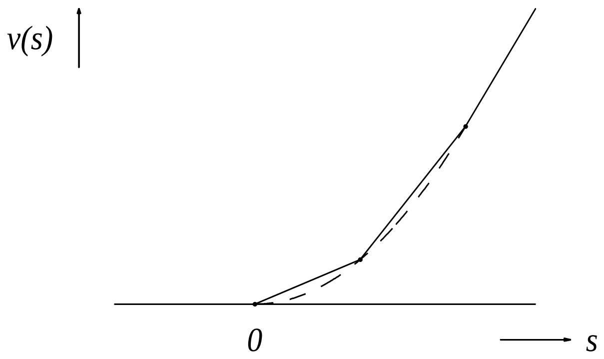

Figure 2.1: Piecewise linear approximation of a smooth penalty function.

In an early computational paper, Dupačová et al. [3] discuss stochastic programming models and solution techniques for a water management problem. For the recourse version of their model, they consider various types of penalty functions for floods, lack of irrigation, and decrease of recreation. They advocate the use of piecewise linear penalty functions as approximations of more general convex functions. However, in the actual computations (one-sided) simple recourse penalty functions are used, presumably to keep the computation time within reasonable bounds. Figure 2.1 shows an example of a piecewise linear approximation of a smooth (one-sided) penalty function.

In [11] we proposed models for optimizing electricity distribution in the Netherlands. One of the problems considered is the daily planning of the power supply to be obtained from so-called small generators (e.g. green houses and hospitals), in addition to the usual supply from power plants. Each contract between the distributor and such a small generator specifies, among a number of other constraints, lower and upper bounds for the total supply in a given year. On day t of that year, it is known how much supply has been obtained from a small generator so far, so that today's supply x_t and the uncertain supply in the rest of the year ω_t should satisfy $l_t \leq x_t + \omega_t \leq u_t$, where l_t and u_t denote the adapted lower and upper bounds, respectively. Unit penalty costs for violations of these bounds are known. In addition, to reserve some flexibility for future decisions,

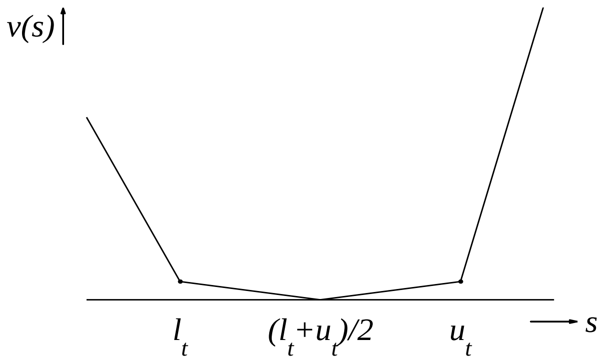

Figure 2.2: Penalty function of electricity distribution model.

the management of the electricity distributor prefers to 'stay away from the bounds'. This preference is modeled by assigning small unit costs to deviations of $x_t + \omega_t$ from $(l_t + u_t)/2$. Figure 2.2 shows an example of the resulting piecewise linear penalty cost function.

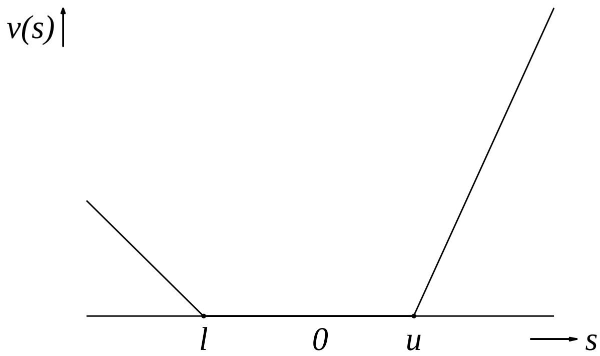

Figure 2.3: Penalty function of ALM model.

In [2] we described a (preliminary version of) a recourse model for the Asset Liability Management problem for pension funds. One of the novel aspects of this model is that we explicitly consider the stability of contribution rates. This is implemented by assigning penalty costs to changes in this rate from one period to the next, if they exceed a certain amount. Both the unit penalty cost and the threshold value may be different for decreases and increases. An example of the corresponding penalty cost function is shown in Figure 2.3.

3. Multiple simple recourse

To set the stage for our results on multiple simple recourse models, we first review some well-known properties of the simple recourse expected value function and related functions.

3.1 Simple recourse functions

As stated in the introduction, the simple recourse model is

$$\min cx + \mathcal{Q}(x) : x \in X$$

with

$$\mathcal{Q}(x) := \mathbb{E}_\omega \left[v(\omega - Tx) \right], \qquad x \in \mathbb{R}^n,$$

$$v(s) := \sum_{i=1}^{m} \left(q_i^+ (s_i)^+ + q_i^- (s_i)^- \right), \qquad s \in \mathbb{R}^m,$$

assuming that $q^+ + q^- \geq 0$.

Because of the separability of the function v, the expected value function $\mathcal{Q}$ is separable in the *tender variables* $z := Tx$:

$$\mathcal{Q}(x) = \sum_{i=1}^{m} Q_i(z_i)$$

with

$$Q_i(z_i) := q_i^+ \mathbb{E}_{\omega_i} \left[(\omega_i - z_i)^+ \right] + q_i^- \mathbb{E}_{\omega_i} \left[(\omega_i - z_i)^- \right], \quad z_i \in \mathbb{R}, \tag{2}$$

which is finite if and only if $\mathbb{E}_{\omega_i} \left[|\omega_i| \right]$ is finite, which we assume from now on.

Since each of these functions $Q_i : \mathbb{R} \mapsto \mathbb{R}$ has the same structure, we analyze the function $\mathcal{Q}$ by studying the generic function

$$Q(x) := \mathbb{E}_\omega \left[v(\omega - x) \right], \quad x \in \mathbb{R},$$

where ω is a one-dimensional random variable, with cumulative distribution function (cdf) denoted by F and mean value μ.

For $x \in \mathbb{R}$, we define the *expected surplus function*

$$G(x) := \mathbb{E}_\omega \left[(\omega - x)^+ \right]$$

and the *expected shortage function*

$$H(x) := \mathbb{E}_\omega \left[(\omega - x)^- \right],$$

so that $Q(x) = q^+ G(x) + q^- H(x)$. Hence, properties of the function Q follow trivially from properties of the functions G and H, which are easily derived using the following well-known formulae. For $x \in \mathbb{R}$,

$$G(x) = \int_x^\infty \big(1 - F(t)\big)\, dt \qquad \text{and} \qquad H(x) = \int_{-\infty}^x F(t) dt. \tag{3}$$

In particular, in the next section we will use the following properties.

Lemma 3.1 *(i) The simple recourse functions G, H, and Q are finite, convex, and Lipschitz continuous on $\mathbb{R}$.*

(ii) The right derivative of the function G equals $G'_+(x) = F(x) - 1$, $x \in \mathbb{R}$. The asymptotes of G are given by $\mu - x$ as $x \to -\infty$, and 0 as $x \to \infty$.

(iii) The right derivative of the function H equals $H'_+(x) = F(x)$, $x \in \mathbb{R}$. The asymptotes of H are given by 0 as $x \to -\infty$, and $x - \mu$ as $x \to \infty$.

(iv) The right derivative of the function Q equals $Q'_+(x) = q^+ \big(F(x) - 1\big) + q^- F(x)$, $x \in \mathbb{R}$. The asymptotes of Q are given by $-q^+(x - \mu)$ as $x \to -\infty$, and $q^-(x - \mu)$ as $x \to \infty$. □

3.2 Multiple simple recourse functions

The recourse structure of the multiple simple recourse (MSR) model is chosen such that its value function v assigns piecewise linear penalty costs to individual shortages and surpluses. Like in the simple recourse case, the value function is separable. To avoid unnecessary notational burden, we restrict the detailed presentation to the one-dimensional case. For $s \in \mathbb{R}$,

$$\begin{aligned} v(s) := \min_{y \geq 0} \quad & \sum_{k=1}^{K} \left(q_k^+ y_k^+ + q_k^- y_k^- \right) \\ \text{s.t.} \quad & \sum_{k=1}^{K} y_k^+ - \sum_{k=1}^{K} y_k^- = s \\ & \left.\begin{array}{l} y_k^+ \leq u_k - u_{k-1}, \\ y_k^- \leq l_k - l_{k-1}, \end{array}\right. \quad k = 1, \ldots, K-1 \end{aligned}$$

with $u_0 = l_0 = 0$ and

$$\begin{array}{l} 0 \leq q_1^+ \leq \ldots \leq q_{K-1}^+ \leq q_K^+ \\ 0 \leq u_1 \leq \ldots \leq u_{K-1} \\ 0 \leq q_1^- \leq \ldots \leq q_{K-1}^- \leq q_K^- \\ 0 \leq l_1 \leq \ldots \leq l_{K-1} \end{array} \tag{4}$$

That is, corresponding to each linear part of this penalty cost function, an (upper bounded) variable is defined. Due to the conditions on the cost coefficients, the resulting function is convex. See Figure 3.1 for an example of such a function v.

REMARK. For completeness, we state that the recourse matrix of the m-dimensional MSR second-stage problem is given by

$$\begin{pmatrix} e_1 & & & & -e_1 & & & \\ & e_2 & & & & -e_2 & & \\ & & \ddots & & & & \ddots & \\ & & & e_m & & & & -e_m \end{pmatrix},$$

where e_i is a K_i-vector of ones, $i = 1, \ldots, m$.

It is obvious that the MSR structure is *complete*, i.e., the value function v satisfies $v(s) < \infty$ for all $s \in \mathbb{R}^m$. MSR problems can therefore be solved using algorithms for general complete recourse problems. This would be ill-advised, however, since such algorithms do not take advantage of the special structure of MSR problems.

REMARK. Without loss of generality we define v in terms of the same number of variables/intervals for both shortage and surplus. (Use e.g. $q_{k+1}^- = q_k^-$ for $k \geq K^-$ to

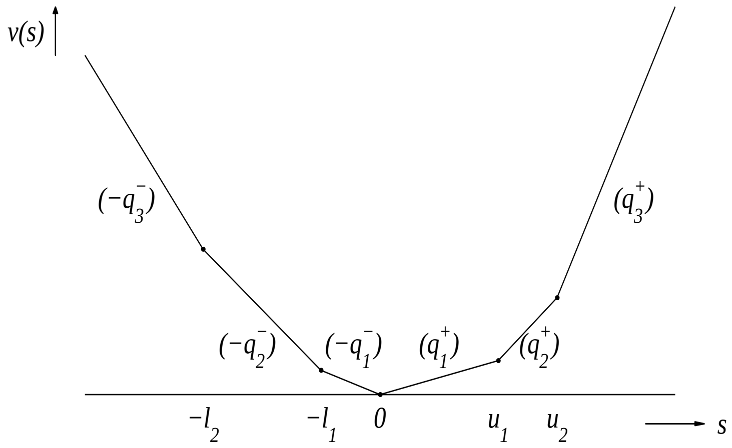

Figure 3.1: Example of a multiple simple recourse value function ($m = 1$, $K = 3$).

obtain an MSR value function with $K^- < K$ intervals corresponding to shortages.)

REMARK. Taking $K = 1$, we obtain the simple recourse value function as a special case of multiple simple recourse.

For a fixed $s \in \mathbb{R}$, it is easy to determine $v(s)$ by constructing an optimal solution of the defining second-stage problem. For example, if $s > 0$ then $y_k^- = 0$ for all k, and the variables y_k^+ are chosen such that $\sum_k y_k^+ = s$, with $y_{k+1}^+ > 0$ only if y_k^+ is equal to its upper bound. For $s \leq 0$, an optimal solution can be constructed analogously.

Using straightforward computation, we find the following closed form for v. For $s \in \mathbb{R}$,

$$v(s) = \sum_{k=0}^{K-1} \left[\left(q_{k+1}^+ - q_k^+\right)(s - u_k)^+ + \left(q_{k+1}^- - q_k^-\right)(s + l_k)^- \right], \qquad (5)$$

where we conveniently define $q_0^+ = q_0^- = 0$.

By definition, the expected value function $Q(x)$, $x \in \mathbb{R}$, equals the expectation with respect to ω of $v(\omega - x)$. Thus, using (5) we find that

$$Q(x) = \sum_{k=0}^{K-1} \Big[\left(q_{k+1}^+ - q_k^+\right) G(x + u_k) + \left(q_{k+1}^- - q_k^-\right) H(x - l_k) \Big], \quad x \in \mathbb{R}, \quad (6)$$

with G and H defined by (3), is a closed form for the (one-dimensional) multiple simple recourse expected value function.

For later reference, we now present several properties of the function Q.

Lemma 3.2 *Consider the multiple simple recourse expected value function Q, given in closed form by (6).*

(i) The function Q is finite, convex, and Lipschitz continuous on $\mathbb{R}$.

(ii) The right derivative of Q exists everywhere. For $x \in \mathbb{R}$, it is given by

$$Q'_+(x) = \sum_{k=0}^{K-1} \Big[\left(q_{k+1}^+ - q_k^+\right) F(x + u_k) + \left(q_{k+1}^- - q_k^-\right) F(x - l_k) \Big] - q_K^+.$$

Moreover,

$$\lim_{x \to -\infty} Q'_+(x) = -q_K^+ \qquad \textit{and} \qquad \lim_{x \to \infty} Q'_+(x) = q_K^-.$$

(iii) The asymptotes for Q are given by

$$q_K^+(\mu - x) - \sum_{k=0}^{K-1} \left(q_{k+1}^+ - q_k^+\right) u_k, \qquad \textit{as } x \to -\infty,$$

$$q_K^-(x - \mu) - \sum_{k=0}^{K-1} \left(q_{k+1}^- - q_k^-\right) l_k, \qquad \textit{as } x \to \infty.$$

PROOF. Immediate from applying Lemma 3.1 to the individual terms of (6). $\square$

Example 1. Consider the MSR value function v, defined by the parameters $K = 3$, $u = [2,\ 5]$, $q^+ = [1,\ 2,\ 3]$, $l = [1,\ 4]$, and $q^- = [1,\ 3,\ 4]$,

$$\begin{aligned} v(s) = \min_{y \geq 0} \quad & y_1^+ + 2y_2^+ + 3y_3^+ + y_1^- + 3y_2^- + 4y_3^- \\ \text{s.t.} \quad & y_1^+ + y_2^+ + y_3^+ - y_1^- - y_2^- - y_3^- = s \quad , \qquad s \in \mathbb{R}, \\ & y_1^+ \leq 2,\ y_2^+ \leq 5 \\ & y_1^- \leq 1,\ y_2^- \leq 4 \end{aligned}$$

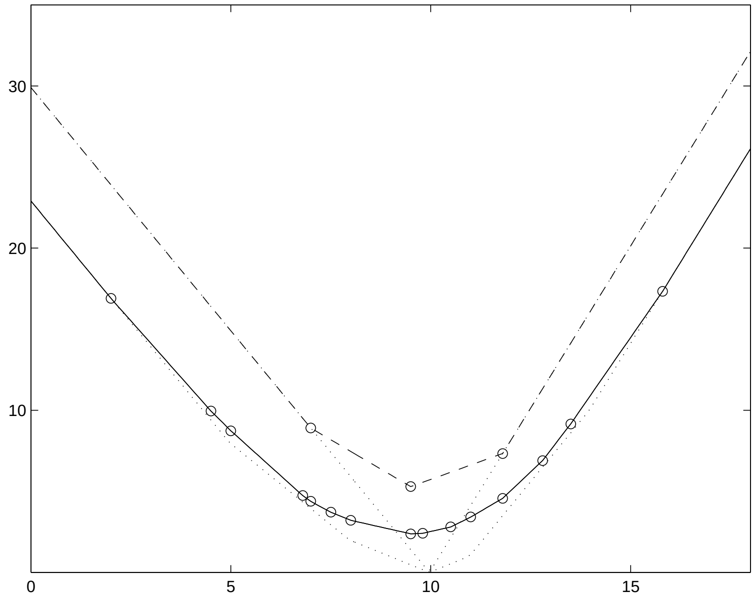

Figure 3.2: MSR (solid) and SR (dashed) expected value functions and value functions $v(\mu - s)$ (dotted) of Example 1.

and the corresponding expected value function $Q(x) = \mathbb{E}_\omega[v(\omega - x)]$, where ω is a discrete random variable with $\Pr\{\omega = 7\} = 2/9$, $\Pr\{\omega = 9.5\} = 3/9$, and $\Pr\{\omega = 11.8\} = 4/9$, so that $\mu = 299/30$. The functions $v(s - \mu)$ and $Q(x)$ are shown in Figure 3.2.

Figure 3.2 also shows the SR value function defined by the cost parameters $q^+ = q_K^+ = 3$ and $q^- = q_K^- = 4$, and the corresponding expected value function. The MSR structure defined above can be seen as a refinement of this SR structure: smaller deviations are penalized with lower unit costs, whereas large deviations are treated in the same way in both models.

Besides obvious differences, there are some structural similarities between the two expected value functions: both are piecewise linear and convex, and, moreover, their asymptotes have the same slopes $-q_K^+$ and q_K^-. $\triangleleft$

As suggested by Example 1, the mathematical properties of the multiple simple recourse (MSR) expected value function are very similar to those of its simple recourse (SR) counterpart. This strong relationship is exploited in the next section.

4. Relation between MSR and SR

The following result is proven in [7], and repeated here for easy reference.

Theorem 4.1 (Theorem 3.1 in [7]) *Let φ be a finite, convex, Lipschitz continuous function on $\mathbb{R}$. Define*

$$a_1 = -\lim_{z\to-\infty} \varphi'_+(z), \qquad a_2 = \lim_{z\to\infty} \varphi'_+(z).$$

Assume that $a_1 + a_2 \neq 0$. Then

$$\Phi(s) = \frac{\varphi'_+(s) + a_1}{a_1 + a_2}$$

is a cdf.

If φ has an asymptote for $z \to \infty$, say $\varphi(z) \sim a_2 z + c_2$ as $z \to \infty$, then

$$\varphi(z) = a_2 z + c_2 + (a_1 + a_2) \int_z^\infty \big(1 - \Phi(s)\big)\, ds.$$

If φ has an asymptote for $z \to -\infty$, say $\varphi(z) \sim -a_1 z + c_1$ as $z \to -\infty$, then

$$\varphi(z) = -a_1 z + c_1 + (a_1 + a_2) \int_{-\infty}^z \Phi(s)\, ds.$$

If both asymptotes exist, then

$$\varphi(z) = a_1 \int_z^\infty \big(1 - \Phi(s)\big)\, ds + a_2 \int_{-\infty}^z \Phi(s)\, ds + \frac{a_1 c_2 + a_2 c_1}{a_1 + a_2}.$$

□

Loosely speaking, Theorem 4.1 states that every function that is similar to an SR expected value function can be represented as such a function. Next, we apply this result to the MSR expected value function.

Corollary 4.1 *Consider the multiple simple recourse expected value function Q, given in closed form by (6). Then*

$$Q(x) = q_K^+ \mathbb{E}_\xi \left[(\xi - x)^+ \right] + q_K^- \mathbb{E}_\xi \left[(\xi - x)^- \right] - C, \quad x \in \mathbb{R},$$

where ξ is a random variable with cdf V,

$$V(t) = \frac{\sum_{k=0}^{K-1} \left[\left(q_{k+1}^+ - q_k^+ \right) F(t + u_k) + \left(q_{k+1}^- - q_k^- \right) F(t - l_k) \right]}{q_K^+ + q_K^-}, \quad t \in \mathbb{R},$$

with F the cdf of ω. The constant C is given by

$$C = \frac{q_K^+ \sum_{k=1}^{K-1} \left(q_{k+1}^- - q_k^- \right) l_k + q_K^- \sum_{k=1}^{K-1} \left(q_{k+1}^+ - q_k^+ \right) u_k}{q_K^+ + q_K^-}.$$

PROOF. The result follows from Theorem 4.1 and the properties of Q presented in Lemma 3.2. $\square$

REMARK. Corollary 4.1 can also be stated in terms of random variables. To this end, define the discrete random variable η, independent of ω, with

$$\Pr\{\eta = l_k\} = \frac{q_{k+1}^- - q_k^-}{q_K^+ + q_K^-}, \quad k = 1, \ldots, K-1,$$

$$\Pr\{\eta = 0\} = \frac{q_1^+ + q_1^-}{q_K^+ + q_K^-},$$

$$\Pr\{\eta = -u_k\} = \frac{q_{k+1}^+ - q_k^+}{q_K^+ + q_K^-}, \quad k = 1, \ldots, K-1.$$

Then, in the setting of Corollary 4.1, $\xi = \omega + \eta$ with cdf V and

$$C = q_K^+ \mathbb{E}_\eta \left[(\eta)^+ \right] + q_K^- \mathbb{E}_\eta \left[(\eta)^- \right].$$

The interpretation of Corollary 4.1 is that every (one-dimensional) MSR expected value function is *equivalent to an SR expected value function*, which is *explicitly given* in terms of its parameters q^+, q^-, and the distribution of the random variable ξ, plus a known constant.

Recall that the full-dimensional expected value function $\mathcal{Q}$ is defined as $\mathcal{Q}(x) = \sum_{i=1}^{m} Q_i(T_i x)$, $x \in \mathbb{R}^n$, where each Q_i is a one-dimensional expected value function as considered in Corollary 4.1. Extending this result to the function $\mathcal{Q}$ in the obvious way, it follows that *MSR models can be solved by existing algorithms for SR models*, which are very efficient (see e.g. [1]). Indeed, only the following preprocessing steps are needed: for $i = 1, \dots, m$,

(i) Compute the constant C_i;

(ii) Construct the distribution of ξ_i, by applying the specified transformation of the distribution of ω_i.

If all components of ω are discretely distributed, the resulting distribution of (the components of) ξ is also discrete, and can be specified directly, without reference to the distribution function. Because this special case is important for applications, this result is stated separately.

Corollary 4.2 *Assume the setting of Corollary 4.1.*

Consider the case that ω is a discrete random variable with support Ω. Then the random variable ξ is discretely distributed, with support

$$\Xi = \bigcup_{k=0}^{K-1} \Big\{ \{\Omega - u_k\} \cup \{\Omega + l_k\} \Big\}$$

and probabilities

$$\Pr\{\xi = \bar{\xi}\} = \frac{\sum_{k=0}^{K-1} \Big[\big(q_{k+1}^{+} - q_k^{+}\big) \Pr\{\omega = \bar{\xi} + u_k\} + \big(q_{k+1}^{-} - q_k^{-}\big) \Pr\{\omega = \bar{\xi} - l_k\} \Big]}{q_K^{+} + q_K^{-}}.$$

Note that if Ω is finite, then the cardinality of Ξ is bounded by $|\Omega|(2K-1)$.

PROOF. The cdf F of ω is piecewise constant, with discontinuities in Ω. Hence, by Corollary 4.1, the cdf V of ξ is piecewise constant with discontinuities in Ξ, so that ξ is a discrete random variable with support Ξ. The corresponding probabilities are given by $V(\bar{\xi}) - \lim_{s \uparrow \bar{\xi}} V(s)$, $\bar{\xi} \in \Xi$. $\square$

Example 2. Consider the MSR value function v, defined by the parameters $K = 2$,

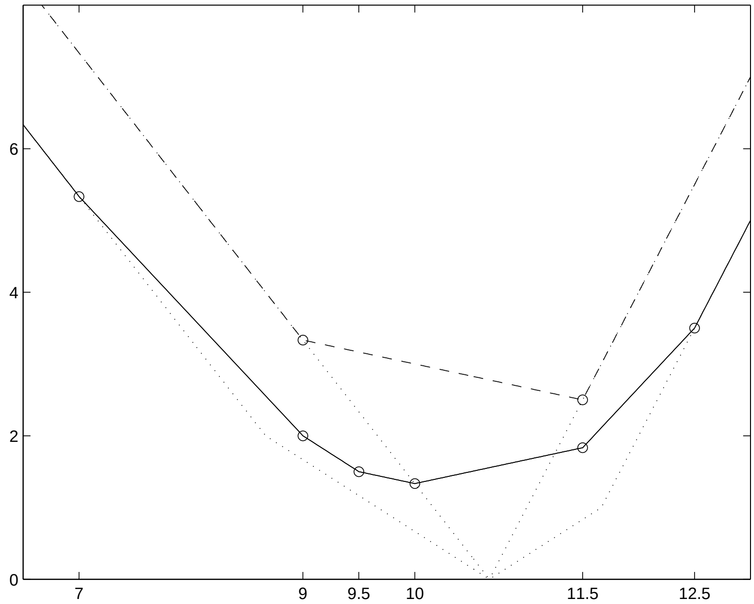

Figure 4.1: The functions Q and $v(\mu - s)$ of Example 2. Also shown are the SR expected value function (with respect to ω; dashed) and the corresponding value function.

$u_1 = 2$, $q^+ = [1,\ 2]$, $l_1 = 1$, and $q^- = [1,\ 3]$,

$$v(s) = \begin{array}{ll} \min\limits_{y \geq 0} & y_1^+ + 2y_2^+ + y_1^- + 3y_2^- \\ \text{s.t.} & y_1^+ + y_2^+ - y_1^- - y_2^- = s \\ & y_1^+ \leq 2 \\ & y_1^- \leq 1 \end{array} \quad , \qquad s \in \mathbb{R},$$

and the corresponding expected value function $Q(x) = \mathbb{E}_\omega [v(\omega - x)]$, where ω is a discrete random variable with realizations $\omega_1 = 9$ and $\omega_2 = 11.5$, with $\Pr\{\omega = \omega_1\} = 1/3$, $\Pr\{\omega = \omega_2\} = 2/3$, so that $\mu = 32/3$. Figure 4.1 shows these functions Q and $v(\mu - s)$, $s \in \mathbb{R}$.

According to Corollaries 4.1 and 4.2

$$Q(x) = 2\mathbb{E}_\xi \left[(\xi - x)^+\right] + 3\mathbb{E}_\xi \left[(\xi - x)^-\right] - 2, \quad x \in \mathbb{R},$$

where ξ is the discrete random variable with support

$$\begin{aligned}\Xi &= \{9,\ 11.5\} \cup \{9 - u_1,\ 11.5 - u_1\} \cup \{9 + l_1,\ 11.5 + l_1\} \\ &= \{7,\ 9,\ 9.5,\ 10,\ 11.5,\ 12.5\}.\end{aligned}$$

To compute e.g. $\Pr\{\xi = 10\}$ we observe that 10 can be written as $\omega_1 + l_1$ (no other suitable combinations of ω and $-u$ or l), so that

$$\Pr\{\xi = 10\} = \frac{q_2^- - q_1^-}{q_2^+ + q_2^-} \Pr\{\omega = \omega_1\} = 2/15.$$

In the same way, we find that the respective probabilities for $\xi \in \Xi$ are given by $p_\xi = [1,\ 2,\ 2,\ 2,\ 4,\ 4]/15$. $\triangleleft$

The algorithm as described above is implemented as Mscr2Scr 1.0 (Multiple simple continuous recourse to Simple continuous recourse, M.H. van der Vlerk and J. Mayer, 2001) in the model management system SLP-IOR [4]. The current version of Mscr2Scr is restricted to MSR problems with discrete random variables.

5. Multiple simple integer recourse

We now turn to the integer version of the MSR model, which we will denote as multiple simple integer recourse (MSIR).

5.1 Definition and closed forms

Like in the continuous recourse case, MSIR is a generalization of the simple integer recourse (SIR) model, allowing for a refined penalty cost structure for (individual) surpluses and shortages.

If the integrality restrictions in the SIR model are relaxed, then it is equal to a continuous SR model. Similarly, the LP relaxation of the MSIR model is an MSR model. Thus, not surprisingly, the MSIR value function is separable so that we can again restrict the analysis to the one-dimensional version.

For $s \in \mathbb{R}$, the one-dimensional MSIR value function is defined as

$$
\begin{aligned}
v(s) := \min_{y \geq 0} \quad & \sum_{k=1}^{K} \left(q_k^+ y_k^+ + q_k^- y_k^- \right) \\
\text{s.t.} \quad & \sum_{k=1}^{K} y_k^+ \geq s, \quad \sum_{k=1}^{K} y_k^- \geq -s \\
& \left.\begin{array}{l} y_k^+ \leq u_k - u_{k-1}, \\ y_k^- \leq l_k - l_{k-1}, \end{array}\right. \quad k = 1, \dots, K-1 \\
& y \in \mathbb{Z}^{2K}
\end{aligned}
$$

with $u_0 = l_0 = 0$, the vectors u and l integer, and the elements of q^+, u, q^-, and l satisfying the same monotonicity assumptions (4) as in the continuous recourse setting.

REMARK. For $K = 1$, we obtain the value function of the (one-dimensional) SIR problem.

REMARK. Note that this second-stage problem is defined using two inequalities instead of the single equality used in the continuous recourse case. This is necessary, since the right-hand side parameter s can be any real number, whereas the left-hand side is integral by definition.

Using the monotonicity of the cost coefficients, it is straightforward to determine $v(s)$ for any fixed $s \in \mathbb{R}$. This leads to the following closed form for v:

$$
v(s) = \sum_{k=0}^{K-1} \Big[\left(q_{k+1}^+ - q_k^+\right) \lceil s - u_k \rceil^+ + \left(q_{k+1}^- - q_k^-\right) \lfloor s + l_k \rfloor^- \Big], \quad s \in \mathbb{R},
$$

where, as before, we define $q_0^+ = q_0^- = 0$. For $t \in \mathbb{R}$, $\lceil t \rceil^+$ denotes the positive part of the integer round up of t, and $\lfloor t \rfloor^-$ is the negative part of the integer round down of t.

By definition, the (one-dimensional) MSIR expected value function Q is obtained as the expectation of $v(\omega - x)$, $x \in \mathbb{R}$, giving

$$
Q(x) = \sum_{k=0}^{K-1} \Big[\left(q_{k+1}^+ - q_k^+\right) \mathcal{G}(x + u_k) + \left(q_{k+1}^- - q_k^-\right) \mathcal{H}(x - l_k) \Big], \quad x \in \mathbb{R}, \quad (7)
$$

where

$$\begin{aligned}
\mathcal{G}(x) &:= \mathbb{E}_\omega\left[\lceil \omega - x \rceil^+\right] = \sum_{j=0}^{\infty} \left(1 - F(x+j)\right), \\
\mathcal{H}(x) &:= \mathbb{E}_\omega\left[\lfloor \omega - x \rfloor^-\right] = \sum_{j=0}^{\infty} F(x-j),
\end{aligned}$$

with F cdf of ω.

The formulae above and structural properties of the SIR functions $\mathcal{G}$, $\mathcal{H}$, and Q (with $K = 1$) are derived in [12] and [15]. In particular, it holds that these functions are non-convex in general, so that Theorem 4.1 can not be applied to obtain a simple recourse representation of the MSIR function Q.

Instead, we will show that reasonable convex approximations of the MSIR function Q can be constructed, and that every such approximation is equivalent to a *continuous* SR expected value function. Consequently, MSIR problems can be solved approximately by solving a continuous SR problem.

5.2 Convex approximations of MSIR

Consider the LP relaxation of the MSIR second-stage problem. Obviously, the expected value function of the resulting MSR problem is a convex lower bound for the MSIR function Q. It is also not difficult to see that a convex upper bound is obtained by adding $q_K^+ + q_K^-$ to this MSR function. Since any reasonable convex approximation of Q should be in between these trivial approximations, it follows that such an approximation has asymptotes with the same slopes, as specified by Lemma 3.2. Since every such approximation clearly satisfies the assumptions of Theorem 4.1, it follows that every reasonable convex approximation of the MSIR function Q can be represented as a continuous SR expected value function (plus a constant). For the convex approximations presented below, an explicit representation will be presented in the next section.

To construct non-trivial convex approximations of the MSIR function Q, we follow the ideas that were developed for the corresponding SIR function (i.e., for $K = 1$).

In [8] a strongly polynomial algorithm is presented for the construction of the convex hull of the SIR expected value function for the case that ω is a *discrete random variable*. The algorithm is based on structural properties of the function Q (lower semicontinuous, piecewise constant) which are shared by the MSIR version, so that in principle the same approach can be used to construct the convex hull of the latter function. It is also clear, however, that many technicalities need to be handled to obtain this result. This matter will not be worked out here.

If ω is *continuously distributed*, the function $\mathcal{Q}$ (both SIR and MSIR version) is continuous, but non-convex in general. It is convex if and only if the probability density function (pdf) of ω belongs to a certain class, which is completely specified in [10]. The main idea of [9] is to approximate the original pdf of ω by a specific (family of) density functions from this class, thus constructing a convex approximation of the SIR function $\mathcal{Q}$. Below we apply this approach to the MSIR function $\mathcal{Q}$.

As shown in [10], the SIR functions $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{Q}$ are convex if ω has a pdf that is piecewise constant on every interval $(\alpha + j, \alpha + j + 1)$, $j \in \mathbb{Z}$, for some $\alpha \in [0, 1)$. It follows trivially that also the MSIR function $\mathcal{Q}$ is convex in this case. Such piecewise constant densities, depending on the shift parameter α, will be used to approximate arbitrary density functions. In this way, we obtain convex approximations of the MSIR function $\mathcal{Q}$.

Definition 5.1 *Let F be the cdf of a continuous random variable. For any $\alpha \in [0, 1)$, the α-approximation of the pdf f of F is defined as*

$$f_\alpha(s) := F(\lfloor s \rfloor_\alpha + 1) - F(\lfloor s \rfloor_\alpha), \quad s \in \mathbb{R},$$

where $\lfloor s \rfloor_\alpha = \lfloor s - \alpha \rfloor + \alpha$ is the round down with respect to $\alpha + \mathbb{Z}$.

Denoting by ω_α the continuous random variable with pdf f_α, the α-approximation of the function $\mathcal{G}$ is defined as

$$\mathcal{G}_\alpha(x) := \mathbb{E}_{\omega_\alpha}\left[\lceil \omega_\alpha - x \rceil^+\right], \quad x \in \mathbb{R},$$

which is convex by construction. The convex function $\mathcal{H}_\alpha$ is defined analogously.

It follows that the α-approximation of the MSIR function $\mathcal{Q}$, defined as

$$\begin{aligned} \mathcal{Q}_\alpha(x) \;&:=\; \sum_{k=0}^{K-1} \Big[\left(q_{k+1}^+ - q_k^+\right) \mathbb{E}_{\omega_\alpha}\left[\lceil \omega_\alpha - x \rceil^+\right] \\ &\qquad\qquad + \left(q_{k+1}^- - q_k^-\right) \mathbb{E}_{\omega_\alpha}\left[\lfloor \omega_\alpha - x \rfloor^-\right] \Big] \qquad (8) \\ &= \sum_{k=0}^{K-1} \Big[\left(q_{k+1}^+ - q_k^+\right) \mathcal{G}_\alpha(x + u_k) + \left(q_{k+1}^- - q_k^-\right) \mathcal{H}_\alpha(x - l_k) \Big], \quad x \in \mathbb{R}, \end{aligned}$$

is a convex function.

Note that α-approximations can be defined without making any assumption about the distribution of ω. Thus, convex approximations Q_α of the MSIR function Q can also be constructed in case ω is discretely distributed. However, in that case the following uniform bound on the approximation error does not apply; in fact, it can be shown that the error then equals $q_K^+ + q_K^-$ in the worst case.

Corollary 5.1 *Assume that ω has a pdf f that is of bounded variation. Then, for all $\alpha \in [0, 1)$,*

$$\|Q_\alpha - Q\|_\infty \leq \frac{q_K^+ + q_K^-}{4} |\Delta| f,$$

where $|\Delta| f$ denotes the total variation of f on $\mathbb{R}$.

PROOF. In [9] it is shown that both $\|\mathcal{G}_\alpha - \mathcal{G}\|_\infty$ and $\|\mathcal{H}_\alpha - \mathcal{H}\|_\infty$ are bounded by $|\Delta| f/4$. The result follows by summation. □

5.3 Representation of Q_α as SR expected value function

In the previous section we defined α-approximations for multiple simple integer recourse models. We conclude our discussion on MSIR models by providing a continuous simple recourse representation of such approximations.

Corollary 5.2 *For a fixed but arbitrary $\alpha \in [0, 1)$, consider the α-approximation Q_α of the MSIR expected value function Q, as defined in (8). Then*

$$Q_\alpha(x) = q_K^+ \mathbb{E}_{\xi_\alpha}\left[(\xi_\alpha - x)^+\right] + q_K^- \mathbb{E}_{\xi_\alpha}\left[(\xi_\alpha - x)^-\right] + D, \quad x \in \mathbb{R},$$

where ξ_α is a random variable with cdf V_α,

$$V_\alpha(t) = \frac{\sum_{k=0}^{K-1} \left[\left(q_{k+1}^+ - q_k^+\right) F(\lfloor t \rfloor_\alpha + u_k) + \left(q_{k+1}^- - q_k^-\right) F(\lfloor t \rfloor_\alpha + 1 - l_k)\right]}{q_K^+ + q_K^-}, \quad t \in \mathbb{R},$$

with F the cdf of ω. That is, ξ_α is discretely distributed, with support contained in $\alpha + \mathbb{Z}$ and probabilities

$$\begin{aligned}\Pr\{\xi_\alpha = \alpha + j\} \quad = \quad & \frac{1}{q_K^+ + q_K^-} \sum_{k=0}^{K-1} \Big[\left(q_{k+1}^+ - q_k^+\right) \Pr\{\omega \in \Omega_\alpha^j + u_k\} \\ & + \left(q_{k+1}^- - q_k^-\right) \Pr\{\omega \in \Omega_\alpha^{j+1} - l_k\}\Big], \quad j \in \mathbb{Z},\end{aligned}$$

where $\Omega_\alpha^j := (\alpha + j - 1, \alpha + j]$.

The constant D *is given by*

$$D = \frac{q_K^+ \, q_K^-}{q_K^+ + q_K^-} - C,$$

where C *is the constant given in Corollary 4.1.*

PROOF. The result follows from applying Theorem 4.1 to the function Q_α. To this end, we first derive the required information on properties of Q_α by studying the constituting functions $\mathcal{G}_\alpha$ and $\mathcal{H}_\alpha$.

As shown in [9], the function $\mathcal{G}_\alpha$ can be written as

$$\mathcal{G}_\alpha(x) = \mathbb{E}_{\xi_\alpha^1}\left[\left(\xi_\alpha^1 - x\right)^+\right], \quad x \in \mathbb{R},$$

where ξ_α^1 is a random variable with cdf $F(\lfloor t \rfloor_\alpha)$. Let μ_α^1 denote the mean value of ξ_α^1. Similarly,

$$\mathcal{H}_\alpha(x) = \mathbb{E}_{\xi_\alpha^2}\left[\left(\xi_\alpha^2 - x\right)^-\right], \quad x \in \mathbb{R},$$

where ξ_α^2 is a random variable with cdf $F(\lfloor t \rfloor_\alpha + 1)$, so that the mean value μ_α^2 of ξ_α^2 equals $\mu_\alpha^1 - 1$.

By Lemma 3.1 we have, for $x \in \mathbb{R}$,

$$(\mathcal{G}_\alpha)'_+(x) = F(\lfloor x \rfloor_\alpha) - 1 \quad \text{and} \quad (\mathcal{H}_\alpha)'_+(x) = F(\lfloor x \rfloor_\alpha + 1),$$

so that

$$\begin{aligned}(Q_\alpha)'_+(x) \quad = \quad \sum_{k=0}^{K-1} \Big[& \left(q_{k+1}^+ - q_k^+\right)\left(F(\lfloor x \rfloor_\alpha + u_k) - 1\right) \\ & + \left(q_{k+1}^- - q_k^-\right) F(\lfloor x \rfloor_\alpha + 1 - l_k)\Big],\end{aligned}$$

and

$$\lim_{x \to -\infty} (Q_\alpha)'_+(x) = -q_K^+, \qquad \lim_{x \to \infty} (Q_\alpha)'_+(x) = q_K^-.$$

Moreover, since $\mathcal{G}_\alpha$ has asymptotes $\mu_\alpha^1 - x$ as $x \to -\infty$ and 0 as $x \to \infty$, and $\mathcal{H}_\alpha$ has asymptotes 0 as $x \to -\infty$ and $x - \mu_\alpha^2$ as $x \to \infty$, it follows that the asymptotes for

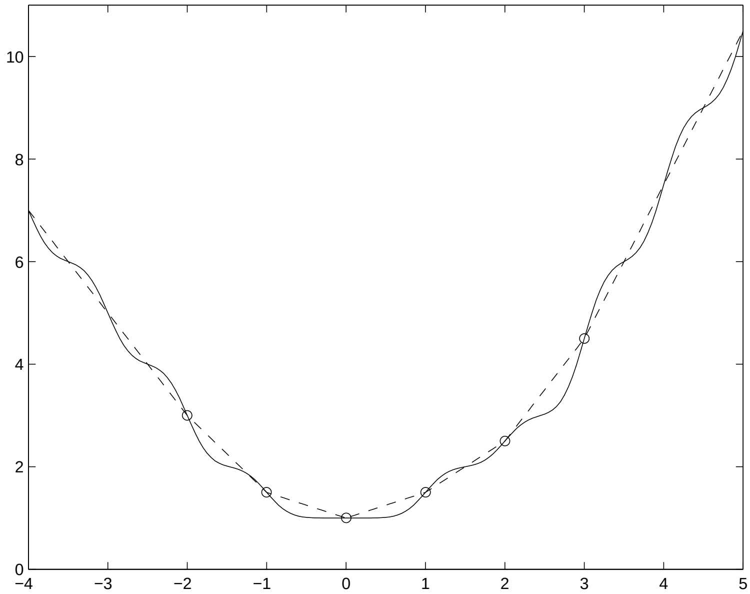

Figure 5.1: The functions Q (solid) and Q_α (dashed) of Example 3.

Q_α are given by

$$q_K^+(\mu_\alpha^1 - x) - \sum_{k=0}^{K-1} \left(q_{k+1}^+ - q_k^+\right) u_k, \qquad \text{as } x \to -\infty,$$

$$q_K^-(x - \mu_\alpha^2) - \sum_{k=0}^{K-1} \left(q_{k+1}^- - q_k^-\right) l_k, \qquad \text{as } x \to \infty.$$

Using this information, the result follows from Theorem 4.1 by straightforward computation. $\square$

Example 3. Consider the MSIR value function v, defined by the parameters $K = 2$, $u_1 = 2$, $q^+ = [1,\ 2]$, $l_1 = 3$, and $q^- = [1,\ 3]$,

$$
\begin{aligned}
v(s) = \min_{y\geq 0} \quad & y_1^+ + 2y_2^+ + y_1^- + 3y_2^- \\
\text{s.t.} \quad & y_1^+ + y_2^+ - y_1^- - y_2^- = s \quad , \qquad s \in \mathbb{R}, \\
& y_1^+ \leq 2 \\
& y_1^- \leq 3 \\
& y \in \mathbb{Z}^4
\end{aligned}
$$

the corresponding expected value function $Q(x) = \mathbb{E}_\omega\left[v(\omega - x)\right]$, where ω is a normal random variable with mean value μ and variance σ^2, and the α-approximation Q_α.

Figure 5.1 shows the functions Q and Q_α for the case that $\mu = 0$, $\sigma^2 = 0.05$, and $\alpha = 0$. The random variable ξ_α, as defined in Corollary 5.2, has support $\{-2, -1, \dots, 3\}$ with respective probabilities $(1, 2, 2, 1, 2, 2)/10$ (computation based on truncation of the support of ω to $[-4\sigma, 4\sigma] \subset (-1, 1)$, so that $\Pr\{\omega \in \Omega_\alpha^j\} = 1/2$ for $j = 0, 1$). The constant $D = -2.4$. $\triangleleft$

6. Summary and conclusion

Starting from the well-known (integer) simple recourse (SR) model, we developed efficient solution methods for the corresponding multiple simple recourse (MSR) models. Such MSR models are generalizations of SR models in that they allow for a refinement of the penalty cost structure, which makes them attractive from an application point of view.

We have shown that MSR models (or convex approximations in the integer case) can be represented as explicitly specified continuous SR models, and thus can be solved efficiently by existing algorithms for continuous SR models.

Apart from a trivial adaptation of the cost parameters, the continuous SR representation of MSR models is obtained by a particular transformation of the underlying distribution of the random right-hand side parameters. It is common practice in stochastic programming to replace a given distribution by a suitable approximation. The main example is of course the use of discrete approximations (e.g., empirical distributions) of continuous distributions in recourse models. However, the approach used in this paper is conceptually different, because it transforms problems of a more general or difficult type into a well-solved, easy problem type.

Recently we have successfully applied this approach to models with a more general (non-separable) integer recourse structure [16]. In the near future, we will extend this line of research to models with non-linear separable penalty cost functions (e.g. piecewise quadratic, see [14]).

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